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NUMERICAL SOLUTION OF INITIAL-BOUNDARY SYSTEM OF NONLINEAR HYPERBOLIC EQUATIONS

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In this article, we present a numerical approximation of the initial-boundary system of nonlinear hyperbolic equations based on spectral Jacobi-Gauss-Radau collocation (J-GR-C) method. A J-GR-C method in combination with the implicit Runge-Kutta scheme are employed to obtain a highly accurate approximation to the mentioned problem. J-GR-C method, based on Jacobi polynomials and Gauss-Radau quadrature integration, reduces solving the system of nonlinear hyperbolic equations to solve a system of nonlinear ordinary differential equations (SNODEs). In the examples given, numerical results by the J-GR-C method are compared with the exact solutions. In fact, by selecting relatively few J-GR-C points, we are able to get very accurate approximations. In this way, the results show that this method has a good accuracy and efficiency for solving coupled partial differential equations.

Key words : System of nonlinear hyperbolic equations; collocation method; Jacobi-Gauss-Radau quadrature; implicit Runge-Kutta method.

1. INTRODUCTION

Hyperbolic partial differential equation (HPDE) is a partial differential equation (PDE) that has initial value conditions for the first $\mu - 1$ derivatives, where μ is the order of PDE. A simple example of a HPDE is the wave equation. HPDEs describe a wide range of problems in various fields of science and engineering such as the phenomena of turbulence and supersonic flow, flow of a shock wave traveling in a non-viscous and viscous fluid [1], sedimentation of two kinds of particles in fluid suspensions under the effect of gravity, acoustic transmission [2], traffic and aerofoil flow theory [3], hypoelastic solids [4], process engineering [5], population based modeling and batch crystallization [6], geophysics [7], astrophysics [8], and many other disciplines. In the few past decades, analytical and numerical solutions of HPDEs [9-12], parabolic PDEs [13-16] and evolution equations [17, 18] have more interesting of scientific research.

There are great efforts to develop approximate numerical methods to solve PDE. The preference between the numerical methods depends firstly on the accuracy yields by the used method. Spectral method is one of the more accurate numerical computational methods for solving linear and non-linear initial-boundary PDEs [19, 20]. There exist three primary types of spectral methods, namely, the Galerkin [21, 22], tau [23, 24] and collocation [25-27] methods. A significant advantage of the spectral collocation method over the others spectral methods is that, it was easily implemented for approximating different problems such as nonlinear differential equations [28-30], integral equations [31, 32], integro-differential equations [33, 34], fractional orders differential equations [35, 39], function approximation and variational problems. According to exponential rate of convergence obtained by collocation method, it is very useful in providing highly accurate solutions.

The Jacobi polynomials are the eigenfunctions of the singular Sturm-Liouville problem. There are several particular cases of Jacobi polynomials such as Gegenbauer, Legendre and Chebyshev polynomials [40]. Also, the Jacobi polynomials have been used in a variety of applications due to their ability to approximate general classes of functions, some of which are the resolution of the Gibbs' phenomenon [41], electrocardiogram data compression [42] and the solution to differential equations [43, 44].

The main purpose of this paper is to extend the application of spectral Jacobi-Gauss-Radau collocation method to solve two systems of nonlinear Hyperbolic equations. We apply the J-GR-C method to reduce such problems to systems of ordinary differential equations in time. Implicit Runge-Kutta scheme is then employed for obtaining the numerical solution for the mentioned system of ordinary differential equations. To the best of our knowledge, there are no results on Jacobi-Gauss-Radau collocation method for solving such problems.

Throughout the paper, we present some properties of Jacobi polynomials in the next section. Section 3 is devoted to develop the numerical analysis of the Jacobi collocation method for solving the coupled nonlinear hyperbolic system of first order. Three test problems are proposed in Section 4 to show the accuracy of the method. In the final section, we present some observations and conclusions.

2. Some Properties of Jacobi Polynomials

We present in this section some basic knowledge of Jacobi polynomials that are most relevant to spectral approximations [45]. It includes Legendre and Chebyshev polynomials as two special cases, so it is worthwhile work with a general Jacobi polynomials. A basic property of the Jacobi polynomials is that they are the eigenfunctions to a singular Sturm-Liouville problem:

$$(1-x^2)\phi^{''}(x)+[\vartheta-\theta+(\theta+\vartheta+2)x]\phi^{'}(x)+n(n+\theta+\vartheta+1)\phi(x)=0. \tag{1}$$

The Jacobi polynomials are generated from the three-term recurrence relation:

$$\begin{aligned}
&P_{k+1}^{(\theta,\vartheta)}(x)=(a_k^{(\theta,\vartheta)}x-b_k^{(\theta,\vartheta)})P_k^{(\theta,\vartheta)}(x)-c_k^{(\theta,\vartheta)}P_{k-1}^{(\theta,\vartheta)}(x), \quad k\geq 1,\\
&P_0^{(\theta,\vartheta)}(x)=1, \quad P_1^{(\theta,\vartheta)}(x)=\frac{1}{2}(\theta+\vartheta+2)x+\frac{1}{2}(\theta-\vartheta),
\end{aligned}$$

where

$$\begin{aligned}
a_k^{(\theta,\vartheta)}&=\frac{(2k+\theta+\vartheta+1)(2k+\theta+\vartheta+2)}{2(k+1)(k+\theta+\vartheta+1)},\\
b_k^{(\theta,\vartheta)}&=\frac{(\vartheta^2-\theta^2)(2k+\theta+\vartheta+1)}{2(k+1)(k+\theta+\vartheta+1)(2k+\theta+\vartheta)},\\
c_k^{(\theta,\vartheta)}&=\frac{(k+\theta)(k+\vartheta)(2k+\theta+\vartheta+2)}{(k+1)(k+\theta+\vartheta+1)(2k+\theta+\vartheta)}.
\end{aligned}$$

The Jacobi polynomials satisfy the following relations

$$P_k^{(\theta,\vartheta)}(-x)=(-1)^kP_k^{(\theta,\vartheta)}(x), \quad P_k^{(\theta,\vartheta)}(-1)=\frac{(-1)^k\Gamma(k+\vartheta+1)}{k!\Gamma(\vartheta+1)}. \tag{2}$$

Moreover, the q derivative of Jacobi polynomials of degree k $P_k^{(\theta,\vartheta)}(x)$, can be obtained from:

$$D^{(q)}P_k^{(\theta,\vartheta)}(x) = \frac{\Gamma(j+\theta+\vartheta+q+1)}{2^q\Gamma(j+\theta+\vartheta+1)}P_{k-q}^{(\theta+q,\vartheta+q)}(x). \tag{3}$$

Let $w^{(\theta,\vartheta)}(x) = (1-x)^\theta(1+x)^\vartheta$, then we define the weighted space $L^2_{w^{(\theta,\vartheta)}}$ as usual. The inner product and the norm of $L^2_{w^{(\theta,\vartheta)}}$ with respect to the weight function are defined as follows:

$$(u,v)_{w^{(\theta,\vartheta)}} = \int_{-1}^{1} u(x)\,v(x)w^{(\theta,\vartheta)}(x)dx, \qquad \|u\|_{w^{(\theta,\vartheta)}} = (u,u)^{\frac{1}{2}}_{w^{(\theta,\vartheta)}}. \tag{4}$$

The set of Jacobi polynomials forms a complete $L^2_{w^{(\theta,\vartheta)}}$-orthogonal system, and

$$\| P_k^{(\theta,\vartheta)} \|_{w^{(\theta,\vartheta)}} = h_k = \frac{2^{\theta+\vartheta+1}\Gamma(k+\theta+1)\Gamma(k+\vartheta+1)}{(2k+\theta+\vartheta+1)\Gamma(k+1)\Gamma(k+\theta+\vartheta+1)}. \tag{5}$$

It is well known that the Legendre polynomials, the Chebyshev polynomials of the first, second, third and fourth kinds, and the ultraspherical polynomials are special cases of the Jacobi polynomials. Therefore, this work covers all the previous mentioned polynomials. More specifically, Legendre, Chebyshev and ultraspherical spectral collocation methods can be obtained as special cases from the proposed method.

