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ON MULTIPLICITY OF TRIANGLES

S. S. Rukmani, J. Prema[1] and V. Vijayalakshmi

Department of Mathematics, Anna University, MIT Campus,
Chennai 600 044, *India*
e-mails: ssrukku@gmail.com, prema1586@gmail.com,
vijayalakshmi@annauniv.edu

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Let $\{G(n,l)\}$ be the family of graphs where $G(n,l)$ is obtained from the complete graph K_{n+l} by removing the edges of a complete subgraph on l vertices. In this paper, we determine the multiplicity of triangles in any two coloring of the edges of $G(n,l)$, for sufficiently large values of n.

Key words : Complete graphs; edge coloring; multiplicity; triangles

1. Introduction

Let G be any graph. The multiplicity of triangles in G, denoted by $M(K_3, G)$, is defined as the minimum number of monochromatic copies of K_3 that occur in any two coloring of the edges of G. $M(K_3, K_n)$ was determined by Goodman [1] for all values of n. Theoretically, it should be possible to find $M(K_3, G)$ for any graph G. But when G is sparse, the problem becomes difficult. In 2000, Vijayalakshmi [3] determined $M(K_3, G)$ where G is the graph obtained from the complete graph K_n by removing any set of k parallel edges (k and n are positive integers and $k \leq \frac{n}{2}$). In this paper we determine the multiplicity of triangles in the family of graphs $\{G(n,l)\}$, where $G(n,l)$ is obtained from the complete graph K_{n+l} by removing the edges of a complete subgraph on l vertices, for sufficiently large values of n. First we obtain a lower bound for the multiplicity of triangles in the family of graphs $\{G(n,l)\}$ for all values of n and l using the weight method given by Sauve [2] and then we show that this lower bound is sharp for sufficiently large values of n by giving explicit coloring scheme.

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2. Method of Weights

Let $\{G(n,l)\}$ be the family of graphs where $G(n,l)$ is obtained from the complete graph K_{n+l} by removing the edges of a complete subgraph on l vertices. Our aim is to determine the minimum number of monochromatic triangles that occur in any two coloring (say red and blue) of the edges of $G(n,l)$. We assign weight for every pair of edges at each vertex p of $G(n,l)$. Let $A(p)$ be the set of all pairs of edges at the vertex p in $G(n,l)$. Suppose $a \in A(p)$. We define $W(a) = 2$, if both the edges are of the same color and $W(a) = -1$ otherwise. For every vertex p of $G(n,l)$ we define $W(p)$, the weight at the vertex p to be $\sum_{a\in A(p)} W(a)$. We denote the weight of the graph $G(n,l)$ as $W(G(n,l))$. Define $W(G(n,l)) = \sum_{p\in V(G(n,l))} W(p)$, where $V(G(n,l))$ is the vertex set of $G(n,l)$. Let S be the set of all subraphs of $G(n,l)$ induced by any three vertices of $G(n,l)$. Hence, the weight of the graph $W(G(n,l)) = \sum_{T\in S} W(T)$.

The six different types of subgraphs in S are as follows:

S_1, the set of all subgraphs induced by 3 vertices in which all the edges are of the same color.

S_2, the set of all subgraphs induced by 3 vertices in which all the edges are not of the same color.

S_3, the set of all subgraphs induced by 3 vertices such that the subgraphs have 2 edges of the same color and a non edge.

S_4, the set of all subgraphs induced by 3 vertices such that the subgraphs have 2 edges of different colors and a non edge.

S_5, the set of all subgraphs induced by 3 vertices such that the subgraphs have two non edges and a single edge of any color.

S_6, the set of all subgraphs induced by 3 vertices such that the subgraphs have no edges.

Thus, the weight of a subgraph T induced by any three vertices of $G(n,l)$ is given by

$$W(T) = \begin{cases} 6 & \text{if } \mathrm{T} \in \mathrm{S}_1 \\ 0 & \text{if } \mathrm{T} \in \mathrm{S}_2 \\ 2 & \text{if } \mathrm{T} \in \mathrm{S}_3 \\ -1 & \text{if } \mathrm{T} \in \mathrm{S}_4 \\ 0 & \text{if } \mathrm{T} \in \mathrm{S}_5 \\ 0 & \text{if } \mathrm{T} \in \mathrm{S}_6 \end{cases}$$

Hence, $W(G(n,l)) = 6|S_1| + 2|S_3| - |S_4|$ where $|S_i|$ is the cardinality of the set S_i.

$$\Rightarrow |S_1| = \frac{1}{6}\big(W(G(n,l)) - 2|S_3| + |S_4|\big).$$

Let $S_3(p)$ be the number of subgraphs of the set S_3 where the two edges of the same color are incident at p and $S_4(p)$ be the number of subgraphs of the set S_4 where the two edges of different colors are incident at p.

It is easy to say that

$|S_3| = \sum_{p\in V(G(n,l))} S_3(p)$ and $|S_4| = \sum_{p\in V(G(n,l))} S_4(p)$

Therefore, we get

$$|S_1| = \frac{1}{6}\Bigg(\sum_{p\in V(G(n,l))} W(p) - 2\sum_{p\in V(G(n,l))} S_3(p) + \sum_{p\in V(G(n,l))} S_4(p)\Bigg) \qquad (1)$$

We call the above equation (1) as **Weight Equation**. We will use this weight equation in the following Sections. In any two coloring of the edges of $G(n,l)$, $S_3(p)+S_4(p)$ is a constant, as this is precisely the number of pairs of edges $\{pv, pw\}$ such that $vw \notin E(G(n,l))$, where $E(G(n,l))$ is the edge set of $G(n,l)$. Therefore at any vertex p, maximizing $S_3(p)$ is equivalent to minimizing $S_4(p)$. From equation (1), the graph $G(n,l)$ will have the minimum number of monochromatic triangles if it satisfies the following conditions.

