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ON CHEVALLEY'S $\mathbb{Z}$-FORM

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We give a new proof of the existence of a Chevalley basis for a semisimple Lie algebra over $\mathbb{C}$.

Key words : Chevalley basis; semisimple Lie algebras.

1. Introduction

Let $\mathfrak{g}$ be a complex semisimple Lie algebra. In a path-breaking paper [2], Chevalley exhibitted a basis $\mathbb{B}$ of $\mathfrak{g}$ (known as Chevalley basis since then) such that the structural constants of $\mathfrak{g}$ with respect to the basis $\mathbb{B}$ are integers. The basis moreover consists of a basis of a Cartan subalgebra $\mathfrak{t}$ (on which the roots take integral values) together with root vectors of $\mathfrak{g}$ with respect to $\mathfrak{t}$. The structural constants were determined explicitly (up to signs) in terms of the structure of the root system of $\mathfrak{g}$. Tits [3] provided a more elegant approach to obtain Chevalley's results which essentially exploited geometric properties of the root system. Casselman [1] used the methods of Tits to extend the Chevalley theorem to the Kac-Moody case. In this note we prove Chevalley's theorem through an approach different from those of Chevalley and Tits.

Let $\mathbf{G}$ be the simply connected algebraic group corresponding to $\mathfrak{g}$. Let $\mathbf{T}$ be a maximal torus (the Lie subalgebra $\mathfrak{t}$ corresponding to it is a Cartan subalgebra of $\mathbf{G}$). Let Φ be the root system of $\mathbf{G}$ with respect to $\mathbf{T}$ and Δ a simple root system. For $\varphi \in \Phi$, let $\mathbf{G}(\varphi)$ be the 3-dimensional subgroup (isomorphic to $SL(2)$) corresponding to φ. For an element $v \neq 0$ in the root space $\mathfrak{g}^{\varphi}$, there is a unique natural isomorphism $f_{\varphi,v}$ of $SL(2)$ on $\mathbf{G}(\varphi)$ taking the diagonal group into $\mathbf{T}$ and the induced Lie algebra map taking e_{12} to v. Set $f_{(\varphi,v)}(e_{12} - e_{21}) = s_{\varphi,v}$. The key result we prove in this note is the following.

For each $\alpha \in \Delta$, let e_α be a non-zero element in $\mathfrak{g}^\alpha$ and set $s_\alpha = s_{\alpha, e_\alpha}$. Let $\tilde{\mathbf{W}}$ be the sugroup of $\mathbf{G}$ generated by $\{s_\alpha | \alpha \in \Delta\}$. It is well known that $\tilde{\mathbf{W}} \subset N(\mathbf{T})$, the normalizer of $\mathbf{T}$. Then $\tilde{\mathbf{W}} \cap \mathbf{T} = \mathbf{T}(2)$, the group of 2-torsion elements in $\mathbf{T}$. Further if $w, w' \in \tilde{\mathbf{W}}$ and $\alpha, \alpha' \in \Delta$ with $w(\alpha) = w(\alpha')$, $Ad(w)(e_\alpha) = \pm Ad(w')(e_{\alpha'})$.

One deduces then the existence of a Chevalley basis from the above two assertions. The second assertion above is in fact equivalent to the existence of the Chevalley basis and the first can be deduced from the existence. The first assertion seems to be of some interest in itself. It says that while the sequence

$$\{1\} \to \mathbf{T} \to N(\mathbf{T}) \to N(\mathbf{T})/\mathbf{T} \to \{1\}$$

does not split it comes close to to doing so.