3. Jacobi Spectral Collocation Method

The main objective of this section is to develop the J-GR-C method to numerically solve the system of nonlinear hyperbolic equations.

Firstly, we are interested in using the J-GR-C method to transform the following PDE into SNODEs

$$\begin{aligned} &\frac{\partial u(y,t)}{\partial t} + \frac{\partial u(y,t)}{\partial y} + \eta u(y,t)v(y,t) = 0, \\ &\frac{\partial v(y,t)}{\partial t} - \frac{\partial v(y,t)}{\partial y} + \eta u(y,t)v(y,t) = 0, \quad (y,t) \in [A,B]\times[0,T], \end{aligned} \tag{6}$$

where η is a constant parameter, with the boundary-initial conditions

$$\begin{aligned} &u(A,t) = g_1(t), \quad v(A,t) = g_2(t), \quad t\in[0,T], \\ &u(y,0) = f_1(y), \quad v(y,0) = f_2(y), \quad y\in[A,B]. \end{aligned} \tag{7}$$

Now, suppose the change of variables $x = \frac{2}{B-A}y + \frac{A+B}{A-B}$, $\quad u(y,t) = r(x,t)$, $\quad v(y,t) = s(x,t)$, which will be used to transform problem (6-7) into another one in the classical interval, $[-1,1]$, for the space variable, to directly implement collocation method based on Jacobi family defined on $[-1,1]$,

$$\begin{aligned}
&\frac{\partial r(x,t)}{\partial t} + \left(\frac{2}{B-A}\right)\frac{\partial r(x,t)}{\partial x} + \eta r(x,t)s(x,t) = 0,\\
&\frac{\partial s(x,t)}{\partial t} - \left(\frac{2}{B-A}\right)\frac{\partial s(x,t)}{\partial x} + \eta r(x,t)s(x,t) = 0, \quad (x,t) \in [-1,1]\times[0,T],
\end{aligned} \tag{8}$$

with the boundary-initial conditions

$$\begin{aligned}
&r(-1,t) = g_3(t), \quad s(-1,t) = g_4(t), \quad t \in [0,T],\\
&r(x,0) = f_3(x), \quad s(x,0) = f_4(x), \quad x \in [-1,1].
\end{aligned} \tag{9}$$

Now, we outline the main step of the J-GR-C method for solving system of nonlinear hyperbolic equations. In the Jacobi collocation method, the two approximate solutions can be expressed as a truncated Jacobi series:

$$\begin{aligned}
r(x,t) &= \sum_{j=0}^{N} a_j(t) P_j^{(\theta,\vartheta)}(x),\\
s(x,t) &= \sum_{j=0}^{N} b_j(t) P_j^{(\theta,\vartheta)}(x),
\end{aligned} \tag{10}$$

and in virtue of (4-5), we deduce that

$$\begin{aligned}
a_j(t) &= \frac{1}{h_j}\int_{-1}^{1} r(x,t)\, w^{(\theta,\vartheta)}(x)\, P_j^{(\theta,\vartheta)}(x)dx,\\
b_j(t) &= \frac{1}{h_j}\int_{-1}^{1} s(x,t)\, w^{(\theta,\vartheta)}(x)\, P_j^{(\theta,\vartheta)}(x)dx.
\end{aligned} \tag{11}$$

To evaluate the previous integrals accurately, we present the Jacobi-Gauss-Radau quadrature. For any $\phi \in S_{2N+1}[-1,1]$,

$$\int_{-1}^{1} w^{(\theta,\vartheta)}(x)\phi(x)dx = \sum_{j=0}^{N} \varpi_{N,j}^{(\theta,\vartheta)} \phi(x_{N,j}^{(\theta,\vartheta)}), \tag{12}$$

where $S_N[-1,1]$ is the set of polynomials of degree less than or equal to N, $x_{N,j}^{(\theta,\vartheta)}$ $(0 \leq j \leq N)$ and $\varpi_{N,j}^{(\theta,\vartheta)}$ $(0 \leq j \leq N)$ are the nodes and the corresponding Christoffel numbers of the Jacobi-Gauss-Radau quadrature formula on the interval $[-1,1]$, respectively. In accordance to (4) the coefficients $a_j(t)$ in terms of the solution at the collocation points can be approximated by

$$\begin{aligned} a_j(t) &= \frac{1}{h_j}\sum_{i=0}^{N} P_j^{(\theta,\vartheta)}(x_{N,i}^{(\theta,\vartheta)})\varpi_{N,i}^{(\theta,\vartheta)}\, r(x_{N,i}^{(\theta,\vartheta)},t),\\ b_j(t) &= \frac{1}{h_j}\sum_{i=0}^{N} P_j^{(\theta,\vartheta)}(x_{N,i}^{(\theta,\vartheta)})\varpi_{N,i}^{(\theta,\vartheta)}\, s(x_{N,i}^{(\theta,\vartheta)},t). \end{aligned} \tag{13}$$

Therefore, (10) can be rewritten as

$$\begin{aligned} r(x,t) &= \sum_{i=0}^{N}\left(\sum_{j=0}^{N}\frac{1}{h_j}P_j^{(\theta,\vartheta)}(x_{N,i}^{(\theta,\vartheta)})P_j^{(\theta,\vartheta)}(x)\varpi_{N,i}^{(\theta,\vartheta)}\right) r(x_{N,i}^{(\theta,\vartheta)},t),\\ s(x,t) &= \sum_{i=0}^{N}\left(\sum_{j=0}^{N}\frac{1}{h_j}P_j^{(\theta,\vartheta)}(x_{N,i}^{(\theta,\vartheta)})P_j^{(\theta,\vartheta)}(x)\varpi_{N,i}^{(\theta,\vartheta)}\right) s(x_{N,i}^{(\theta,\vartheta)},t). \end{aligned} \tag{14}$$