Condition (A): At every vertex v of $G(n,l)$, almost equal number of red and blue edges are incident with v.

Condition (B): At every vertex v of $G(n,l)$, whenever uw is a non edge, the edges vu and vw are of the same color.

Notation : The degree pair at any vertex v is denoted by (r,b) where r and b are the number of red edges and blue edges incident at v respectively.

3. A Lower Bound for the Multiplicity of Triangles in *G(n,l)*

Consider the complete graph K_{n+l} with vertex set $\{v_1, v_2, ..., v_n, u_1, u_2, ..., u_l\}$. Partition these vertices into two sets $V_1 = \{v_1, v_2, ..., v_n\}$ and $V_2 = \{u_1, u_2, ..., u_l\}$. Remove the edges of the complete subgraph induced by the vertices of V_2. Let the resultant graph be $G(n,l)$. We shall color the edges of $G(n,l)$ with two colors say red and blue. In the graph $G(n,l)$, the degree of each vertex in V_1 is $n-1+l$ and the degree of each vertex in V_2 is n. Clearly, $S_3(p)$ can take maximum value $\binom{l}{2}$ for any vertex p in V_1. When $S_3(p)$ is maximum, $S_4(p)$ is minimum and is equal to zero for each vertex p.

We begin by determining a lower bound for the multiplicity of triangles in $G(n,l)$ and we will show that this lower bound is sharp for sufficiently large values of n by giving explicit coloring scheme in Section 4.

***Theorem* 3.1** —

$$M(K_3, G(n,l)) \geq \begin{cases} \frac{1}{24}[n^3 + n^2(3l-6) + n(8-6l-3l^2)] \\ \qquad\qquad if\ n\ is\ even\ and\ l\ is\ even \\ \\ \frac{1}{24}[n^3 + n^2(3l-6) + n(5-6l-3l^2)] \\ \qquad\qquad if\ n\ is\ even,\ \frac{n}{2}\ is\ even\ and\ l\ is\ odd \\ \\ \frac{1}{24}[n^3 + n^2(3l-6) + n(5-6l-3l^2) + 12] \\ \qquad\qquad if\ n\ is\ even,\ \frac{n}{2}\ is\ odd\ and\ l\ is\ odd \\ \\ \frac{1}{24}[n^3 + n^2(3l-6) + n(5-6l-3l^2) + 3l] \\ \qquad\qquad if\ n\ is\ odd,\ \frac{n+l-1}{2}\ is\ even\ and\ l\ is\ even \\ \\ \frac{1}{24}[n^3 + n^2(3l-6) + n(5-6l-3l^2) + 3l + 12] \\ \qquad\qquad if\ n\ is\ odd,\ \frac{n+l-1}{2}\ is\ odd\ and\ l\ is\ even \\ \\ \frac{1}{24}[n^3 + n^2(3l-6) + n(8-6l-3l^2) + 3l] \\ \qquad\qquad if\ n\ is\ odd\ and\ l\ is\ odd \end{cases}$$

where n and l are non negative integers.

PROOF : We prove this theorem by considering four cases. We use the weight equation (1) to calculate the minimum number of monochromatic triangles in $G(n,l)$.

Case 1 : n is even and l is even.

By conditions (A) and (B), the weight of the graph $G(n,l)$ is minimum when each vertex in V_1 has degree pair $(\frac{n+l}{2}, \frac{n+l-2}{2})$ or $(\frac{n+l-2}{2}, \frac{n+l}{2})$ and each vertex in V_2 has degree pair $(\frac{n}{2}, \frac{n}{2})$. So, from the weight equation

$$\begin{aligned} 6|S_1| &\geq n\left[2\binom{\frac{n+l-2}{2}}{2} + 2\binom{\frac{n+l}{2}}{2} - \left(\frac{n+l-2}{2}\right)\left(\frac{n+l}{2}\right)\right] + l\left[4\binom{\frac{n}{2}}{2} - \left(\frac{n}{2}\right)\left(\frac{n}{2}\right)\right] - 2n\binom{l}{2} \\ &= \frac{1}{4}\left[n^3 + n^2(3l-6) + n(8-6l-3l^2)\right]. \end{aligned}$$

Thus, $\quad |S_1| \geq \frac{1}{24}\left[n^3 + n^2(3l-6) + n(8-6l-3l^2)\right].$

Case 2 : n is even and l is odd.

By conditions (A) and (B), the weight of the graph $G(n,l)$ is minimum when each vertex in V_1 has degree pair $(\frac{n+l-1}{2}, \frac{n+l-1}{2})$ and each vertex in V_2 has degree pair $(\frac{n}{2}, \frac{n}{2})$. So, from the weight equation

$$\begin{aligned} 6|S_1| &\geq n\left[4\binom{\frac{n+l-1}{2}}{2} - \left(\frac{n+l-1}{2}\right)\left(\frac{n+l-1}{2}\right)\right] + l\left[4\binom{\frac{n}{2}}{2} - \left(\frac{n}{2}\right)\left(\frac{n}{2}\right)\right] - 2n\binom{l}{2} \\ &= \frac{1}{4}\left[n^3 + n^2(3l-6) + n(5-6l-3l^2)\right] \end{aligned}$$

Thus, $|S_1| \geq \frac{1}{24}\left[n^3 + n^2(3l-6) + n(5-6l-3l^2)\right]$.