2. Representations of $SL(2, \mathbb{C})$

We recall in this section some basic facts about finite dimensional (holomorphic) representations of the $SL(2, \mathbb{C})$ (or simply $SL(2)$), the group of (2×2)-matrices over $\mathbb{C}$ of determinant 1 (under multiplication). D will denote the group of diagonal matrices in $SL(2)$. For $t \in \mathbb{C}^*$, $d(t)$ denotes the diagonal matrix with $d(t)_{11} = t$ and $d(t)_{22} = t^{-1}$. We set $\mathbb{C}^2 = V$ and denote by ρ the natural representation of $SL(2)$ on V. We denote by $\mathfrak{sl}(2, \mathbb{C})$ (or simply $\mathfrak{sl}(2)$), the Lie algebra $\{X \epsilon M(2, \mathbb{C}) | trace(X) = 0\}$ of $SL(2)$. We let $\dot{\rho}$ denote the natural representation of $\mathfrak{sl}(2)$ on V (induced by ρ). For $1 \leq i, j \leq 2$, let e_{ij} be the matrix whose $(k\ l)$-th entry is $\delta_{ik} \cdot \delta_{jl}$. Set $s = e_{12} - e_{21}$, $e_+ = e_{12}$, $e_- = e_{21}$ and $h = e_{11} - e_{22}$. Then $s \in SL(2)$ is the "Weyl" element; it normalizes D and $s^2 = -$(identity). Further, $\{e_+, h, e_-\}$ is a basis of $\mathfrak{sl}(2)$; and we have $[h, e_+] = 2 \cdot e_+$, $[h, e_-] = -2 \cdot e_-$ and $[e_+, e_-] = h$.

For an integer $m \geq 0$ we denote by ρ_m (resp. $\dot{\rho}_m$) the representation of $SL(2)$ (resp. $\mathfrak{sl}(2)$) induced by ρ on the symmetric m-th power $S^m(V)$ of V. If $\{u, v\}$ is the standard basis of $V = \mathbb{C}^2$, $\{u^p \cdot v^q |\ p + q = m\}$ is a basis of $S^m(V)$. Let $t \epsilon \mathbb{C}^*$ and $s \epsilon SL(2)$ be the matrix $e_+ - e_-$. One then has, as is easily seen by inspection:

Lemma 2.1 —

$$\rho_m(d(t))(u^p \cdot v^q) = t^{p-q} \cdot u^p \cdot v^q$$

$$\dot{\rho}_m(e_-)(u^p \cdot v^q) = q \cdot u^{p+1} \cdot v^{q-1}$$

$$\dot{\rho}_m(e_+)(u^p \cdot v^q) = p \cdot u^{p-1} \cdot v^{q+1}$$

$$\dot{\rho}_m(h)(u^p \cdot v^q) = 2 \cdot (p - q) \cdot u^p \cdot v^q$$

$$\rho_m(s)(u^p \cdot v^q) = (-1)^p \cdot u^q \cdot v^p = (-1)^p \cdot (q!/p!)^{\pm 1} \cdot \dot{\rho}_m(e_\mp)^{\pm(p-q)}(u^p \cdot v^q)$$

according as $p \geq q$ or $q \geq p$.

For any integer $m \geq 0$, ρ_m is an irreducible representation of $SL(2)$ of dimension $m+1$ and, up to equivalence, it is the unique irreducible representation of dimension $m+1$. Moreover every representation τ of $SL(2)$ is a direct sum of irreducible representations; thus τ is a direct sum of copies of the "ρ_m"s. One deduces from the lemma above the following:

Corollary 2.2 — Let τ be an irreducible representation of $SL(2)$ on a vector space E of dimension $m+1$. If z is an eigen vector for $\tau(h)$ in E with eigen value k,

$$\tau(s)(z) = (-1)^{(m+k)/2} \cdot ((m-k)/2)! \cdot \tau(e_\mp)^k(z)/((m+k)/2)!$$

In particular, if $z \epsilon E$ is a D-fixed vector, $\tau(s)(z) = \pm z$.

