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ON POSITIVE SOLUTIONS OF m-POINT BOUNDARY VALUE PROBLEMS FOR p-LAPLACIAN IMPULSIVE DYNAMIC EQUATIONS ON TIME SCALES

Fatma Tokmak Fen* and Ilkay Yaslan Karaca**

**Department of Mathematics, Gazi University,* 06500 *Teknikokullar, Ankara, Turkey*
***Department of Mathematics, Ege University,* 35100 *Bornova, Izmir, Turkey*
e-mails: fatma.tokmakk@gmail.com, ilkay.karaca@ege.edu.tr

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In this paper, we consider a second-order m-point impulsive boundary value problem on time scales. We establish the criteria for the existence of at least two positive solutions of a non-eigenvalue problem. Later, we study the existence of at least one positive solution of an eigenvalue problem. We also give two examples to illustrate our results.

Key words : Positive solutions; double fixed-point theorem; fixed point index theorem; time scales; impulsive boundary value problems.

1. INTRODUCTION

It is known that the theory of impulsive differential equations have become more important in recent years in some mathematical models of real processes and phenomena. For the introduction of the basic theory of impulsive equations, see [1, 4, 12, 17] and the references therein.

The theory of dynamic equations on time scales has been developing rapidly and have received much attention in recent years. The study unifies existing result in differential and finite difference equations and provides powerful new tools for exploring connections between the traditionally separated fields. We refer to the books by Bohner and Peterson [6, 7].

However, the corresponding theory of such equations is still in the beginning stages of its development, especially the impulsive dynamic system on time scales, see [3, 5, 10, 11, 14, 18, 19]. Few works have been done on the existence of multiple positive solutions for p-Laplacian impulsive boundary value problems on time scales, see [8, 9, 16]. Moreover, multiplicity of positive solutions for m-point boundary value problems for the p-Laplacian impulsive dynamic equations has seldom been studied; see [20, 15].

In [20], when $\mathbb{T} = \mathbb{R}$, Zhang and Ge studied the multi-point boundary value problems with one dimensional p-Laplacian and impulse effects

$$\begin{cases} -\left(\phi_p(u'(t))\right)' = f(t, u(t)),\ t \neq t_k,\ t \in (0,1), \\ -\Delta u|_{t=t_k} = I_k(u(t_k)),\ k = 1, 2, ..., n, \\ u'(0) = \sum_{i=1}^{m-2} a_i u(\xi_i), \quad u'(1) = 0. \end{cases} \tag{1.1}$$

Utilizing fixed-point index theorem, the authors concerned the existence and multiplicity of positive solutions for problem (1.1).

In [15], Li *et al*., studied the following m-point boundary value problems for p-Laplacian impulsive dynamic equations on time scales

$$\begin{cases} [\phi_p(y^{\triangle}(t))]^{\nabla} + w(t)f(t,y(t)) = 0,\ t \in [0,T]_{\mathbb{T}},\ t \neq t_k,\ k = 1,2,...,n \\ y(t_k^+) - y(t_k^-) = I_k(u(t_k)), \\ y(0) = \sum_{i=1}^{m-2} a_i y(\xi_i), \quad y^{\triangle}(T) = 0. \end{cases} \tag{1.2}$$

By using the Leray-Schauder fixed point theorem and the nonlinear alternative of Leray-Schauder type, they get the existence of at least one positive solution. They also considered the existence of at least three positive solutions by the help of a new fixed point theorem.

Motivated by the above results, in this study, we consider the following second-order impulsive boundary value problem (BVP) on time scales

$$-[\phi_p(u^{\triangle}(t))]^{\nabla} = f(t,u(t)),\ t \in J := [0,1]_{\mathbb{T}},\ t \neq t_k,\ k = 1,2,...,n \tag{1.3}$$

$$-\Delta u|_{t=t_k} = I_k(u(t_k)), \tag{1.4}$$

$$u^{\triangle}(0) = 0, \quad \alpha u(1) + \beta u^{\triangle}(1) = \sum_{i=1}^{m-2} a_i u(\xi_i), \tag{1.5}$$

and the eigenvalue problem

$$-[\phi_p(u^{\triangle}(t))]^{\nabla} = \lambda f(t,u(t)) \tag{1.6}$$

with the same boundary conditions where $\lambda > 0,\ \mathbb{T}$ is a time scale, $0,1 \in \mathbb{T},\ [0,1]_{\mathbb{T}} = [0,1] \cap \mathbb{T},\ t_k \in (0,1)_{\mathbb{T}},\ k = 1,2,...,n$ with $0 < t_1 < t_2 < ... < t_n < 1,\ \phi_p(s)$ is a p-Laplacian operator, i.e., $\phi_p(s) = |s|^{p-2}s,\ p > 1,\ (\phi_p)^{-1} = \phi_q,\ \frac{1}{p} + \frac{1}{q} = 1,\ \Delta u|_{t=t_k}$ denotes the jump of $u(t)$ at $t = t_k$, i.e.,

$$\Delta u|_{t=t_k} = u(t_k^+) - u(t_k^-),$$

where $u(t_k^+) = \lim_{h\to 0} u(t_k + h)$ and $u(t_k^-) = \lim_{h\to 0} u(t_k - h)$ represent the right-hand limit and left-hand limit, respectively, of $u(t)$ at $t = t_k$. In addition, $a_i,\ f$ and I_k satisfy