This bound is not attainable when $\frac{n}{2}$ is odd since the number of vertices of odd degree in any graph is even. So, the next possible minimum would be when exactly one vertex $v_1 \in V_1$ has the degree pair $(\frac{n+l-1}{2} - 1, \frac{n+l-1}{2} + 1)$ or $(\frac{n+l-1}{2} + 1, \frac{n+l-1}{2} - 1)$, the remaining vertices in V_1 have degree pair $(\frac{n+l-1}{2}, \frac{n+l-1}{2})$ and each vertex in V_2 has degree pair $(\frac{n}{2}, \frac{n}{2})$.

So, from the weight equation

$$\begin{aligned} 6|S_1| &\geq (n-1)\left[4\binom{\frac{n+l-1}{2}}{2} - \left(\frac{n+l-1}{2}\right)\left(\frac{n+l-1}{2}\right)\right] + 2\binom{\frac{n+l+1}{2}}{2} + 2\binom{\frac{n+l-3}{2}}{2} \\ &\quad - \left(\frac{n+l+1}{2}\right)\left(\frac{n+l-3}{2}\right) + l\left[4\binom{\frac{n}{2}}{2} - \left(\frac{n}{2}\right)\left(\frac{n}{2}\right)\right] - 2n\binom{l}{2} \\ &= \frac{1}{4}\left[n^3 + n^2(3l-6) + n(5-6l-3l^2) + 12\right] \end{aligned}$$

Thus, $|S_1| \geq \frac{1}{24}\left[n^3 + n^2(3l-6) + n(5-6l-3l^2) + 12\right]$.

Case 3 : n is odd and l is even.

By conditions (A) and (B), the weight of the graph $G(n,l)$ is minimum when each vertex in V_1 has degree pair $(\frac{n+l-1}{2}, \frac{n+l-1}{2})$ and each vertex in V_2 has same degree pair $(\frac{n-1}{2}, \frac{n+1}{2})$ or $(\frac{n+1}{2}, \frac{n-1}{2})$.

So, from the weight equation

$$6|S_1| \geq n\left[4\binom{\frac{n+l-1}{2}}{2} - (\tfrac{n+l-1}{2})(\tfrac{n+l-1}{2})\right] + l\left[2\binom{\frac{n+1}{2}}{2} + 2\binom{\frac{n-1}{2}}{2} - (\tfrac{n+1}{2})(\tfrac{n-1}{2})\right] - 2n\binom{l}{2}$$

$$= \tfrac{1}{4}\left[n^3 + n^2(3l-6) + n(5-6l-3l^2) + 3l\right].$$

Thus, $|S_1| \geq \frac{1}{24}[n^3 + n^2(3l-6) + n(5-6l-3l^2) + 3l]$.

This bound is not attainable when $\frac{n+l-1}{2}$ is odd since the number of vertices of odd degree in any graph is even. So, the next possible minimum would be when exactly one vertex $v_1 \in V_1$ has the degree pair $(\frac{n+l-1}{2}-1, \frac{n+l-1}{2}+1)$ or $(\frac{n+l-1}{2}+1, \frac{n+l-1}{2}-1)$, the remaining vertices in V_1 has degree pair $(\frac{n+l-1}{2}, \frac{n+l-1}{2})$ and each vertex in V_2 has same degree pair $(\frac{n+1}{2}, \frac{n-1}{2})$ or $(\frac{n-1}{2}, \frac{n+1}{2})$.

So, from the weight equation

$$6|S_1| \geq (n-1)\left[4\binom{\frac{n+l-1}{2}}{2} - (\tfrac{n+l-1}{2})(\tfrac{n+l-1}{2})\right] + 2\binom{\frac{n+l+1}{2}}{2} + 2\binom{\frac{n+l-3}{2}}{2}$$

$$-(\tfrac{n+l+1}{2})(\tfrac{n+l-3}{2}) + l\left[2\binom{\frac{n+1}{2}}{2} + 2\binom{\frac{n-1}{2}}{2} - (\tfrac{n+1}{2})(\tfrac{n-1}{2})\right] - 2n\binom{l}{2}$$

$$= \tfrac{1}{4}\left[n^3 + n^2(3l-6) + n(5-6l-3l^2) + 3l + 12\right].$$

Thus, $|S_1| \geq \frac{1}{24}\left[n^3 + n^2(3l-6) + n(5-6l-3l^2) + 3l + 12\right]$.

Case 4 : n is odd and l is odd.

By conditions (A) and (B), the weight of the graph $G(n,l)$ is minimum when each vertex in V_1 has degree pair $(\frac{n+l-2}{2}, \frac{n+l}{2})$ or $(\frac{n+l}{2}, \frac{n+l-2}{2})$ and each vertex in V_2 has same degree pair $(\frac{n-1}{2}, \frac{n+1}{2})$ or $(\frac{n+1}{2}, \frac{n-1}{2})$.