Furthermore, if we differentiate (14) once, and evaluate it at all J-GR-C points, it is easy to compute the first spatial partial derivative in terms of the values at theses collocation points as

$$\begin{aligned} r_x\left(x_{N,n}^{(\theta,\vartheta)},t\right) &= \sum_{i=0}^{N} A_{ni} r\left(x_{N,i}^{(\theta,\vartheta)},t\right),\\ s_x\left(x_{N,n}^{(\theta,\vartheta)},t\right) &= \sum_{i=0}^{N} B_{ni} s\left(x_{N,i}^{(\theta,\vartheta)},t\right), \quad n = 0,1,\cdots,N, \end{aligned} \tag{15}$$

where

$$\begin{aligned} A_{ni} &= \sum_{j=0}^{N}\frac{j+\theta+\vartheta+1}{2h_j}P_j^{(\theta,\vartheta)}\left(x_{N,i}^{(\theta,\vartheta)}\right)P_{j-1}^{(\theta+1,\vartheta+1)}\left(x_{N,n}^{(\theta,\vartheta)}\right)\varpi_{N,i}^{(\theta,\vartheta)},\\ B_{ni} &= \sum_{j=0}^{N}\frac{j+\theta+\vartheta+1}{2h_j}P_j^{(\theta,\vartheta)}\left(x_{N,i}^{(\theta,\vartheta)}\right)P_{j-1}^{(\theta+1,\vartheta+1)}\left(x_{N,n}^{(\theta,\vartheta)}\right)\varpi_{N,i}^{(\theta,\vartheta)}. \end{aligned} \tag{16}$$

In the proposed J-GR-C method the residual of (8) is set to zero at N of Jacobi-Gauss-Radau points. Moreover, the boundary conditions (9) will be enforced at the collocation points -1. Therefore, adopting (14-16), enable one to write (8-9) in the form:

$$\begin{aligned} \dot{\mathrm{r}}_n(t) &= -\left(\frac{2}{B-A}\right)\sum_{i=0}^{N} A_{ni} r_i(t) - \eta r_n(t) s_n(t), \\ \dot{\mathrm{s}}_n(t) &= \left(\frac{2}{B-A}\right)\sum_{i=0}^{N} B_{ni} s_i(t) - \eta r_n(t) s_n(t), \end{aligned} \tag{17}$$

where

$$r_k(t) = r\left(x_{N,k}^{(\theta,\vartheta)}, t\right), \quad s_k(t) = s\left(x_{N,k}^{(\theta,\vartheta)}, t\right), \quad k = 1, \cdots, N, \quad n = 1, \cdots, N.$$

This provides a $2N$ system of first order ordinary differential equations in the expansion coefficients $a_j(t)$, namely

$$\begin{aligned} \dot{\mathrm{r}}_n(t) &= -\left(\frac{2}{B-A}\right)\left(\sum_{i=1}^{N} A_{ni} r_i(t) + A_{n0} r_0(t)\right) - \eta r_n(t) s_n(t), \\ \dot{\mathrm{s}}_n(t) &= \left(\frac{2}{B-A}\right)\left(\sum_{i=1}^{N} B_{ni} s_i(t) + B_{n0} s_0(t)\right) - \eta r_n(t) s_n(t). \end{aligned} \tag{18}$$

This means that problem (6-7) is transformed to the following SNODEs

$$\begin{aligned} \dot{\mathrm{r}}_n(t) &= -\left(\frac{2}{B-A}\right)\left(\sum_{i=1}^{N} A_{ni} r_i(t) + A_{n0} r_0(t)\right) - \eta r_n(t) s_n(t), \\ \dot{\mathrm{s}}_n(t) &= \left(\frac{2}{B-A}\right)\left(\sum_{i=1}^{N} B_{ni} s_i(t) + B_{n0} s_0(t)\right) - \eta r_n(t) s_n(t). \end{aligned} \tag{19}$$

subject to the initial values

$$r_n(0) = f_3\left(x_{N,n}^{(\theta,\vartheta)}\right), \quad s_n(0) = f_4\left(x_{N,n}^{(\theta,\vartheta)}\right), \quad n = 1, \cdots, N. \tag{20}$$

Finally, (19-20) can be rewritten into a matrix form of $2N$ system of first order ODEs with their vectors of initial values:

$$\begin{aligned} \dot{\mathbf{w}}(t) &= \mathbf{F}(t, r(t), s(t)), \\ \mathbf{w}(0) &= \mathbf{f}, \end{aligned} \tag{21}$$

where

$$\dot{\mathbf{w}}(t) = [\dot{\mathrm{r}}_1(t), \dot{\mathrm{r}}_2(t), \ldots, \dot{\mathrm{r}}_N(t), \dot{\mathrm{s}}_1(t), \dot{\mathrm{s}}_2(t), \ldots, \dot{\mathrm{s}}_N(t)]^T,$$

$$\mathbf{f} = [f_3(x_{N,1}^{(\theta,\vartheta)}), f_3(x_{N,2}^{(\theta,\vartheta)}), \ldots, f_3(x_{N,N}^{(\theta,\vartheta)}), f_4(x_{N,1}^{(\theta,\vartheta)}), f_4(x_{N,2}^{(\theta,\vartheta)}), \ldots, f_4(x_{N,N}^{(\theta,\vartheta)})]^T,$$

$$\mathbf{F}(t,u(t)) = [F_1(t,u(t)), F_2(t,u(t)), \ldots, F_N(t,u(t)), G_1(t,u(t)), G_2(t,u(t)), \ldots, G_N(t,u(t))]^T,$$

where

$$F_n(t,u(t)) = -(\frac{2}{B-A})(\sum_{i=1}^{N} A_{ni}r_i(t) + A_{n0}r_0(t)) - \eta r_n(t)s_n(t), \tag{22}$$

$$G_n(t,u(t)) = (\frac{2}{B-A})(\sum_{i=1}^{N} B_{ni}s_i(t) + B_{n0}s_0(t)) - \eta r_n(t)s_n(t). \tag{23}$$

SNODEs (21) can be solved by using implicit Runge Kutta method of fourth order.

4. Numerical Examples

This section considers three numerical examples to demonstrate the accuracy and applicability of the proposed method in the present paper. Comparison of the results obtained by various choices of Jacobi parameters θ and ϑ reveals that the present method is very effective and convenient for all choices of θ and ϑ.

We consider the following three examples.

Example 1 : Let us first consider the following system of nonlinear hyperbolic equations:

$$\begin{aligned} &\frac{\partial u(y,t)}{\partial t} + \frac{\partial u(y,t)}{\partial y} + 100u(y,t)v(y,t) = 0, \\ &\frac{\partial v(y,t)}{\partial t} - \frac{\partial v(y,t)}{\partial y} + 100u(y,t)v(y,t) = 0, \quad (y,t) \in [-1.5, 1.5] \times [0,1], \end{aligned} \tag{24}$$

with the boundary-initial conditions

$$\begin{aligned} &u(-1.5,t) = -0.009(1+\tanh(-1.5+0.1t)), \\ &v(-1.5,t) = 0.011(-1+\tanh(-1.5+0.1t)), \\ &u(y,0) = -0.009(1+\tanh(y)), \\ &v(y,0) = 0.011(-1+\tanh(y)). \end{aligned} \tag{25}$$

If we apply the generalized tanh method [46]. then we find that the exact solutions of Eqs. (24) are

$$u(y,t) = -0.009(1+\tanh(y+0.1t)), \quad v(y,t) = 0.011(-1+\tanh(y+0.1t)). \tag{26}$$