$(H1)$ $a_i \in [0,+\infty)$ satisfy $0 < \sum_{i=1}^{m-2} a_i < \alpha,\ \beta > 0$;

$(H2)$ $f \in \mathcal{C}([0,1]_{\mathbb{T}} \times [0,+\infty), [0,+\infty))$;

$(H3)$ $I_k \in \mathcal{C}([0,+\infty), [0,+\infty))$.

By means of the double fixed point theorem mentioned by Avery and Henderson [2], we establish the existence of at least two positive solutions for the impulsive BVP (1.3)-(1.5). In accordance with the fixed point index theory in the cone [13], we get the existence of at least one positive solution for the eigenvalue problem (1.6), (1.4) and (1.5). Our results are also new when $\mathbb{T} = \mathbb{R}$ (the differential case) and $\mathbb{T} = \mathbb{Z}$ (the discrete case). Therefore, the results can be considered as a contribution to this field.

This paper is organized as follows. In Section 2, we provide some necessary background about time scales, the theory of cones in Banach space and some preliminary lemmas. We give and prove our main results in Section 3. Finally, in Section 4, we give two examples to demonstrate our results.

2. Preliminaries

In this section, we present some definitions and theorems, which will be needed in the proof of the main results.

Definition 2.1 — Let $(\mathbb{B}, \|.\|)$ be a real Banach space. A nonempty closed set $K \subset \mathbb{B}$ is said to be a cone provided that

(i) $ax + by \in K$ for all $x, y \in K$ and $a, b \geq 0$;

(ii) $y, -y \in K$ implies $y = 0$.

Every cone $K \subset \mathbb{B}$ induces an ordering in $\mathbb{B}$ given by $x \leq y$ if and only if $y - x \in K$.

Definition 2.2 — The map α is said to be a nonnegative continuous concave functional on a cone K of a real Banach space $\mathbb{B}$, provided that $\alpha : K \to [0, \infty)$ is continuous and

$$\alpha(tu + (1-t)v) \geq t\alpha(u) + (1-t)\alpha(v),$$

for all $u, v \in K,\ 0 \leq t \leq 1$.

Similarly, we say the map γ a nonnegative continuous convex functional on a cone K of a real Banach space $\mathbb{B}$ provided that

$$\gamma(tu + (1-t)v) \leq t\gamma(u) + (1-t)\gamma(v),$$

for all $u, v \in K,\ 0 \leq t \leq 1$.

For a nonnegative continuous functional γ on a cone K in a real Banach space $\mathbb{B}$, and each $d > 0$, we set

$$K(\gamma, d) = \{x \in K | \gamma(x) < d\}.$$

The following fixed point theorems are fundamental and important to the proofs of our main results.

***Theorem* 2.1** — *(Double Fixed Point Theorem) [2] — Let K be a cone in a real Banach space $\mathbb{B}$. Let α and γ be increasing, nonnegative, continuous functionals on K, and let θ be a nonnegative, continuous functional on K with $\theta(0) = 0$ such that, for some $c > 0$ and $M > 0$,*

$$\gamma(x) \leq \theta(x) \leq \alpha(x) \text{ and } \|x\| \leq M\gamma(x),$$

for all $x \in \overline{K(\gamma, c)}$. Suppose that there exist positive numbers a and b with $a < b < c$ such that

$$\theta(\lambda x) \leq \lambda\theta(x), \quad for \quad 0 \leq \lambda \leq 1 \quad and \quad x \in \partial K(\theta, b),$$

and

$$T : \overline{K(\gamma, c)} \to K$$

is a completely continuous operator such that:

(i) $\gamma(Tx) > c$, *for all* $x \in \partial K(\gamma, c)$;

(ii) $\theta(Tx) < b$, *for all* $x \in \partial K(\theta, b)$;

(iii) $K(\alpha, a) \neq \emptyset$, *and* $\alpha(Tx) > a$, *for all* $x \in \partial K(\alpha, a)$.