3. Semisimple Lie Groups, Maximal Tori, Roots

3.1 *Maximal torus, roots and root-spaces*

Let $\mathbf{G}$ be a connected simply connected semisimple algebraic group over $\mathbb{C}$ and $\mathfrak{g}$ its Lie algebra. Let $\mathbf{T}$ be a maximal torus in $\mathbf{G}$ and $\mathfrak{t} \subset \mathfrak{g}$ the Lie subalgebra of $\mathfrak{g}$ corresponding to $\mathfrak{t}$. A root of $\mathbf{G}$ (with respect to $\mathbf{T}$) is a *non-trivial eigen character* $\varphi : \mathbf{T} \to \mathbb{C}^*$ for the adjoint action of $\mathbf{T}$ on the Lie algebra $\mathfrak{g}$. We denote by $X(\mathbf{T})$ the group of characters on $\mathbf{T}$ and by $\Phi \subset X(\mathbf{T})$ the set of roots. For $\varphi \in \Phi$, $\mathfrak{g}^\varphi = \{v \in \mathfrak{g} | Ad(t)(v) = \varphi(t) \cdot v, \forall t \in \mathbf{T}\}$, the eigenspace corresponding to φ, "the root-space of φ" is of dimension 1 and $\mathfrak{g} = \mathfrak{t} \oplus \coprod_{\varphi \in \Phi} \mathfrak{g}^\varphi$. The Killing form $< .,. >$ on $\mathfrak{g}$, restricted to $\mathfrak{t}$ is non-degenerate. Hence it defines an isomorphism A of $\mathfrak{t}$ on its dual $\mathfrak{t}^*$. For $\lambda \in \mathfrak{t}^*$, set $\mathsf{A}^{-1}(\lambda) = \mathsf{H}_\lambda$. We have then for $H \in \mathfrak{t}$ and $\lambda \in \mathfrak{t}^*$, $< H_\lambda, H >$ = $\lambda(H)$. For $\lambda, \mu \in \mathfrak{t}^*$, set $< \lambda, \mu > = < H_\lambda, H_\mu >$.

3.2 *Simple roots; the Weyl group*

Let $N(\mathbf{T})$ be the normalizer of $\mathbf{T}$ (in $\mathbf{G}$) and $\mathbf{W}$ = $N(\mathbf{T})/\mathbf{T}$, the Weyl group. The inner conjugation action of $N(\mathbf{T})$ on $\mathbf{T}$ factors through to an action of $\mathbf{W}$ on $\mathbf{T}$ and hence on $X(\mathbf{T})$; the set Φ is stable under this action. The set $\{< \varphi, \varphi >^{1/2} \mid \varphi \in \Phi\}$ of "root-lengths" has at most two elements and the Weyl group acts transitively on the set of all roots of the same length. We fix a lexicographic (total) order in $X(\mathbf{T})$ and denote by Φ^+ (resp. Φ^-, resp. Δ) the system of positive (resp. negative, resp. simple) roots. Let Ω be the set of roots dominant for the simple system Δ. The cardinality of Ω is the same as that of the set of root-lengths. For $\beta \in \Omega$, $\mathbf{P}_\beta$ is the parabolic subgroup determined by β: it is the sugroup of $\mathbf{G}$ corresponding to the Lie sub-algebra generated by $\{\mathfrak{g}^\varphi | \varphi \in \Phi^+\}$, $\mathfrak{t}$ and $\{\mathfrak{g}^\varphi | \varphi \in \Phi, < \varphi, \beta > = 0\}$.