Table 1: Maximum absolute errors with various choices of $N,\ \theta$ and ϑ for Example 1

N	θ	ϑ	M_1	M_2	θ	ϑ	M_1	M_2
4	0	0	3.80×10^{-3}	7.30×10^{-3}	$-\frac{1}{2}$	$-\frac{1}{2}$	3.21×10^{-3}	5.68×10^{-3}
8			4.08×10^{-5}	1.08×10^{-4}			5.84×10^{-5}	7.18×10^{-5}
12			3.96×10^{-6}	1.05×10^{-5}			2.07×10^{-6}	4.67×10^{-6}
16			5.39×10^{-8}	2.08×10^{-7}			5.64×10^{-8}	9.97×10^{-8}
20			9.20×10^{-9}	1.14×10^{-8}			1.48×10^{-8}	4.87×10^{-9}
4	$-\frac{1}{2}$	$\frac{1}{2}$	1.09×10^{-3}	2.52×10^{-3}	$\frac{1}{2}$	$\frac{1}{2}$	4.10×10^{-3}	8.42×10^{-3}
8			3.87×10^{-5}	4.63×10^{-5}			2.82×10^{-5}	3.38×10^{-4}
12			1.20×10^{-6}	1.74×10^{-6}			6.63×10^{-6}	2.15×10^{-5}
16			3.36×10^{-8}	4.69×10^{-8}			1.12×10^{-7}	5.27×10^{-7}
20			1.55×10^{-8}	1.79×10^{-9}			1.19×10^{-8}	2.43×10^{-8}

The difference between the measured or inferred value of approximate solution and its actual value (absolute error), given by

$$E_1(y,t) = |u(y,t) - \widetilde{u}(y,t)|, \quad E_2(y,t) = |v(y,t) - \widetilde{v}(y,t)|, \tag{27}$$

where $u(y,t)(v(y,t))$ and $\widetilde{u}(y,t)(\widetilde{v}(y,t))$ are the exact solutions and the approximate solutions at the point (y,t), respectively. Moreover, the maximum absolute errors are given by

$$M_1 = \text{Max}\{E_1(y,t) : \forall (y,t) \in D \times [0,T]\}, \quad M_2 = \text{Max}\{E_2(y,t) : \forall (y,t) \in D \times [0,T]\}. \tag{28}$$

Maximum absolute errors of $u(y,t)$ and $v(y,t)$ related to (24-25) are introduced in Table 1 using J-GR-C method with various choices of $N,\ \theta$ and ϑ.

The approximate solutions $\widetilde{u}(x,t)$ and $\widetilde{v}(x,t)$ of problem (24) where $\theta = \frac{1}{2}$, $\vartheta = -\frac{1}{2}$ and $N = 20$ are displayed in Figs. 1 and 2, respectively. While, in Figs. 3 and 4, we present the absolute errors $E_1(x,t)$ and $E_2(x,t)$ with $N = 20$. Moreover, we plot the approximate solutions $\widetilde{u}(x,t)$ and $\widetilde{v}(x,t)$, and the exact solutions $u(x,t)$ and $v(x,t)$ for different values of t in Figs. 5, 6, respectively, with values of parameters listed in their captions.

Table 2: Maximum absolute errors with various choices of $N,\ \theta$ and ϑ for Example 2

N	θ	ϑ	M_1	M_2	θ	ϑ	M_1	M_2
2	$-\frac{1}{2}$	$-\frac{1}{2}$	3.32×10^{-2}	7.19×10^{-2}	$\frac{1}{2}$	$\frac{1}{2}$	9.97×10^{-2}	2.15×10^{-1}
4			1.71×10^{-4}	3.71×10^{-4}			6.43×10^{-4}	1.39×10^{-3}
6			3.56×10^{-7}	7.71×10^{-7}			1.66×10^{-6}	3.59×10^{-6}
8			7.60×10^{-8}	1.68×10^{-8}			1.84×10^{-8}	1.33×10^{-8}
2	0	0	6.65×10^{-2}	1.43×10^{-1}	$\frac{1}{2}$	$-\frac{1}{2}$	1.29×10^{-1}	2.22×10^{-1}
4			3.92×10^{-4}	8.49×10^{-4}			9.66×10^{-4}	1.90×10^{-3}
6			9.21×10^{-7}	1.99×10^{-6}			2.71×10^{-6}	5.43×10^{-6}
8			4.13×10^{-8}	2.11×10^{-8}			4.59×10^{-7}	6.78×10^{-7}

Example 2 : We consider the homogeneous linear system

$$\begin{aligned} &\frac{\partial u(y,t)}{\partial t}-\frac{\partial v(y,t)}{\partial y}+u(y,t)+v(y,t)=0,\\ &\frac{\partial v(y,t)}{\partial t}-\frac{\partial u(y,t)}{\partial y}+u(y,t)+v(y,t)=0,\quad (y,t)\in[0,1]\times[0,1], \end{aligned}\tag{29}$$

with the boundary-initial conditions

$$\begin{aligned} &u(0,t)=\sinh(-t),\quad v(0,t)=\cosh(-t),\\ &u(y,0)=\sinh(y),\quad v(y,0)=\cosh(y). \end{aligned}\tag{30}$$

The exact solutions [47] of Eqs. (29) are

$$u(y,t)=\sinh(y-t),\quad v(y,t)=\cosh(y-t),\tag{31}$$

Maximum absolute errors of $u(y,t)$ and $v(y,t)$ related to (29-33) are introduced in Table 2 using J-GR-C method with various choices of $N,\ \theta$ and ϑ. Figs. 7 and 8, show that the approximate solutions $\widetilde{u}(x,t)$ and the exact solutions $u(x,t)$ are coincided for different values of t or x, respectively. Similarly, Figs. 9 and 10, demonstrate that the approximate solutions $\widetilde{v}(x,t)$ and the exact solutions $v(x,t)$ are coincided for different values of t or x, respectively.

Example 3 : Consider the nonhomogeneous nonlinear system

$$\begin{aligned} &\frac{\partial u(y,t)}{\partial t}+v(y,t)\frac{\partial u(y,t)}{\partial y}+u(y,t)-1=0,\\ &\frac{\partial v(y,t)}{\partial t}-u(y,t)\frac{\partial v(y,t)}{\partial y}-v(y,t)-1=0,\quad (y,t)\in[0,1]\times[0,1], \end{aligned}\tag{32}$$

Table 3: Maximum absolute errors with various choices of N, θ and ϑ for Example 3