Then T has at least two fixed points, x_1 *and* x_2 *belonging to* $\overline{K(\gamma, c)}$ *such that*

$$a < \alpha(x_1), \quad with \quad \theta(x_1) < b,$$

and

$$b < \theta(x_2), \quad with \quad \gamma(x_2) < c.$$

Theorem 2.2 — *[13] Let* K *be a cone in a real Banach space* $\mathbb{B}$*. Let* D *be an open bounded subset of* $\mathbb{B}$ *with* $D_K = D \cap K \neq \emptyset$ *and* $\overline{D}_K \neq K$*. Assume that* $T : \overline{D}_K \to K$ *is completely continuous such that* $x \neq Tx$ *for* $x \in \partial D_K$*. Then the following results hold:*

(i) *If* $\|Tx\| \leq x,\ x \in \partial D_K$, *then* $i_K(T, D_K) = 1$.

(ii) *If there exists* $e \in K \setminus \{0\}$ *such that* $x \neq Tx + \lambda e$ *for all* $x \in \partial D_K$ *and all* $\lambda > 0$, *then* $i_K(T, D_K) = 0$.

(iii) *Let* U *be open in* K *such that* $\overline{U} \subset D_K$*. If* $i_K(T, D_K) = 1$ *and* $i_K(T, U_K) = 0$, *then* T *has a fixed point in* $D_K \setminus \overline{U}_K$*. The same result holds if* $i_K(T, D_K) = 0$ *and* $i_K(T, U_K) = 1$.

3. Main Results

In this section, by defining an appropriate Banach space and cone, we impose growth conditions on f and I_k which allow us to apply the theorems in Section 2 to establish the existence results of the positive solutions for the eigenvalue problem (1.6) with boundary and impulse conditions (1.4), (1.5) and BVP (1.3)-(1.5).

Let $J' = [0, 1]_{\mathbb{T}} \setminus \{t_1, t_2, ..., t_n\}$. We define

$$\begin{aligned} \mathbb{B} \;&= \{u | u : [0, 1]_{\mathbb{T}} \to \mathbb{R} \text{ is continuous at } t \neq t_k, \text{ left continuous at } t = t_k, \\ &\text{and there exist } u(t_k^-) \text{ and } u(t_k^+) \text{ with } u(t_k^-) = u(t_k) \text{ for } k = 1, 2, ..., n\}. \end{aligned}$$

Then $\mathbb{B}$ is a real Banach space with the norm $\|u\| = \sup_{t \in [0,1]_{\mathbb{T}}} |u(t)|$. By a solution of (1.3)-(1.5), we mean a function $u \in \mathbb{B} \cap \mathcal{C}^2(J')$ which satisfies (1.3)-(1.5). We define a cone $K \subset \mathbb{B}$ as

$$\begin{aligned} K \;=\; &\{u \in \mathbb{B} : u \text{ is a concave, nonnegative and nonincreasing function,} \\ &\alpha u(1) + \beta u^{\triangle}(1) = \sum_{i=1}^{m-2} a_i u(\xi_i)\}. \end{aligned}$$

We need the following lemmas that will be used to prove our main results.

Lemma 3.1 — Suppose that $(H1)-(H3)$ are satisfied. Then $u \in \mathbb{B} \cap \mathcal{C}^2(J')$ is a solution of the impulsive boundary value problem (1.3)-(1.5) if and only if $u(t)$ is a solution of the following integral equation

$$\begin{aligned} u(t) &= \frac{\beta}{\alpha - \sum_{i=1}^{m-2} a_i} \phi_q\left(\int_0^1 f(r,u(r))\nabla r\right) \\ &\quad + \frac{1}{\alpha - \sum_{i=1}^{m-2} a_i} \sum_{i=1}^{m-2} a_i \left[\int_{\xi_i}^1 \phi_q\left(\int_0^s f(r,u(r))\nabla r\right)\triangle s + \sum_{0<t_k<\xi_i} I_k(u(t_k))\right] \\ &\quad + \int_t^1 \phi_q\left(\int_0^s f(r,u(r))\nabla r\right)\triangle s + \sum_{t_k<t} I_k(u(t_k)), \end{aligned} \tag{3.1}$$

and $u(t) \geq 0$.