So, from the weight equation

$$6|S_1| \geq n\left[2\binom{\frac{n+l}{2}}{2} + 2\binom{\frac{n+l-2}{2}}{2} - (\tfrac{n+l-2}{2})(\tfrac{n+l}{2})\right]$$

$$+l\left[2\binom{\frac{n+1}{2}}{2} + 2\binom{\frac{n-1}{2}}{2} - (\tfrac{n+1}{2})(\tfrac{n-1}{2})\right] - 2n\binom{l}{2}$$

$$= \tfrac{1}{4}\left[n^3 + n^2(3l-6) + n(8-6l-3l^2) + 3l\right]$$

Thus, $|S_1| \geq \frac{1}{24}\left[n^3 + n^2(3l-6) + n(8-6l-3l^2) + 3l\right]$.

Hence the theorem.

4. Coloring Scheme

In this section we show that the lower bound given in Theorem 3.1 is sharp for sufficiently large values of n by giving explicit coloring scheme. Let the vertices of the complete graph K_{n+l} be partitioned into five sets say $X = \{x_i/1 \leq i \leq u\}$, $Y = \{y_i/1 \leq i \leq u\}$, $Z = \{z_i/1 \leq i \leq u\}$, $W = \{w_i/1 \leq i \leq u\}$ and $V = \{v_i/1 \leq i \leq l\}$ where $n = 4u$. Remove the edges of the complete subgraph induced by the vertices of V. Let the resultant graph be $G(n, l)$. Let the complete subgraphs induced by X, Y, Z and W be A_1, A_2, A_3 and A_4 respectively.

Case 1 : $l \equiv 0 \; mod \; 4$.

Let $l = 4k$, $k \geq 1$. The following coloring scheme shows that the lower bound given in Theorem 3.1 is sharp for $n \geq 8k + 4$.

Sub-case (a) : $n \equiv 0 \; mod \; 4$.

Let $n = 4u$ ($u \geq 2k + 1$). Consider the graph $G(n, l)$ on $n + l$ vertices with the partitions X, Y, Z, W and V as explained above. Color the edges x_iv_j, y_iv_j ($1 \leq i \leq u$ and $1 \leq j \leq 4k$), z_iw_j, y_iw_j ($1 \leq i, j \leq u$), x_iz_j ($1 \leq i \neq j \leq u$) with red color and the edges z_iv_j, w_iv_j ($1 \leq i \leq u$ and $1 \leq j \leq 4k$), x_iw_j, x_iy_j, y_iz_j ($1 \leq i, j \leq u$), x_iz_i ($1 \leq i \leq u$) with blue color. It is possible to find at least k edge disjoint Hamilton cycles in each of the complete subgraphs A_i ($1 \leq i \leq 4$). Let these edge disjoint Hamilton cycles of A_i be $H_{i,1}, H_{i,2}, H_{i,3}, ..., H_{i,k}$ ($1 \leq i \leq 4$). Color the edges of the Hamilton cycles $H_{1,1}, H_{1,2}, H_{1,3}, ..., H_{1,k-1}$ of A_1 and the Hamilton cycles $H_{2,1}, H_{2,2}, H_{2,3}, ..., H_{2,k}$ of A_2 with blue color and the remaining edges of both A_1 and A_2 with red color. Also, color the edges of the Hamilton cycles $H_{3,1}, H_{3,2}, H_{3,3}, ..., H_{3,k}$ of A_3 with red color and the remaining edges of A_3 with blue color. Let the k independent edges of the Hamilton cycle $H_{4,1}$ be $(w_1, w_2), (w_3, w_4), (w_5, w_6), ..., (w_{2k-1}, w_{2k})$. Color the edges of the Hamilton cycles $H_{4,1}, H_{4,2}, H_{4,3}, ..., H_{4,k}$ of A_4 other than the above independent edges with red color and the remaining edges of A_4 with blue color.

Sub-case (b) : $n \equiv 1 \; mod \; 4$.

Let $n = 4u + 1$. Add a new vertex p_1 to the graph given in the Sub-case (a) of Case 1. Color the edges p_1y_i, p_1z_i ($1 \leq i \leq u$), p_1w_i ($1 \leq i \leq 2k$) with red color and the edges p_1x_i ($1 \leq i \leq u$), p_1v_j ($1 \leq j \leq 4k$), p_1w_i ($2k + 1 \leq i \leq u$) with blue color.

Sub-case (c) : $n \equiv 2 \; mod \; 4$.

Let $n = 4u + 2$. Add a new vertex p_2 to the graph given in the Sub-case (b) of Case 1. Color the edges p_2x_i ($1 \leq i \leq u$), p_1p_2, p_2v_j ($1 \leq j \leq 4k$), p_2w_i ($2k + 1 \leq i \leq u$) with red color and the edges p_2y_i, p_2z_i $(1 \leq i \leq u)$, p_2w_i $(1 \leq i \leq 2k)$ with blue color.

Sub-case (d) : $n \equiv 3 \ mod \ 4$.

Let $n = 4u + 3$. Add a new vertex p_3 to the graph given in the Sub-case (c) of Case 1. Color the edges p_3y_i, p_3z_i $(1 \leq i \leq u)$, p_3w_i $(1 \leq i \leq 2k)$ with red color and the edges p_3x_i $(1 \leq i \leq u)$, p_3v_j $(1 \leq j \leq 4k)$, p_1p_3, p_2p_3, p_3w_i $(2k + 1 \leq i \leq u)$ with blue color.