3.3 *The 3-dimensional sub-algebras* $\mathfrak{g}^\alpha$

For α in Φ, let $h_\alpha = 2 \cdot H_\alpha / < H_\alpha, H_\alpha >$. Now for any $0 \neq v_+ \in \mathfrak{g}^\alpha$; there is a unique $v_- \in \mathfrak{g}^{-\alpha}$ with $[v_+, v_-] = h_\alpha$. Define $i_{\alpha,v} : \mathfrak{sl}(2) \to \mathfrak{g}$ by setting $i_{\alpha,v}(e_\pm) = v_\pm, i_{\alpha,v}(h) = h_\alpha$: it is a Lie algebra isomorphism of $\mathfrak{sl}(2)$ on the Lie subalgebra $\mathfrak{g}^\alpha \oplus \mathbb{C} \cdot h_\alpha \oplus \mathfrak{g}^{-\alpha}$ = $\mathfrak{g}(\alpha)$; the Lie subgroup $\mathbf{G}(\alpha)$ corresponding to $\mathfrak{g}(\alpha)$ is an algebraic subgroup and $i_{\alpha,v}$ integrates to an isomorphism $f_{\alpha,v}$ of $GL(2)$ on $\mathbf{G}(\alpha)$ - we have assumed that $\mathbf{G}$ is simply connected. We set $s_{\alpha,v} = f_{\alpha,v}(s)$. As $s^2 = -(Identity)$ in $SL(2)$, $s^2_{\alpha,v}$ is the unique non-trivial central element in $\mathbf{G}(\alpha)$ and is of order 2. It is easy to see that $s_{\alpha,v}$ belongs to $N(\mathbf{T})$ and that it acts trivially on $\{t \in \mathbf{T} | \alpha(t) = 1\}$, the kernel of α.

3.4 The group $\tilde{\mathbf{W}}$

For each α in Δ we now fix elements $e_{\pm\alpha}$ with $[e_\alpha, e_{-\alpha}] = h_\alpha$ and set $i_{\alpha,e_\alpha} = i_\alpha$, $f_{\alpha,e_\alpha} = f_\alpha$ and $s_{\alpha,e_\alpha} = s_\alpha$. Let $\tilde{\mathbf{W}}$ be the subgroup of $\mathbf{G}$ generated by $\tilde{S}$ = $\{s_\alpha | \alpha \in \Delta\}$. Then $\tilde{\mathbf{W}}$ maps onto $\mathbf{W}$ under the projection map $\pi : N(\mathbf{T}) \to \mathbf{W}$. Let w_α = $\pi(s_\alpha)$ and $S = \pi(\tilde{S})$. Then $(\mathbf{W}, \mathbf{S})$ is a Coxeter system. Let B be the kernel of $\pi : \tilde{\mathbf{W}} \to \mathbf{W}$; B = $\mathbf{T} \cap \tilde{\mathbf{W}}$. We have with this notation the first key result.

Proposition 3.5 — $B = \mathbf{T}_2$, the subgroup of 2-torsion elements in $\mathbf{T}$.

PROOF : Since $(\mathbf{W}, \mathbf{S})$ is a Coxeter system the group B is generated as a normal subgroup by $\{(s_\alpha \cdot s_\beta)^{p_{\alpha\beta}} | \alpha, \beta \in \Delta\}$: here $p_{\alpha\beta}$ is the order of $w_\alpha \cdot w_\beta$ in $\mathbf{W}$. It suffices therefore to prove that $\{(s_\alpha \cdot s_\beta)^{p_{\alpha\beta}} | \alpha, \beta \in \Delta\}$ are all of order 2: note that when $\alpha = \beta$, $p_{\alpha\beta} = 1$ and the $\{s^2_\alpha | \alpha \in \Delta\}$ generate $\mathbf{T}(2)$ since $\mathbf{G}$ is simply connected. We will deal with each of the cases $p = p_{\alpha\beta} = 1, 2, 3, 4$ and 6 separately:

1. $\underline{p = 1}$. This is the case $\alpha = \beta$ and by its very definition s^2_α has order 2.

2. $\underline{p = 2}$. In this case the groups $G(\alpha)$ and $G(\beta)$ commute, hence so do s_α and s_β; it follows that $(s_\alpha \cdot s_\beta)^2$ has order 2.