N	θ	ϑ	M_1	M_2	θ	ϑ	M_1	M_2
2	$\frac{1}{2}$	$\frac{1}{2}$	3.15×10^{-1}	4.01×10^{-1}	$\frac{1}{2}$	$-\frac{1}{2}$	1.44×10^{-1}	5.92×10^{-1}
4			2.03×10^{-3}	1.99×10^{-3}			9.39×10^{-4}	3.20×10^{-3}
6			5.26×10^{-6}	4.87×10^{-6}			3.06×10^{-6}	6.71×10^{-6}
8			2.92×10^{-8}	6.58×10^{-8}			1.70×10^{-7}	6.69×10^{-7}
2	0	0	2.10×10^{-1}	2.67×10^{-1}	$-\frac{1}{2}$	$\frac{1}{2}$	4.11×10^{-1}	1.39×10^{-1}
4			1.24×10^{-3}	1.23×10^{-3}			3.17×10^{-3}	1.08×10^{-3}
6			2.91×10^{-6}	2.78×10^{-6}			8.80×10^{-6}	3.05×10^{-6}
8			1.01×10^{-7}	1.83×10^{-7}			1.77×10^{-7}	8.86×10^{-8}

with the boundary-initial conditions

$$\begin{aligned} u(0,t) &= e^{-t}, \quad v(0,t) = e^{t}, \\ u(y,0) &= e^{y}, \quad v(y,0) = e^{-y}. \end{aligned} \tag{33}$$

The exact solutions [47] of Eqs. (32) are

$$u(y,t) = e^{y-t}, \quad v(y,t) = e^{-y+t}, \tag{34}$$

Maximum absolute errors of $u(y,t)$ and $v(y,t)$ related to (32) are introduced in Table 3 using J-GL-C method with various choices of N, θ and ϑ. We plot in Figs. 11 and 12, the approximate solutions $\widetilde{u}(x,t)$, and the exact solutions $u(x,t)$ for different values of t or x respectively, with values of parameters listed in their captions. While, the absolute errors $E_1(x,t)$ and $E_2(x,t)$ for problem (32) where $\theta = -\vartheta = \frac{1}{2}$ and $N = 8$ are displayed in Fig. 13 and 14, with values of parameters listed in their captions.

CONCLUSIONS

In this article, we have concerned with an efficient and accurate numerical scheme based on the J-GR-C method for solving the initial-boundary system of nonlinear hyperbolic equations. This method reduced the problem to solve SNODEs. Numerical examples were given to demonstrate the validity and applicability of the method. The results show that the proposed method was accurate. Indeed by selecting limited collocation points, excellent numerical results are obtained.

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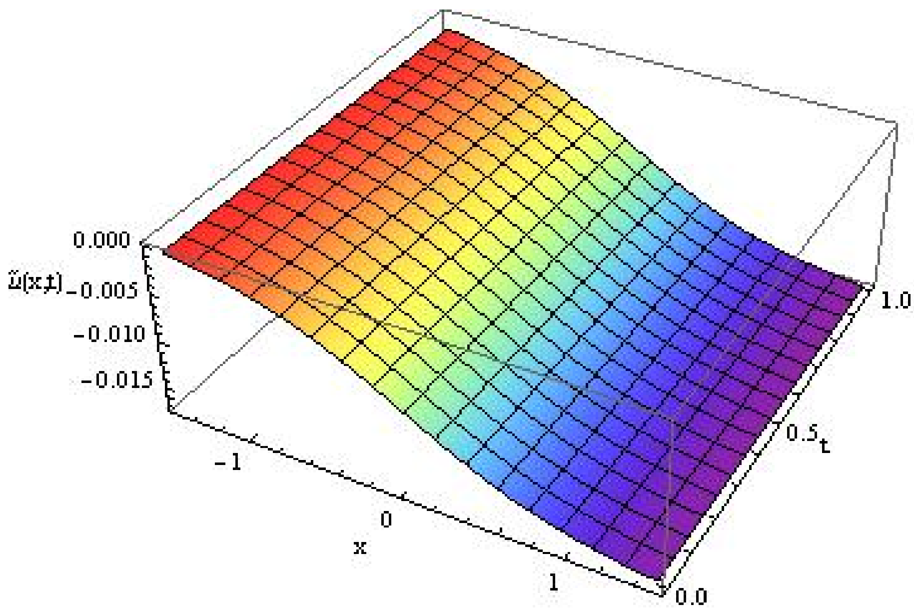

Figure 1: The approximate solution $\widetilde{u}(x,t)$ of problem (24) where $\theta=\frac{1}{2}$, $\vartheta=-\frac{1}{2}$ and $N=20$.

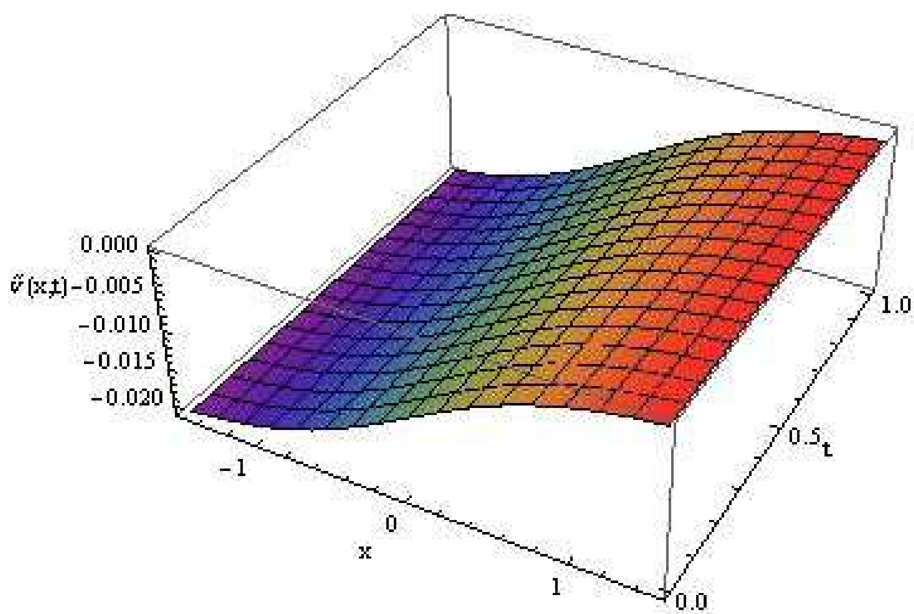

Figure 2: The approximate solution $\widetilde{v}(x,t)$ of problem (24) where $\theta=\frac{1}{2}$, $\vartheta=-\frac{1}{2}$ and $N=20$.

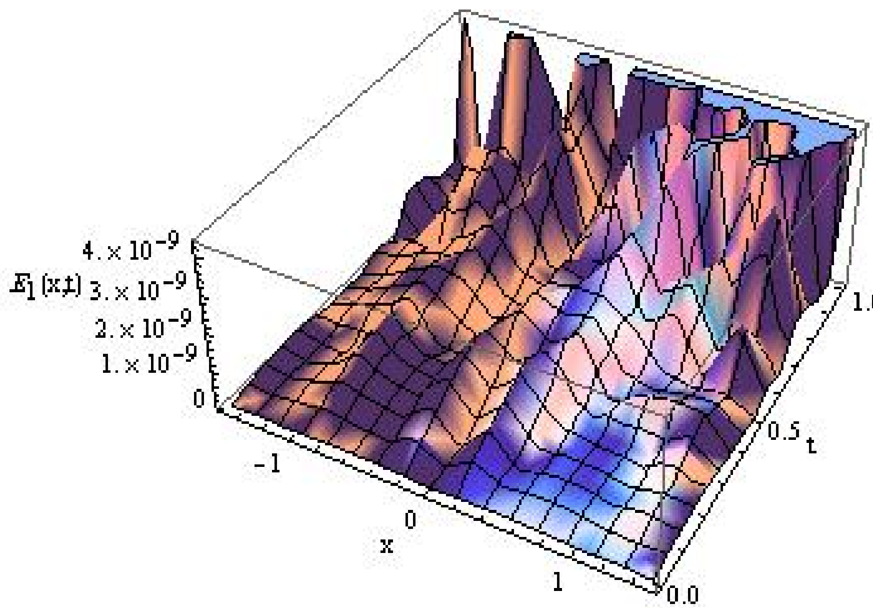

Figure 3: The absolute error $E_1(x,t)$ of problem (24) where $\theta = \frac{1}{2}$, $\vartheta = -\frac{1}{2}$ and $N = 20$.