For the sake of clarity, the red degree at each vertex in the above subcases is given as each column of Table 1 at the end of this section.

Case 2 : $l \equiv 1 \ mod \ 4$.

Let $l = 4k + 1$, $k \geq 0$. The following coloring scheme shows that the lower bound given in Theorem 3.1 is sharp for $n \geq 8k + 4$.

Sub-case (a) : $n \equiv 0 \ mod \ 4$.

Let $n = 4u$ $(u \geq 2k + 1)$. Consider a graph $G(n, l)$ on $n + l$ vertices with the partitions X, Y, Z, W and V as explained. Color the edges x_iv_j, y_iv_j $(1 \leq i \leq u$ and $1 \leq j \leq 4k + 1)$, x_iz_j, z_iw_j, y_iw_j $(1 \leq i, j \leq u)$ with red color and the edges x_iy_j, x_iw_j, y_iz_j $(1 \leq i, j \leq u)$, z_iv_j, w_iv_j $(1 \leq i \leq u \ and \ 1 \leq j \leq 4k + 1)$ with blue color. It is possible to find at least k edge disjoint Hamilton cycles in each of the complete subgraphs A_i ($1 \leq i \leq 4$). Let these edge disjoint Hamilton cycles of A_i be $H_{i,1}, H_{i,2}, H_{i,3}, ..., H_{i,k}$ $(1 \leq i \leq 4)$. Color the edges of the Hamilton cycles $H_{1,1}, H_{1,2}, H_{1,3}, ..., H_{1,k}$ of A_1 and the Hamilton cycles $H_{2,1}, H_{2,2}, H_{2,3}, ..., H_{2,k}$ of A_2 with blue color and the remaining edges of both A_1 and A_2 with red color. Also, color the edges of the Hamilton cycles $H_{3,1}, H_{3,2}, H_{3,3}, ..., H_{3,k}$ of A_3 and the Hamilton cycles $H_{4,1}, H_{4,2}, H_{4,3}, ..., H_{4,k}$ of A_4 with red color and the remaining edges of A_3 and A_4 with blue color.

Sub-case (b) : $n \equiv 1 \ mod \ 4$.

Let $n = 4u + 1$. Add a new vertex p_1 to the graph given in the Sub-case (a) of Case 2. Color the edges p_1x_i, p_1y_i $(1 \leq i \leq k)$, p_1z_i, p_1w_i $(1 \leq i \leq u)$ with red color and the edges p_1x_i, p_1y_i $(k + 1 \leq i \leq u)$, p_1v_j $(1 \leq j \leq 4k + 1)$ with blue color.

Sub-case (c) : $n \equiv 2 \ mod \ 4$.

Let $n = 4u + 2$. Add a new vertex p_2 to the graph given in the Sub-case (b) of Case 2. Color the edges p_1p_2, p_2x_i, p_2y_i $(k + 1 \leq i \leq u)$, $p_2v_j (1 \leq j \leq 4k + 1)$ with red color and the edges p_2x_i, p_2y_i $(1 \leq i \leq k)$, p_2z_i, p_2w_i $(1 \leq i \leq u)$ with blue color.

Sub-case (d) : $n \equiv 3 \ mod \ 4$.

Let $n = 4u + 3$. Add a new vertex p_3 to the graph given in the Sub-case (c) of Case 2. Color the edges p_3x_i, p_3y_i $(1 \leq i \leq k)$, p_3z_i, p_3w_i $(1 \leq i \leq u)$, p_1p_3 with red color and the edges p_3x_i, p_3y_i $(k + 1 \leq i \leq u)$, p_3v_j $(1 \leq j \leq 4k + 1)$, p_2p_3 with blue color.

For the sake of clarity, the red degree at each vertex in the above subcases is given as each column of Table 2 at the end of this section.

Case 3 : $l \equiv 2 \ mod \ 4$.

Let $l = 4k + 2$, $k \geq 0$. The following coloring scheme shows that the lower bound given in Theorem 3.1 is sharp for $n \geq 8k + 12$.

Sub-case (a) : $n \equiv 0 \ mod \ 4$.

Let $n = 4u$ $(u \geq 2k + 3)$. Consider a graph $G(n, l)$ on $n + l$ vertices with the partitions X, Y, Z, W and V as explained. Color the edges x_iv_j, y_iv_j $(1 \leq i \leq u$ and $1 \leq j \leq 4k + 2)$, x_iz_j, z_iw_j $(1 \leq i, j \leq u)$, y_iw_j $(1 \leq i \neq j \leq u)$ with red color and the edges x_iy_j, x_iw_j, y_iz_j $(1 \leq i, j \leq u)$, z_iv_j, w_iv_j $(1 \leq i \leq u$ and $1 \leq j \leq 4k + 2)$, y_iw_i $(1 \leq i \leq u)$ with blue color. It is possible to find at least $k + 1$ edge disjoint Hamilton cycles in each of the complete subgraphs A_i $(1 \leq i \leq 4)$. Let these edge disjoint Hamilton cycles of A_i be $H_{i,1}, H_{i,2}, H_{i,3}, ..., H_{i,k}, H_{i,k+1}$ $(1 \leq i \leq 4)$. Color the edges of the Hamilton cycles $H_{1,1}, H_{1,2}, H_{1,3}, ..., H_{1,k+1}$ of A_1 and the Hamilton cycles $H_{2,1}, H_{2,2}, H_{2,3}, ..., H_{2,k+1}$ of A_2 with blue color and the remaining edges of both A_1 and A_2 with red color. Also, color the edges of the Hamilton cycles $H_{3,1}, H_{3,2}, H_{3,3}, ..., H_{3,k+1}$ of A_3 with red color and the remaining edges of A_3 with blue color. Let the $k + 1$ independent edges of the Hamilton cycle $H_{4,1}$ be $(w_1, w_2), (w_3, w_4), (w_5, w_6), ..., (w_{2k+1}, w_{2k+2})$. Color the edges of the Hamilton cycles $H_{4,1}, H_{4,2}, H_{4,3}, ..., H_{4,k+1}$ of A_4 other than the above independent edges with red color and the remaining edges of A_4 with blue color.