PROOF : Integrating the equation (1.3) from 0 to t, one has

$$-\phi_p(u^{\triangle}(t)) + \phi_p(u^{\triangle}(0)) = \int_0^t f(s,u(s))\nabla s.$$

By the boundary condition (1.5), we have

$$u^{\triangle}(t) = -\phi_q\left(\int_0^t f(s,u(s))\nabla s\right).$$

Integrating the dynamic equation above from t to 1, we get

$$u(t) = u(1) + \int_t^1 \phi_q\left(\int_0^s f(r,u(r))\nabla r\right)\triangle s + \sum_{t_k<t} I_k(u(t_k)). \tag{3.2}$$

Applying the second boundary condition, one has

$$\begin{aligned} u(1) &= \frac{1}{\alpha - \sum_{i=1}^{m-2} a_i} \sum_{i=1}^{m-2} a_i \left[\int_{\xi_i}^1 \phi_q\left(\int_0^s f(r,u(r))\nabla r\right)\triangle s + \sum_{0<t_k<\xi_i} I_k(u(t_k))\right] \\ &\quad + \frac{\beta}{\alpha - \sum_{i=1}^{m-2} a_i} \phi_q\left(\int_0^1 f(r,u(r))\nabla r\right). \end{aligned} \tag{3.3}$$

Therefore, by (3.2) and (3.3), we have

$$\begin{aligned} u(t) &= \frac{\beta}{\alpha - \sum_{i=1}^{m-2} a_i} \phi_q\left(\int_0^1 f(r,u(r))\nabla r\right) \\ &+ \frac{1}{\alpha - \sum_{i=1}^{m-2} a_i} \sum_{i=1}^{m-2} a_i \left[\int_{\xi_i}^1 \phi_q\left(\int_0^s f(r,u(r))\nabla r\right)\triangle s + \sum_{0<t_k<\xi_i} I_k(u(t_k))\right] \\ &+ \int_t^1 \phi_q\left(\int_0^s f(r,u(r))\nabla r\right)\triangle s + \sum_{t_k<t} I_k(u(t_k)). \end{aligned}$$

Conversely, let u be as in (3.1). Taking the delta derivative of $u(t)$ gives

$$u^{\triangle}(t) = -\phi_q(\int_0^t f(s,u(s))\nabla s), \quad \text{i.e.,} \quad \phi_p(u^{\triangle}(t)) = -\int_0^t f(s,u(s))\nabla s. \tag{3.4}$$

So $u^{\triangle}(0) = 0$. Also, it is easy to see that $u(t)$ satisfy (1.4) and (1.5). Furthermore, from $(H1)-(H3)$ and (3.1), it is clear that $u(t) \geq 0$. So, the proof of lemma is completed. □

Lemma 3.2 — If $u \in K$, then $\min_{t\in[0,1]_{\mathbb{T}}} u(t) \geq \gamma\|u\|$, where $\gamma = \frac{\sum_{i=1}^{m-2} a_i(1-\xi_i)}{\alpha - \sum_{i=1}^{m-2} a_i\xi_i}$.

PROOF : Since $u \in K$, nonnegative and nonincreasing

$$\|u\| = u(0), \quad \min_{t\in[0,1]_{\mathbb{T}}} u(t) = u(1).$$

On the other hand, $u^{\triangle}(t)$ is nonincreasing on J'. So, for every $t \in [0,1]_{\mathbb{T}}$, we have

$$\frac{u(t)-u(1)}{1-t} \geq \frac{u(0)-u(1)}{1},$$

i.e., $u(t) \geq (1-t)u(0) + tu(1)$.

Therefore,

$$\sum_{i=1}^{m-2} a_i u(\xi_i) \geq \sum_{i=1}^{m-2} a_i(1-\xi_i)u(0) + \sum_{i=1}^{m-2} a_i\xi_i u(1).$$

This together with $\alpha u(1)+\beta u^{\triangle}(1)=\sum_{i=1}^{m-2}a_iu(\xi_i)$, implies that

$$u(1)\geq\frac{\sum_{i=1}^{m-2}a_i(1-\xi_i)}{\alpha-\sum_{i=1}^{m-2}a_i\xi_i}u(0).$$

So, the proof of Lemma is completed. □

Define $T:K\to\mathbb{B}$ by

$$\begin{aligned}(Tu)(t) &= \frac{\beta}{\alpha-\sum_{i=1}^{m-2}a_i}\phi_q\left(\int_0^1 f(r,u(r))\nabla r\right)\\ &+ \frac{1}{\alpha-\sum_{i=1}^{m-2}a_i}\sum_{i=1}^{m-2}a_i\left[\int_{\xi_i}^1\phi_q\left(\int_0^s f(r,u(r))\nabla r\right)\triangle s+\sum_{0<t_k<\xi_i}I_k(u(t_k))\right]\\ &+ \int_t^1\phi_q\left(\int_0^s f(r,u(r))\nabla r\right)\triangle s+\sum_{t_k<t}I_k(u(t_k)).\end{aligned}\qquad(3.5)$$

From (3.5) and Lemma 3.1, it is easy to obtain the following result.

Lemma 3.3 — Assume that $(H1)-(H3)$ hold. Then $T:K\to K$ is completely continuous.