3. $\underline{p = 3}$. Here $\mathfrak{g}^\alpha \oplus \mathfrak{g}^{(\alpha+\beta)}$ (resp. $\mathfrak{g}^\beta \oplus \mathfrak{g}^{(\alpha+\beta)}$) is an irreducible $\mathbf{G}(\beta)$- (resp. $\mathbf{G}(\alpha)$-) module (of dimension 2). One sees now (from representation theory of $SL(2)$) that we have $s_\beta(e_\alpha) = [e_\beta, e_\alpha] = -[e_\alpha, e_\beta] = -s_\alpha(e_\beta)$. Set $v = [e_\beta, e_\alpha] \in \mathfrak{g}^{(\alpha+\beta)}$. It follows that $s_\beta \cdot s_\alpha \cdot s_\beta = s_\beta \cdot s_\alpha \cdot s^{-1}_\beta \cdot s^2_\beta = s_{(\alpha+\beta),v}$(mod $\mathbf{T}(2)$) = $s^{-1}_{(\alpha+\beta),v}$(mod $\mathbf{T}(2)$) = $s_\alpha \cdot s_\beta \cdot s_\alpha$(mod $\mathbf{T}(2)$); as $s_\alpha = s^{-1}_\alpha$(mod $\mathbf{T}(2)$), $(s_\alpha \cdot s_\beta \cdot s_\alpha)^2 \in \mathbf{T}(2)$.

4. $\underline{p = 4}$. The root system is necessarily of type B_2. We take α to be the long root. We then have $s_\alpha(\beta) = \alpha + \beta$ and $< \beta, \alpha + \beta >= 0$. The subspace $\mathfrak{g}^{\pm\alpha} \oplus \mathfrak{g}^{\pm(\alpha+\beta)} \oplus \mathfrak{g}^{\pm(\alpha \oplus 2 \cdot \beta)}$ of $\mathfrak{g}$ is

then an irreducible $\mathbf{G}(\beta)(\simeq SL(2))$-representation and one concludes from Corollary 2.2 that $Ad(s_\beta)(v) = v$ for all $v \in \mathfrak{g}^{\pm(\alpha+\beta)}$. It follows that s_β and $s_{((\alpha+\beta),v)}$ commute for all $v \neq 0$ in $\mathfrak{g}^{\alpha+\beta}$. Now $s_\alpha \cdot s_\beta \cdot s_\alpha = s_\alpha \cdot s_\beta \cdot s_\alpha^{-1}(mod\ \mathbf{T}(2))$ = $s_{((\alpha+\beta),s_\alpha(e_\beta))}$. Thus $(s_\alpha \cdot s_\beta)^2 \in \mathbf{T}(2)$ and hence $(s_\alpha \cdot s_\beta)^4 = 1$.

5. $\underline{p = 6}$. The root system is of type G_2. We take α to be the long root. Then $\{m \cdot \alpha + n \cdot \beta \mid 0 \leq m \leq 2, 0 \leq n \leq 3, m + n \neq 0\}$ is the set of positive roots. Now s_β (resp. s_α) conjugates $\mathbf{G}(\alpha)$ (resp. $\mathbf{G}(\beta)$) into $\mathbf{G}(\alpha + 3 \cdot \beta)$ (resp. $\mathbf{G}(\beta + \alpha)$) taking s_α (resp. s_β) into an element η (resp. ξ) with η^2 (resp. ξ^2) in $\mathbf{T}(2)$. As $(\alpha + 3 \cdot \beta) \pm (\alpha + \beta)$ are not roots, $\mathbf{G}(\alpha + 3 \cdot \beta)$ and $\mathbf{G}(\beta + \alpha)$ commute. Thus $(s_\alpha \cdot s_\beta)^3$ = $s_\alpha \cdot s_\beta \cdot s_\alpha \cdot s_\beta \cdot s_\alpha \cdot s_\beta$ = $\xi \cdot s_\alpha^2 \cdot \eta \cdot s_\beta^2$ = $\xi \cdot \eta$ (mod $\mathbf{T}(2)$). Hence $(s_\alpha \cdot s_\beta)^6$ = $(\xi)^2 \cdot (\eta)^2$ (mod $\mathbf{T}(2)$) and thus is in $\mathbf{T}(2)$.

This completes the proof of Proposition 3.5.