Sub-case (b) : $n \equiv 1 \ mod \ 4$.

Let $n = 4u + 1$. Add a new vertex p_1 to the graph given in the Sub-case (a) of Case 3. Color the edges p_1y_i, p_1z_i $(1 \leq i \leq u)$, p_1w_i $(1 \leq i \leq 2k + 2)$ with red color and the edges p_1x_i $(1 \leq i \leq u)$, p_1w_i $(2k + 3 \leq i \leq u)$, p_1v_j $(1 \leq j \leq 4k + 2)$ with blue color.

Sub-case (c) : $n \equiv 2 \ mod \ 4$.

Let $n = 4u + 2$. Add a new vertex p_2 to the graph given in the Sub-case (b) of Case 3. Color the edges p_2x_i $(1 \leq i \leq u)$, p_2v_j $(1 \leq j \leq 4k + 2)$, p_2w_i $(2k + 2 \leq i \leq u)$ with red color and the edges p_1p_2, p_2y_i, p_2z_i $(1 \leq i \leq u)$, p_2w_i $(1 \leq i \leq 2k + 1)$ with blue color.

Sub-case (d) : $n \equiv 3 \; mod \; 4$.

Let $n = 4u + 3$. Add a new vertex p_3 to the graph given in the Sub-case (c) of Case 3. Color the edges p_3y_i, p_3z_i $(1 \leq i \leq u)$, p_3w_i $(1 \leq i \leq 2k+1)$, p_2p_3 with red color and the edges p_3v_j $(1 \leq j \leq 4k+2)$, p_3x_i $(1 \leq i \leq u)$, p_3w_i $(2k+2 \leq i \leq u)$, p_1p_3 with blue color.

For the sake of clarity, the red degree at each vertex in the above subcases is given as each column of Table 3 at the end of this section.

Case 4 : $l \equiv 3 \; mod \; 4$.

Let $l = 4k + 3$, $k \geq 0$. The following coloring scheme shows that the lower bound given in Theorem 3.1 is sharp for $n \geq 8k + 12$.

Sub-case (a) : $n \equiv 0 \; mod \; 4$.

Let $n = 4u$ $(u \geq 2k + 3)$. Consider a graph $G(n, l)$ on $n + l$ vertices with the partitions X, Y, Z, W and V as explained. Color the edges x_iv_j, $y_iv_j(1 \leq i \leq u$ and $1 \leq j \leq 4k+3)$, x_iz_j, y_iw_j $(1 \leq i, j \leq u)$, x_iy_i $(1 \leq i \leq u)$, z_iw_j $(1 \leq i \neq j \leq u)$ with red color and the edges x_iw_j, y_iz_j $(1 \leq i, j \leq u)$, z_iv_j, w_iv_j $(1 \leq i \leq u$ and $1 \leq j \leq 4k+3)$, z_iw_i $(1 \leq i \leq u)$, x_iy_j $(1 \leq i \neq j \leq u)$ with blue color. It is possible to find at least $k + 1$ edge disjoint Hamilton cycles in each of the complete subgraphs A_i $(1 \leq i \leq 4)$. Let these edge disjoint Hamilton cycles of A_i be $H_{i,1}, H_{i,2}, H_{i,3}, ..., H_{i,k+1}$ $(1 \leq i \leq 4)$. Color the edges of the Hamilton cycles $H_{1,1}, H_{1,2}, H_{1,3}, ..., H_{1,k+1}$ of A_1 and the Hamilton cycles $H_{2,1}, H_{2,2}, H_{2,3}, ..., H_{2,k+1}$ of A_2 with blue color and the remaining edges of both A_1 and A_2 with red color. Also, color the edges of the Hamilton cycles $H_{3,1}, H_{3,2}, H_{3,3}, ..., H_{3,k+1}$ of A_3 and the Hamilton cycles $H_{4,1}, H_{4,2}, H_{4,3}, ..., H_{4,k+1}$ of A_4 with red color and the remaining edges of both A_3 and A_4 with blue color.

Sub-case (b) : $n \equiv 1 \; mod \; 4$.

Let $n = 4u + 1$. Add a new vertex p_1 to the graph given in the Sub-case (a) of Case 4. Color the edges p_1y_i, p_1z_i $(1 \leq i \leq u)$, p_1w_i $(1 \leq i \leq 2k+1)$ with red color and the edges p_1w_i $(2k+2 \leq i \leq u)$, p_1x_i $(1 \leq i \leq u)$, p_1v_j $(1 \leq j \leq 4k+3)$ with blue color.