Now we consider the existence of at least two positive solutions for the non-eigenvalue BVP (1.3)-(1.5) by the fixed point theorem in [2].

Let us define the increasing, nonnegative, continuous functionals $\gamma,\ \beta$, and α on K by

$$\begin{aligned}\gamma(u) &= \min_{t\in[0,\xi_1]_{\mathbb{T}}}u(t)=u(\xi_1),\\ \beta(u) &= \max_{t\in[\xi_1,\xi_{n-2}]_{\mathbb{T}}}u(t)=u(\xi_1),\\ \alpha(u) &= \max_{t\in[0,\xi_{n-2}]_{\mathbb{T}}}u(t)=u(0).\end{aligned}$$

It is obvious that for each $u\in K$,

$$\gamma(u)\leq\beta(u)\leq\alpha(u).$$

In addition, from by Lemma 3.2, for each $u\in K$,

$$\|u\|\leq\frac{1}{\gamma}\min_{t\in[0,1]_{\mathbb{T}}}u(t)\leq\frac{1}{\gamma}\min_{t\in[0,\xi_1]_{\mathbb{T}}}u(t)=\frac{1}{\gamma}\gamma(u).$$

Thus,

$$\|u\| \leq \frac{1}{\gamma}\gamma(u), \ \ \forall u \in K.$$

For the convenience, we denote

$$\begin{aligned} A &= (1-\xi_1)\phi_q(\xi_1), \\ B &= \frac{(n+1)\alpha+\beta}{\alpha - \sum_{i=1}^{m-2} a_i}. \end{aligned}$$

***Theorem* 3.1** — *Suppose that assumptions $(H1)-(H3)$ are satisfied. Let there exist positive numbers $a < b < c$ such that*

$$0 < a < \frac{A}{B}b < \frac{A\gamma}{B}c,$$

and assume that f satisfies the following conditions

$(H4)$ $f(t,u) > \phi_p\left(\frac{c}{A}\right)$, *for all* $(t,u) \in [0,\xi_1]_{\mathbb{T}} \times [c, \frac{1}{\gamma}c]$,

$(H5)$ $f(t,u) < \phi_p\left(\frac{b}{B}\right), \ \sup\{I_k(u)\} < \frac{b}{B}, \ k = 1,2,...,n,$ *for all* $(t,u) \in [0,1]_{\mathbb{T}} \times [0, \frac{1}{\gamma}b]$,

$(H6)$ $f(t,u) > \phi_p\left(\frac{a}{A}\right)$, *for all* $(t,u) \in [0,1]_{\mathbb{T}} \times [0,a]$.

Then the boundary value problem (1.3)-(1.5) has at least two positive solutions u_1 and u_2 satisfying

$$a < \alpha(u_1) \text{ with } \beta(u_1) < b, \quad b < \beta(u_2) \text{ with } \gamma(u_2) < c.$$

PROOF : We define the completely continuous operator T by (3.5). So, it is easy to check that $T : \overline{K(\gamma,c)} \to K$. We now show that all the conditions of Theorem 2.1 are satisfied. In order to show that condition (i) of Theorem 2.1, we choose $u \in \partial K(\gamma,c)$. Then $\gamma(u) = \min_{t\in[0,\xi_1]_{\mathbb{T}}} u(t) = u(\xi_1) = c$, this implies that $c \leq u(t)$ for $t \in [0,\xi_1]_{\mathbb{T}}$. Recalling that $\|u\| \leq \frac{1}{\gamma}\gamma(u) = \frac{1}{\gamma}c$, we get

$$c \leq u(t) \leq \frac{1}{\gamma}c, \quad t \in [0,\xi_1]_{\mathbb{T}}.$$

Then assumption $(H4)$ implies

$$f(t,u) > \phi_p\left(\frac{c}{A}\right), \quad \text{for all} \quad (t,u) \in [0,\xi_1]_{\mathbb{T}} \times [c, \frac{1}{\gamma}c].$$

Therefore,

$$\begin{aligned} \gamma(Tu) &= \min_{t\in[0,\xi_1]_{\mathbb{T}}} (Tu)(t) = (Tu)(\xi_1) \\ &= \frac{\beta}{\alpha - \sum_{i=1}^{m-2} a_i}\phi_q\left(\int_0^1 f(r,u(r))\nabla r\right) \end{aligned}$$