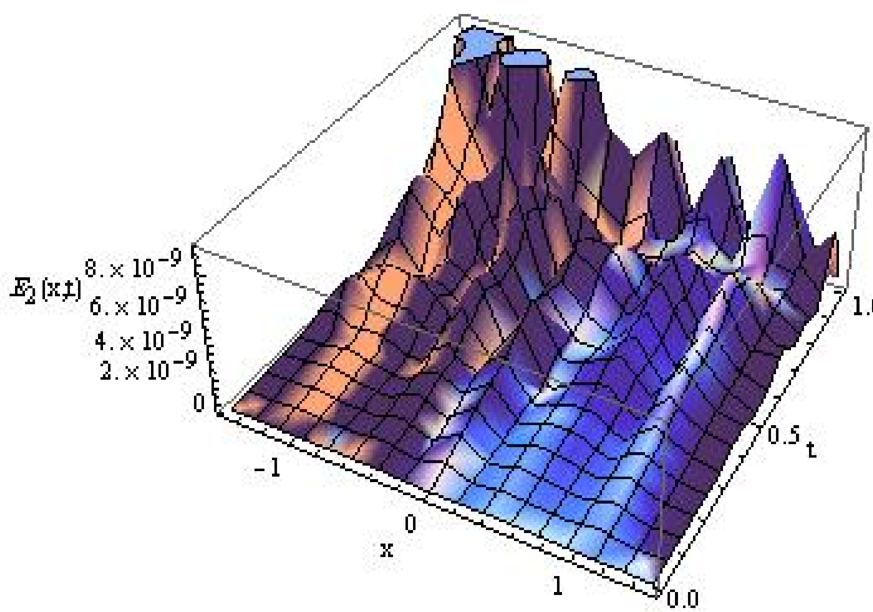

Figure 4: The absolute error $E_2(x,t)$ of problem (24) where $\theta = \frac{1}{2}$, $\vartheta = -\frac{1}{2}$ and $N = 20$.

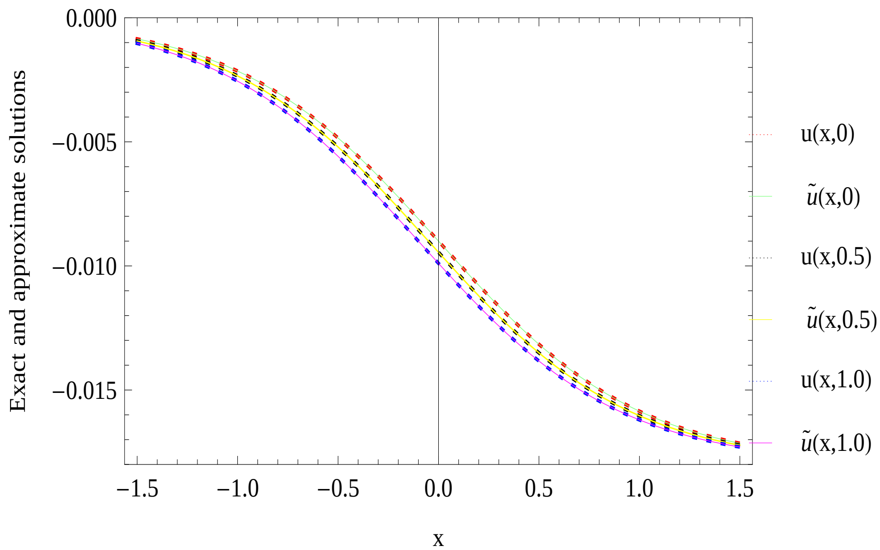

Figure 5: The approximate solution $\widetilde{u}(x,t)$ and the exact solution $u(x,t)$ for $t = 0$, 0.5 and 1.0 of problem (24) where $\theta = \frac{1}{2}$, $\vartheta = -\frac{1}{2}$ and $N = 20$.

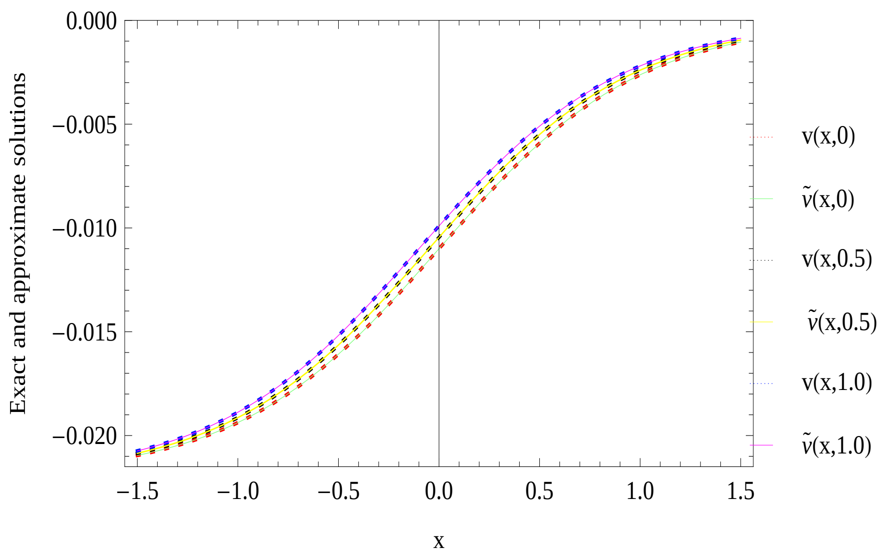

Figure 6: The approximate solution $\widetilde{v}(x,t)$ and the exact solution $v(x,t)$ for $t = 0$, 0.5 and 1.0 of problem (24) where $\theta = \frac{1}{2}$, $\vartheta = -\frac{1}{2}$ and $N = 20$.