4. The Chevalley Basis

The key step we need is:

Proposition 4.1 — If $\alpha, \alpha' \in \Phi$ and $w, w' \in \tilde{\mathbf{W}}$ are such that $w(\alpha) = w'(\alpha')$, then $Ad(w)(e_\alpha)$ $= \pm Ad(w')(e_{\alpha'})$.

Proof : Set $\beta = w(\alpha) = w'(\alpha')$. Then one has $w \cdot s_\alpha \cdot w^{-1} = s_{(\beta, Ad(w)(e_\alpha))}$ and $w' \cdot s_{\alpha'} \cdot w'^{-1} = s_{(\beta, Ad(w')(e'_\alpha))}$. We denote these two elements by τ and τ' respectively. Then $\tau, \tau' \in \mathbf{G}(\beta)$. As they both have the same image w_β in $\mathbf{W}$, we have $\tau' = \tau \cdot \theta$ with $\theta \in \mathbf{G}(\beta) \cap \mathbf{T}(2)$; it follows that θ is $\pm(Identity)$ in $\mathbf{G}(\beta)$. The proposition now follows from the fact that the map $v \to s_{\beta,v}$ of $\mathfrak{g}^\beta \backslash \{0\}$ into $\tilde{\mathbf{W}}$ is injective.

Definition 4.2 — A basis $\mathbb{B}$ for $\mathfrak{g}$ is *adapted to a maximal torus* $\mathbf{T}$ if the following conditions hold.

1. For some simple system Δ in Φ, the root system of $\mathbf{G}$ with respect to $\mathbf{T}$, $\mathbb{B} \cap \mathfrak{t} = \{H_\alpha | \alpha \in \Delta\}$
2. For any root $\varphi \in \Phi$, $\mathbb{B} \cap (\mathfrak{g}^\varphi \backslash \{0\})$ is non-empty =: $\{e_\varphi\}$.
3. For α in Δ, $[h_\alpha, e_{\pm\alpha}] = \pm 2 \cdot e_\alpha$ and $[e_\alpha, e_{-\alpha}] = h_\alpha]$.
4. If $f_\alpha : SL(2) \to \mathbb{G}(\alpha)$ is the isomorphism given by $f_\alpha(\pm e) = e_{\pm\alpha}$ and $f_\alpha(h) = h_\alpha$ and $\tilde{\mathbf{W}}$ is the group generated by the $\{s_\alpha | \alpha \in \Delta\}$, then for $\varphi \in \Phi$ and $w \in \tilde{\mathbb{W}}$, $Ad(w)(e_\varphi) = \epsilon(w, \varphi) \cdot e_{w(\varphi)}$ where $\epsilon(w, \varphi) = \epsilon(w, -\varphi) = \pm 1$.

Remark 4.3 :

1. The discussion preceding the definition shows that a basis adapted to a maximal torus exists.

2. In view of the fact that every root has a $\tilde{\mathbf{W}}$-conjugate in Δ, we have for any $\varphi \in \Phi$, $[h_\varphi, e_{\pm\varphi}] = 2 \cdot e_{\pm\varphi}$ and $[e_\varphi, e_{-\varphi}] = h_\varphi$.
3. If $\{h_\alpha |\ \alpha \in \Delta\} \cup \{e_\varphi \in \mathfrak{g}^\varphi |\ \varphi \in \Phi\}$ is a basis adapted to $\mathbf{T}$ and $\delta : \Phi \to \pm 1$ is a map with $\delta(\varphi) = \delta(-\varphi)$ for all $\varphi \in \Phi$, $\{h_\alpha |\ \alpha \in \Delta\} \cup \{\delta(\varphi) \cdot e_\varphi \mid \varphi \in \Phi\}$ is again a basis adapted to $\mathbf{T}$.
4. One can find a basis adapted to a maximal torus such that for any automorphism σ of the Dynkin diagram of Δ, $\sigma(e_\alpha) = e_{\sigma(\alpha)}$.
5. For $\varphi \in \Phi$, the map $i_\varphi : \mathfrak{sl}(2) \to \mathfrak{g}$ defined by setting $i_\varphi(e_\pm) = e_{\pm\varphi}, i_\varphi(h) = h_\varphi$ is an isomorphism of $\mathfrak{sl}(2)$ on $\mathfrak{g}(\varphi)$, the Lie algebra of $\mathbf{G}(\varphi)$; if $f_\varphi : SL(2) \to \mathbf{G}(\varphi)$ is the morphism induced by i_φ and $s_\varphi = f_\varphi(s)$, then for $w \in \tilde{\mathbf{W}}$ and $\varphi \in \Phi$, $w \cdot s_\varphi \cdot w^{-1} = s_{w(\varphi)}$ (mod $\mathbf{T}(2)$); as $\mathbf{T}(2) \subset \tilde{\mathbf{W}}$ and every φ in Φ is in the $\tilde{\mathbf{W}}$ orbit of an $\alpha \in \Delta$, s_φ is in $\tilde{\mathbf{W}}$.