$$
\begin{aligned}
&+ \quad \frac{1}{\alpha - \sum_{i=1}^{m-2} a_i} \sum_{i=1}^{m-2} a_i \left[\int_{\xi_i}^{1} \phi_q \left(\int_0^s f(r,u(r)) \nabla r \right) \triangle s + \sum_{0<t_k<\xi_i} I_k(u(t_k)) \right] \\
&+ \quad \int_{\xi_1}^{1} \phi_q \left(\int_0^s f(r,u(r)) \nabla r \right) \triangle s + \sum_{t_k<\xi_1} I_k(u(t_k)) \\
&\geq \quad \int_{\xi_1}^{1} \phi_q \left(\int_0^s f(r,u(r)) \nabla r \right) \triangle s \geq \int_{\xi_1}^{1} \phi_q \left(\int_0^{\xi_1} f(r,u(r)) \nabla r \right) \triangle s \\
&= \quad (1-\xi_1)\phi_q \left(\int_0^{\xi_1} f(r,u(r)) \nabla r \right) \\
&> \quad (1-\xi_1)\phi_q(\xi_1) \frac{c}{A} = c.
\end{aligned}
$$

Hence, condition (i) is satisfied.

Secondly, we show that (ii) of Theorem 2.1 is satisfied. For this, we select $u \in \partial K(\beta, b)$. Then, $\beta(u) = \max_{t \in [\xi_1, \xi_{n-2}]_{\mathbb{T}}} u(t) = u(\xi_1) = b$, this means $0 \leq u(t) \leq b$, for all $t \in [\xi_1, 1]_{\mathbb{T}}$. Noticing that $\|u\| \leq \frac{1}{\gamma}\gamma(u) = \frac{1}{\gamma}\beta(u) = \frac{1}{\gamma}b$, we get

$$
0 \leq u(t) \leq \frac{1}{\gamma} b, \text{ for } 0 \leq t \leq 1.
$$

Then, assumption $(H5)$ implies

$$
f(t,u) < \phi_p \left(\frac{b}{B} \right), \quad \sup\{I_k(u)\} < \frac{b}{B}.
$$

Therefore

$$
\begin{aligned}
\beta(Tu) \quad &= \quad \max_{t \in [\xi_1, \xi_{m-2}]_{\mathbb{T}}} (Tu)(t) = (Tu)(\xi_1) \\
&\leq \quad \frac{1}{\alpha - \sum_{i=1}^{m-2} a_i} \left\{ \beta \phi_q \left(\int_0^1 f(r,u(r)) \nabla r \right) + \sum_{i=1}^{m-2} a_i \left(\int_0^1 \phi_q \left(\int_0^1 f(r,u(r)) \nabla r \right) \triangle s \right) \right\} \\
&+ \quad \frac{1}{\alpha - \sum_{i=1}^{m-2} a_i} \sum_{i=1}^{n-2} a_i \left(\sum_{k=1}^{n} I_k(u(t_k)) \right) + \int_0^1 \phi_q \left(\int_0^1 f(r,u(r)) \nabla r \right) \triangle s + \sum_{k=1}^{n} I_k(u(t_k)) \\
&= \quad \left(\frac{\alpha + \beta}{\alpha - \sum_{i=1}^{m-2} a_i} \right) \phi_q \left(\int_0^1 f(r,u(r)) \nabla r \right) + \frac{\alpha}{\alpha - \sum_{i=1}^{m-2} a_i} \sum_{k=1}^{n} I_k(u(t_k))
\end{aligned}
$$

$$
\begin{aligned}
&< \left(\frac{\alpha+\beta}{\alpha-\sum_{i=1}^{m-2} a_i}\right)\frac{b}{B}+\frac{\alpha}{\alpha-\sum_{i=1}^{m-2} a_i}\frac{b}{B}n \\
&= \frac{b}{B}\left(\frac{(n+1)\alpha+\beta}{\alpha-\sum_{i=1}^{m-2} a_i}\right) \\
&= b.
\end{aligned}
$$

So, we get $\beta(Tu) < b$. Hence, condition (ii) is satisfied.

Finally, we show that the condition (iii) of Theorem 2.1 is satisfied. We note that $u(t) = \frac{1}{2}a,\ 0 \leq t \leq 1$ is a member of $K(\alpha, a)$, and so $K(\alpha, a) \neq \emptyset$.

Now, let $u \in \partial K(\alpha, a)$. Then $\alpha(u) = \max_{t\in[0,\xi_{n-2}]_{\mathbb{T}}} u(t) = u(0) = a$. This implies

$$0 \leq u(t) \leq a,\ t \in [0, 1]_{\mathbb{T}}.$$

By assumption $(H6)$,

$$f(t, u) > \phi_p\left(\frac{a}{A}\right).$$

Then,

$$
\begin{aligned}
\alpha(Tu) &= \max_{t\in[0,\xi_{n-2}]_{\mathbb{T}}} (Tu)(t) = (Tu)(0) \\
&\geq \int_0^1 \phi_q\left(\int_0^s f(r, u(r))\nabla r\right)\triangle s \geq \int_{\xi_1}^1 \phi_q\left(\int_0^s f(r, u(r))\nabla r\right)\triangle s \\
&\geq (1-\xi_1)\phi_q\left(\int_0^{\xi_1} f(r, u(r))\nabla r\right) > (1-\xi_1)\frac{a}{A}\phi_q\left(\int_0^{\xi_1} \nabla r\right) \\
&= (1-\xi_1)\frac{a}{A}\phi_q(\xi_1) = a.
\end{aligned}
$$