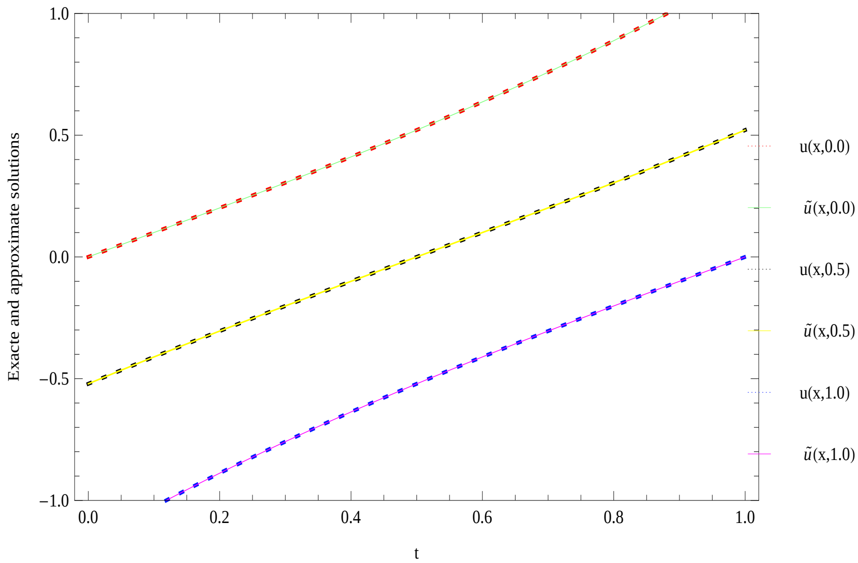

Figure 7: The approximate solution $\widetilde{u}(x,t)$ and the exact solution $u(x,t)$ for $t = 0.0$, 0.4 and 0.8 of problem (29) where $\theta = \vartheta = 1$ and $N = 8$.

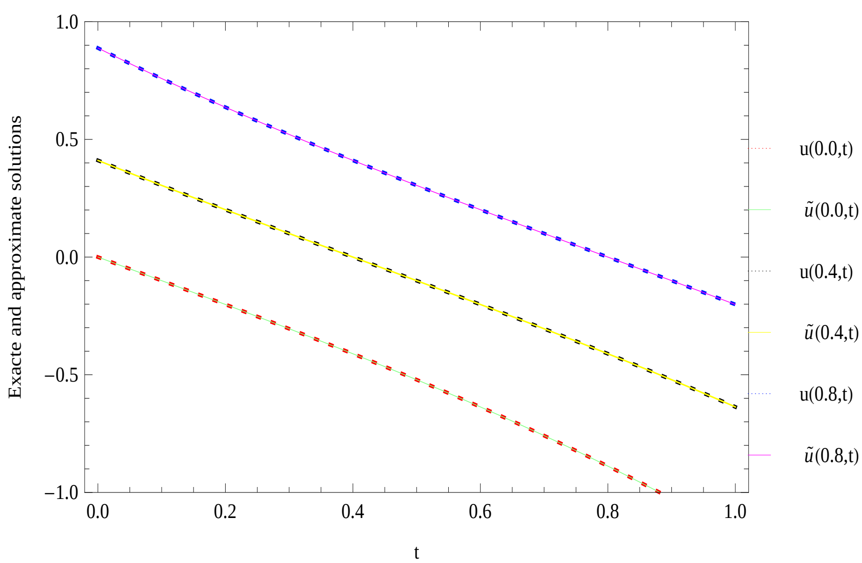

Figure 8: The approximate solution $\widetilde{u}(x,t)$ and the exact solution $u(x,t)$ for $x = 0.0$, 0.4 and 0.8 of problem (29) where $\theta = \vartheta = 1$ and $N = 8$.

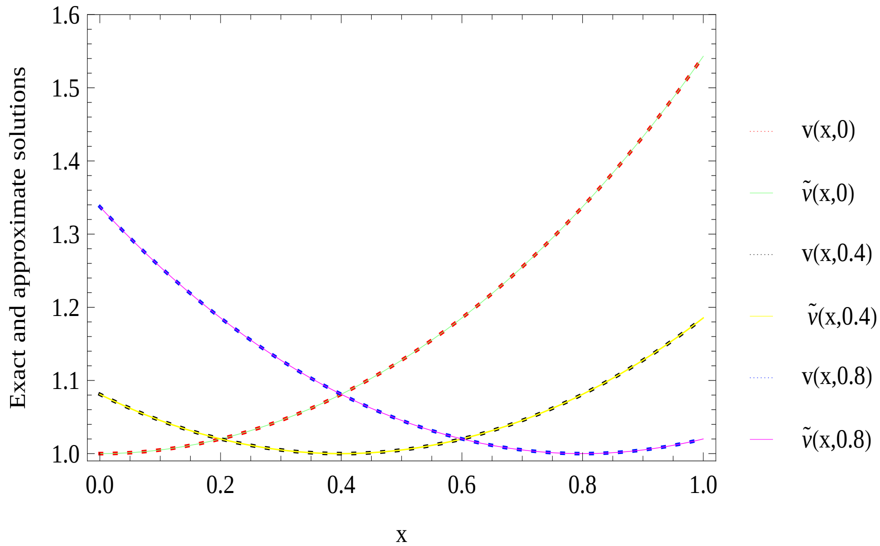

Figure 9: The approximate solution $\widetilde{v}(x,t)$ and the exact solution $v(x,t)$ for $t = 0.0$, 0.4 and 0.8 of problem (29) where $\theta = \vartheta = 1$ and $N = 8$.

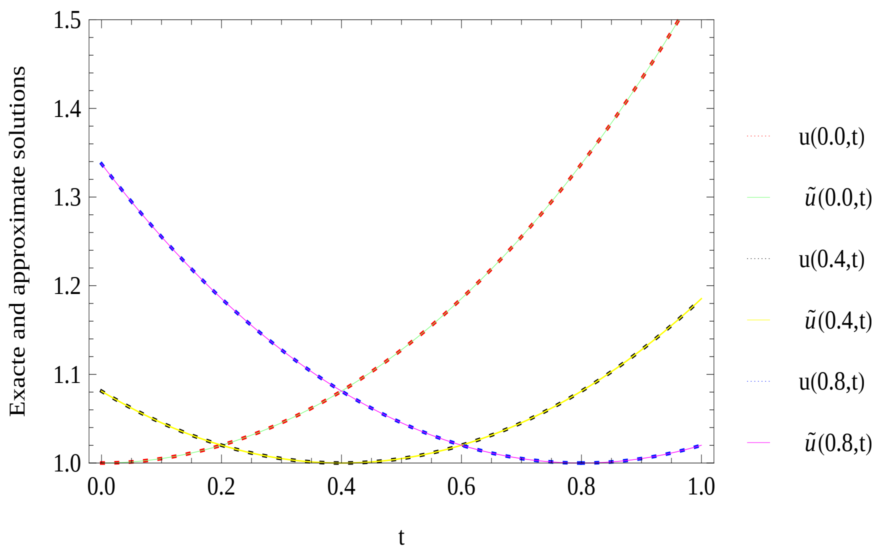

Figure 10: The approximate solution $\widetilde{v}(x,t)$ and the exact solution $v(x,t)$ for $x = 0.0$, 0.4 and 0.8 of problem (29) where $\theta = \vartheta = 1$ and $N = 8$.