Chevalley's theorem — We will now establish the following result which asserts that a basis adapted to a maximal torus is indeed a Chevalley basis.

***Theorem* 4.4** — *Let $h_\alpha | \alpha \in \Delta\} \cup \{e_\varphi | \varphi \in \Phi\}$ be a basis adapted to a maximal torus of $\mathfrak{g}$. Then for φ, ψ, $\varphi + \psi$ in Φ, $[e_\varphi, e_\psi] = \pm N_{\varphi,\psi} e_{(\varphi+\psi)}$ where $N_{\varphi,\psi} - 1 = max\{k |\ \varphi - k \cdot \psi \in \Phi\}$.*

PROOF : We begin with the observation that for $w \in \tilde{\mathbf{W}}$ and $\varphi \in \Phi$, $w \cdot s_\varphi \cdot w^{-1} = s_{w(\varphi)}$ (mod $\mathbf{T}(2)$). Let φ, ψ and $\varphi + \psi$ be in Φ. Let $\mathfrak{m}$ be the subalgebra of $\mathfrak{g}$ (of rank 2) generated by $\{\mathfrak{g}^{k \cdot \varphi + l \cdot \psi} |\ k, l \in \mathbb{Z}, (k \cdot \varphi + l \cdot \psi) \in \Phi\}$. Then $\mathfrak{m}$ is a semi-simple Lie algebra of rank 2. In the light of Proposition 4.1, we see now that it suffices to prove the theorem for groups of rank 2. Thus we assume that $\mathfrak{g} = \mathfrak{m}$ and $\mathfrak{h} = \mathbb{C} \cdot h_\varphi \oplus \mathbb{C} \cdot h_\psi$. Let $h \in \mathfrak{h}$ be an element such that $\varphi(h) = 0$ while $\psi(h) > 0$. We take on the dual of $\mathfrak{h}_\mathbb{R} = \mathbb{R} \cdot h_\varphi \oplus \mathbb{R} \cdot h_\psi$, the lexiographic order determined by the ordered basis (h, h_ϕ). Then one sees easily that φ is a simple root i.e, $\phi \in \Delta$ while $\psi \in \Phi^+$. Let E be the smallest Ad($\mathbf{G}(\psi)$)-stable subspace of $\mathfrak{g}$ containing $\mathfrak{g}^\varphi$. Then $E = \coprod_r \mathfrak{g}^{(\varphi + r \cdot \psi)}$ is easily seen to be an irreducible representation of $\mathbf{G}(\psi) (\simeq SL(2))$. The desired results now follow Lemma 2.2. Since $\varphi + \psi$ is a root we have only three possibilities: (i) $\varphi - \psi$ is not a root or (ii) $\varphi - \psi$ is a root, but $\varphi - 2 \cdot \psi$ is not a root or (iii) $\varphi - \psi$ as well as $\varphi - 2 \cdot \psi$ are roots root. Correspondingly the dimension of E is 2, 3 or 4.

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