So, we get $\alpha(Tu) > a$. Thus, (iii) of Theorem 2.1 is satisfied. Hence, the impulsive boundary value problem (1.3)-(1.5) has at least two positive solutions u_1 and u_2 satisfying

$$a < \alpha(u_1) \text{ with } \beta(u_1) < b,$$

and

$$b < \beta(u_2) \text{ with } \gamma(u_2) < c.$$

The proof is complete. □

Next we consider the existence of at least one positive solution for the eigenvalue problem (1.6) under the conditions (1.4) and (1.5), by the fixed point theorem in [13].

We define

$$\begin{aligned} K_\rho &= \{u \in K : \|u\| < \rho\}, \\ \Omega_\rho &= \{u \in K : \min_{t\in[0,1]_{\mathbb{T}}} u(t) < \gamma\rho\} = \{u \in K : \gamma\|u\| \le \min_{t\in[0,1]_{\mathbb{T}}} u(t) < \gamma\rho\}. \end{aligned}$$

Define an operator T_λ by

$$\begin{aligned} T_\lambda u(t) &= \frac{\beta}{\alpha - \sum_{i=1}^{m-2} a_i}\phi_q\left(\int_0^1 \lambda f(r,u(r))\nabla r\right) \\ &\quad + \frac{1}{\alpha - \sum_{i=1}^{m-2} a_i} \sum_{i=1}^{m-2} a_i \left[\int_{\xi_i}^1 \phi_q\left(\int_0^s \lambda f(r,u(r))\nabla r\right)\triangle s + \sum_{0<t_k<\xi_i} I_k(u(t_k))\right] \\ &\quad + \int_t^1 \phi_q\left(\int_0^s \lambda f(r,u(r))\nabla r\right)\triangle s + \sum_{t_k<t} I_k(u(t_k)). \end{aligned} \tag{3.6}$$

Lemma 3.4 — [13] Ω_ρ has the following properties:

(a) Ω_ρ is open relative to K.

(b) $K_{\gamma\rho} \subset \Omega_\rho \subset K_\rho$.

(c) $u \in \Omega_\rho$ if and only if $\min_{t\in[0,1]_{\mathbb{T}}} u(t) = \gamma\rho$.

(d) If $u \in \Omega_\rho$, then $\gamma\rho \le u(t) \le \rho$ for $t \in [0,1]_{\mathbb{T}}$.

Now for convenience we introduce the following notations. Let

$$\begin{aligned} f_{\gamma\rho}^{\rho} &= \min\left\{\min_{t\in[0,1]_{\mathbb{T}}} \frac{f(t,u)}{\phi_p(\rho)} : u \in [\gamma\rho,\rho]\right\}, \\ f_0^{\rho} &= \max\left\{\max_{t\in[0,1]_{\mathbb{T}}} \frac{f(t,u)}{\phi_p(\rho)} : u \in [0,\rho]\right\}, \\ I_0^{\rho}(k) &= \max\left\{I_k(u) : u \in [0,\rho]\right\}, \\ \frac{1}{l} &= \frac{(n+1)\alpha + \beta}{\alpha - \sum_{i=1}^{m-2} a_i}, \quad \frac{1}{L} = \frac{\beta}{\alpha - \sum_{i=1}^{m-2} a_i}. \end{aligned}$$

***Theorem* 3.2** — *Suppose* $(H1)-(H3)$ *hold.*

$(H7)$ *There exist* $\rho_1, \rho_2 \in (0,\infty)$ *with* $\rho_1 < \gamma\rho_2$ *such that*

$$\frac{\phi_p(L)}{f_{\gamma\rho_2}^{\rho_2}} \leq \lambda \leq \frac{\phi_p(l)}{f_0^{\rho_1}}, \quad I_0^{\rho_1}(k) \leq l\rho_1.$$

Then eigenvalue problem (1.6), (1.4), (1.5) has at least one positive solution u *with* $u \in \Omega_{\rho_2} \backslash \overline{K}_{\rho_1}$.