Sub-case (c) : $n \equiv 2 \; mod \; 4$.

Let $n = 4u + 2$. Add a new vertex p_2 to the graph given in the Sub-case (b) of Case 4. Color the edges p_2x_i $(1 \leq i \leq u)$, p_2v_j $(1 \leq j \leq 4k+3)$, p_2w_i $(2k+2 \leq i \leq u)$ with red color and the edges p_1p_2, p_2y_i, p_2z_i $(1 \leq i \leq u)$, p_2w_i $(1 \leq i \leq 2k+1)$ with blue color.

Sub-case (d) : $n \equiv 3 \; mod \; 4$.

Let $n = 4u+3$. Add a new vertex p_3 to the graph given in the Sub-case (c) of Case 4. Color the edges p_3y_i, p_3z_i ($1 \le i \le u$), p_3w_i ($1 \le i \le 2k+1$), p_2p_3, p_1p_3 with red color and the edges p_3v_j ($1 \le j \le 4k+3$), p_3x_i ($1 \le i \le u$), p_3w_i ($2k+2 \le i \le u$) with blue color.

For the sake of clarity, the red degree at each vertex in the above subcases is given as each column of Table 4 at the end of this section.

In all the above cases the edges are colored in such a way that the conditions (A) and (B) given in Section 2 are satisfied at each vertex of $G(n,l)$.

Hence it is proved that the lower bound given in Theorem 3.1 is sharp for sufficiently large values of n.

Therefore we have the following theorem.

***Theorem* 4.1** — *For sufficiently large values of n and for any non negative integer l,*

$$M(K_3, G(n,l)) = \begin{cases} \frac{1}{24}[n^3 + n^2(3l-6) + n(8-6l-3l^2)] \\ \qquad\qquad if\ n\ is\ even\ and\ l\ is\ even \\ \frac{1}{24}[n^3 + n^2(3l-6) + n(5-6l-3l^2)] \\ \qquad\qquad if\ n\ is\ even,\ \frac{n}{2}\ is\ even\ and\ l\ is\ odd \\ \frac{1}{24}[n^3 + n^2(3l-6) + n(5-6l-3l^2) + 12] \\ \qquad\qquad if\ n\ is\ even,\ \frac{n}{2}\ is\ odd\ and\ l\ is\ odd \\ \frac{1}{24}[n^3 + n^2(3l-6) + n(5-6l-3l^2) + 3l] \\ \qquad\qquad if\ n\ is\ odd,\ \frac{n+l-1}{2}\ is\ even\ and\ l\ is\ even \\ \frac{1}{24}[n^3 + n^2(3l-6) + n(5-6l-3l^2) + 3l + 12] \\ \qquad\qquad if\ n\ is\ odd,\ \frac{n+l-1}{2}\ is\ odd\ and\ l\ is\ even \\ \frac{1}{24}[n^3 + n^2(3l-6) + n(8-6l-3l^2) + 3l] \\ \qquad\qquad if\ n\ is\ odd\ and\ l\ is\ odd \end{cases}$$

Table 1: Case 1: $l \equiv 0 \; mod \; 4$

Vertices	Red degree of vertices when n is equal to			
	4u	4u+1	4u+2	4u+3
x_1	2u+2k	2u+2k	2u+2k+1	2u+2k+1
x_2	2u+2k	2u+2k	2u+2k+1	2u+2k+1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_u	2u+2k	2u+2k	2u+2k+1	2u+2k+1
y_1	2u+2k-1	2u+2k	2u+2k	2u+2k+1
y_2	2u+2k-1	2u+2k	2u+2k	2u+2k+1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
y_u	2u+2k-1	2u+2k	2u+2k	2u+2k+1
z_1	2u+2k-1	2u+2k	2u+2k	2u+2k+1
z_2	2u+2k-1	2u+2k	2u+2k	2u+2k+1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
z_u	2u+2k-1	2u+2k	2u+2k	2u+2k+1
w_1	2u+2k-1	2u+2k	2u+2k	2u+2k+1
w_2	2u+2k-1	2u+2k	2u+2k	2u+2k+1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
w_{2k}	2u+2k-1	2u+2k	2u+2k	2u+2k+1
w_{2k+1}	2u+2k	2u+2k	2u+2k+1	2u+2k+1
w_{2k+2}	2u+2k	2u+2k	2u+2k+1	2u+2k+1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
w_u	2u+2k	2u+2k	2u+2k+1	2u+2k+1
v_1	2u	2u	2u+1	2u+1
v_2	2u	2u	2u+1	2u+1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
v_{4k}	2u	2u	2u+1	2u+1
p_1	-	2u+2k	2u+2k+1	2u+2k+1
p_2	-	-	2u+2k+1	2u+2k+1
p_3	-	-	-	2u+2k