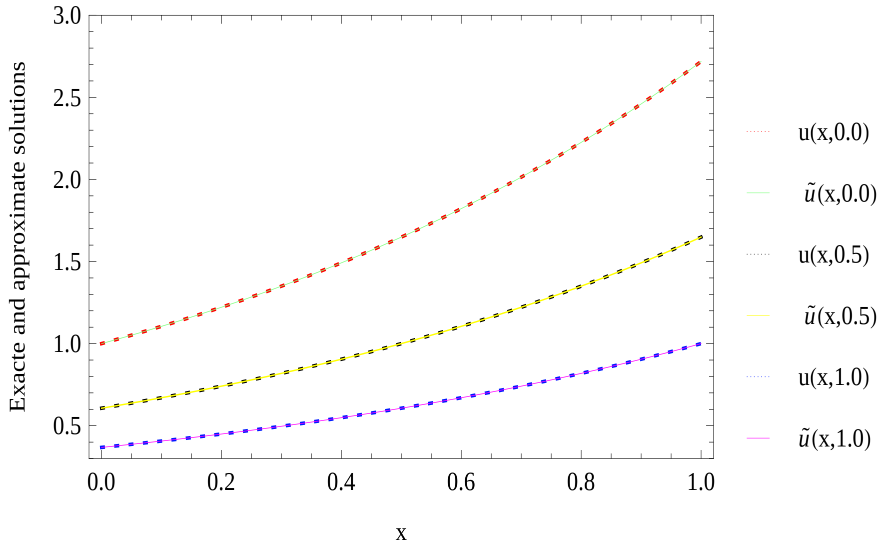

Figure 11: The approximate solution $\widetilde{u}(x,t)$ and the exact solution $u(x,t)$ for $t = 0.0$, 0.5 and 1.0 of problem (32) where $\theta = \vartheta = -\frac{1}{2}$ and $N = 8$.

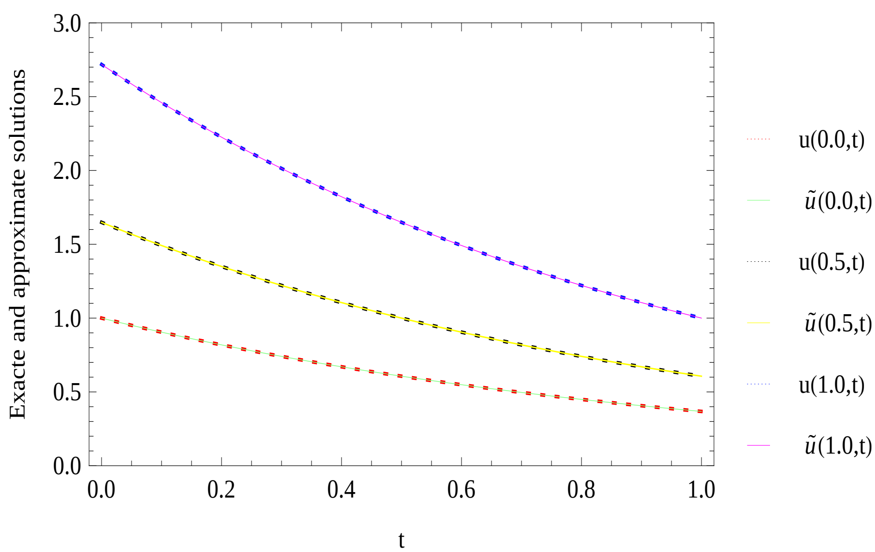

Figure 12: The approximate solution $\widetilde{u}(x,t)$ and the exact solution $u(x,t)$ for $x = 0.0$, 0.5 and 1.0 of problem (32) where $\theta = \vartheta = -\frac{1}{2}$ and $N = 8$.

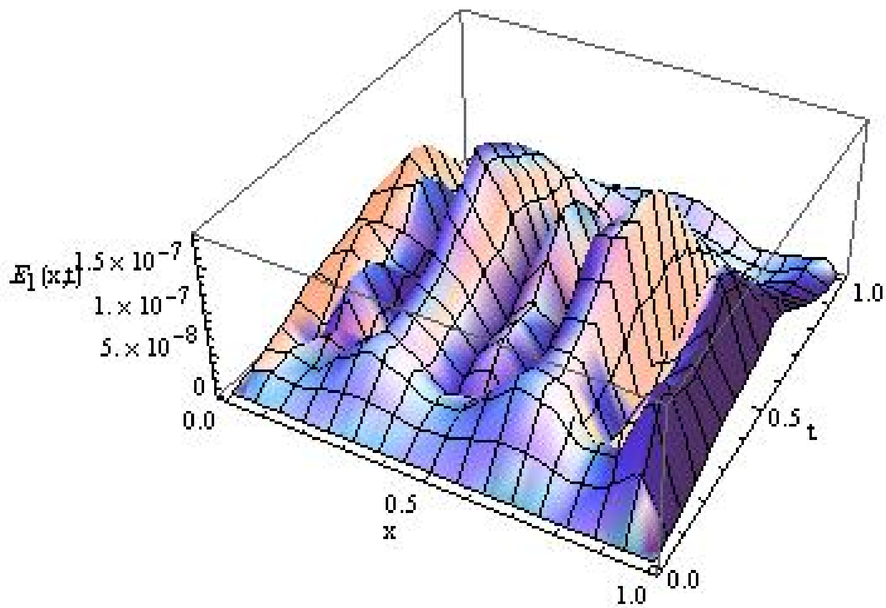

Figure 13: The absolute error $E_1(x,t)$ for problem (32) where $\theta = \vartheta = -\frac{1}{2}$ and $N = 8$.

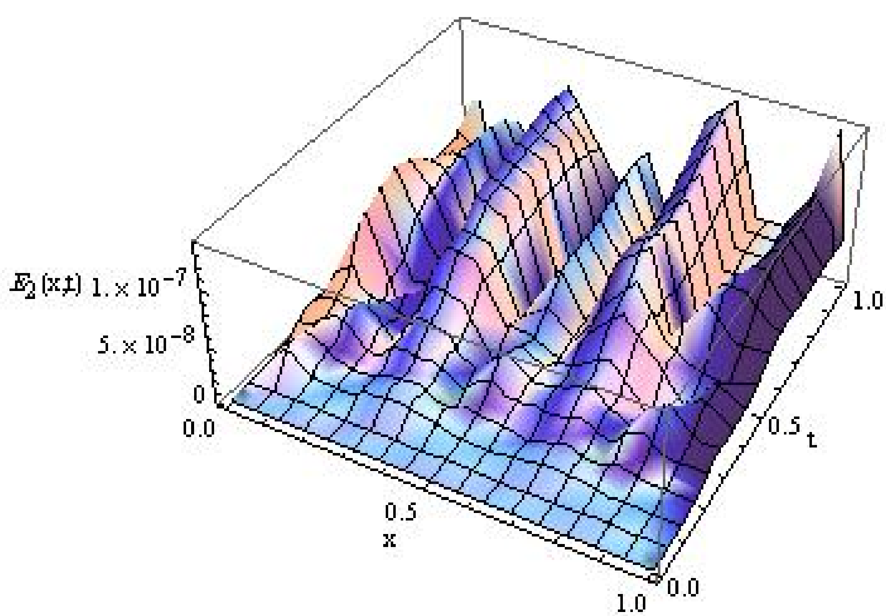

Figure 14: The absolute error $E_2(x,t)$ for problem (32) where $\theta = \vartheta = -\frac{1}{2}$ and $N = 8$.