Table 2: Case 2: $l \equiv 1 \; mod \; 4$

Vertices	Red degree of vertices when n is equal to			
	4u	4u+1	4u+2	4u+3
x_1	2u+2k	2u+2k+1	2u+2k+1	2u+2k+2
x_2	2u+2k	2u+2k+1	2u+2k+1	2u+2k+2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_k	2u+2k	2u+2k+1	2u+2k+1	2u+2k+2
x_{k+1}	2u+2k	2u+2k	2u+2k+1	2u+2k+1
x_{k+2}	2u+2k	2u+2k	2u+2k+1	2u+2k+1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_u	2u+2k	2u+2k	2u+2k+1	2u+2k+1
y_1	2u+2k	2u+2k+1	2u+2k+1	2u+2k+2
y_2	2u+2k	2u+2k+1	2u+2k+1	2u+2k+2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
y_k	2u+2k	2u+2k+1	2u+2k+1	2u+2k+2
y_{k+1}	2u+2k	2u+2k	2u+2k+1	2u+2k+1
y_{k+2}	2u+2k	2u+2k	2u+2k+1	2u+2k+1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
y_u	2u+2k	2u+2k	2u+2k+1	2u+2k+1
z_1	2u+2k	2u+2k+1	2u+2k+1	2u+2k+2
z_2	2u+2k	2u+2k+1	2u+2k+1	2u+2k+2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
z_u	2u+2k	2u+2k+1	2u+2k+1	2u+2k+2
w_1	2u+2k	2u+2k+1	2u+2k+1	2u+2k+2
w_2	2u+2k	2u+2k+1	2u+2k+1	2u+2k+2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
w_u	2u+2k	2u+2k+1	2u+2k+1	2u+2k+2
v_1	2u	2u	2u+1	2u+1
v_2	2u	2u	2u+1	2u+1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
v_{4k}	2u	2u	2u+1	2u+1
v_{4k+1}	2u	2u	2u+1	2u+1
p_1	-	2u+2k	2u+2k+1	2u+2k+2
p_2	-	-	2u+2k+2	2u+2k+2
p_3	-	-	-	2u+2k+1

Table 3: Case 3: $l \equiv 2 \ mod \ 4$

Vertices	Red degree of vertices when n is equal to			
	4u	4u+1	4u+2	4u+3
x_1	2u+2k+1	2u+2k+1	2u+2k+2	2u+2k+2
x_2	2u+2k+1	2u+2k+1	2u+2k+2	2u+2k+2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_u	2u+2k+1	2u+2k+1	2u+2k+2	2u+2k+2
y_1	2u+2k	2u+2k+1	2u+2k+1	2u+2k+2
y_2	2u+2k	2u+2k+1	2u+2k+1	2u+2k+2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
y_u	2u+2k	2u+2k+1	2u+2k+1	2u+2k+2
z_1	2u+2k	2u+2k+1	2u+2k+1	2u+2k+2
z_2	2u+2k	2u+2k+1	2u+2k+1	2u+2k+2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
z_u	2u+2k	2u+2k+1	2u+2k+1	2u+2k+2
w_1	2u+2k	2u+2k+1	2u+2k+1	2u+2k+2
w_2	2u+2k	2u+2k+1	2u+2k+1	2u+2k+2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
w_{2k+1}	2u+2k	2u+2k+1	2u+2k+1	2u+2k+2
w_{2k+2}	2u+2k	2u+2k+1	2u+2k+2	2u+2k+2
w_{2k+3}	2u+2k+1	2u+2k+1	2u+2k+2	2u+2k+2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
w_u	2u+2k+1	2u+2k+1	2u+2k+2	2u+2k+2
v_1	2u	2u	2u+1	2u+1
v_2	2u	2u	2u+1	2u+1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
v_{4k+1}	2u	2u	2u+1	2u+1
v_{4k+2}	2u	2u	2u+1	2u+1
p_1	-	2u+2k+2	2u+2k+2	2u+2k+2
p_2	-	-	2u+2k+1	2u+2k+2
p_3	-	-	-	2u+2k+2

Table 4: Case 4: $l \equiv 3 \; mod \; 4$

Vertices	Red degree of vertices when n is equal to			
	4u	4u+1	4u+2	4u+3
x_1	2u+2k+1	2u+2k+1	2u+2k+2	2u+2k+2
x_2	2u+2k+1	2u+2k+1	2u+2k+2	2u+2k+2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_u	2u+2k+1	2u+2k+1	2u+2k+2	2u+2k+2
y_1	2u+2k+1	2u+2k+2	2u+2k+2	2u+2k+3
y_2	2u+2k+1	2u+2k+2	2u+2k+2	2u+2k+3
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
y_u	2u+2k+1	2u+2k+2	2u+2k+2	2u+2k+3
z_1	2u+2k+1	2u+2k+2	2u+2k+2	2u+2k+3
z_2	2u+2k+1	2u+2k+2	2u+2k+2	2u+2k+3
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
z_u	2u+2k+1	2u+2k+2	2u+2k+2	2u+2k+3
w_1	2u+2k+1	2u+2k+2	2u+2k+2	2u+2k+3
w_2	2u+2k+1	2u+2k+2	2u+2k+2	2u+2k+3
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
w_{2k+1}	2u+2k+1	2u+2k+2	2u+2k+2	2u+2k+3
w_{2k+2}	2u+2k+1	2u+2k+1	2u+2k+2	2u+2k+2
w_{2k+3}	2u+2k+1	2u+2k+1	2u+2k+2	2u+2k+2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
w_u	2u+2k+1	2u+2k+1	2u+2k+2	2u+2k+2
v_1	2u	2u	2u+1	2u+1
v_2	2u	2u	2u+1	2u+1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
v_{4k+3}	2u	2u	2u+1	2u+1
p_1	-	2u+2k+1	2u+2k+1	2u+2k+2
p_2	-	-	2u+2k+2	2u+2k+3
p_3	-	-	-	2u+2k+3

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