$(H8)$ *There exist* $\rho_1, \rho_2 \in (0,\infty)$ *with* $\rho_1 < \rho_2$ *such that*

$$\frac{\phi_p(L)}{f_{\gamma\rho_1}^{\rho_1}} \leq \lambda \leq \frac{\phi_p(l)}{f_0^{\rho_2}}, \quad I_0^{\rho_2}(k) \leq l\rho_2.$$

Then eigenvalue problem (1.6), (1.4), (1.5) has at least one positive solution u *with* $u \in K_{\rho_2} \backslash \overline{\Omega}_{\rho_1}$.

PROOF : We only consider the condition $(H7)$. If $(H8)$ holds, then the proof is similar to that of the case when $(H7)$ holds. By Lemma 3.3, we know that the operator $T_\lambda : K \to K$ is completely continuous.

First, we show that $i_K(T, K_{\rho_1}) = 1$. In fact, by (3.6), $\lambda f_0^{\rho_1} \leq \phi_p(l)$ and $I_0^{\rho_1}(k) \leq l\rho_1$, we have for $u \in \partial K_{\rho_1}$,

$$\begin{aligned}
T_\lambda u(t) &= \frac{\beta}{\alpha - \sum_{i=1}^{m-2} a_i}\phi_q\left(\int_0^1 \lambda f(r,u(r))\nabla r\right) \\
&\quad + \frac{1}{\alpha - \sum_{i=1}^{m-2} a_i}\sum_{i=1}^{m-2} a_i\left[\int_{\xi_i}^1 \phi_q\left(\int_0^s \lambda f(r,u(r))\nabla r\right)\triangle s + \sum_{0<t_k<\xi_i} I_k(u(t_k))\right] \\
&\quad + \int_t^1 \phi_q\left(\int_0^s \lambda f(r,u(r))\nabla r\right)\triangle s + \sum_{t_k<t} I_k(u(t_k)) \\
&\leq \frac{1}{\alpha - \sum_{i=1}^{m-2} a_i}\left\{\beta\phi_q\left(\int_0^1 \lambda f(r,u(r))\nabla r\right) + \sum_{i=1}^{m-2} a_i\left(\int_0^1 \phi_q\left(\int_0^1 \lambda f(r,u(r))\nabla r\right)\triangle s\right)\right\} \\
&\quad + \frac{1}{\alpha - \sum_{i=1}^{m-2} a_i}\sum_{i=1}^{m-2} a_i\left(\sum_{k=1}^{n} I_k(u(t_k))\right) + \int_0^1 \phi_q\left(\int_0^1 \lambda f(r,u(r))\nabla r\right)\triangle s + \sum_{k=1}^{n} I_k(u(t_k))
\end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{\alpha+\beta}{\alpha-\sum_{i=1}^{m-2} a_i}\right)\phi_q\left(\int_0^1 \lambda f(r,u(r))\nabla r\right)+\frac{\alpha}{\alpha-\sum_{i=1}^{m-2} a_i}\sum_{k=1}^{n} I_k(u(t_k))\\
&\le \left(\frac{\alpha+\beta}{\alpha-\sum_{i=1}^{m-2} a_i}\right)\phi_q\left(\int_0^1 \phi_p(l\rho_1)\nabla r\right)+\frac{\alpha}{\alpha-\sum_{i=1}^{m-2} a_i}\sum_{k=1}^{n} l\rho_1\\
&= \left(\frac{(n+1)\alpha+\beta}{\alpha-\sum_{i=1}^{m-2} a_i}\right) l\rho_1=\rho_1,
\end{aligned}
$$

i.e., $\|T_\lambda u\| < \|u\|$ for $u \in \partial K_{\rho_1}$. By (i) of Theorem 2.2, we obtain that $i_K(T_\lambda, K_{\rho_1}) = 1$.

Secondly, we show that $i_K(T_\lambda, \Omega_{\rho_2}) = 0$. Let $e(t) \equiv 1$. Then $e \in \partial K_1$. We claim that

$$u \neq T_\lambda u + \mu e,\ u \in \partial\Omega_{\rho_2},\ \mu > 0.$$

In fact, if not, there exists $u_0 \in \partial\Omega_{\rho_2}$ and $\mu_0 > 0$ such that

$$u_0 = T_\lambda u_0 + \mu_0 e. \tag{3.7}$$

Then, Lemma 3.1, Lemma 3.2 and (3.7) imply that for $t \in [0,1]_{\mathbb{T}}$

$$
\begin{aligned}
u_0 &= T_\lambda u_0 + \mu_0 e \ge \gamma\|T_\lambda u_0\| + \mu_0\\
&\ge \gamma\frac{\beta}{\alpha-\sum_{i=1}^{m-2} a_i}\phi_q\left(\int_0^1 \lambda f(s,u(s))\nabla s\right)+\mu_0\\
&\ge \gamma\frac{\beta}{\alpha-\sum_{i=1}^{m-2} a_i} L\rho_2+\mu_0=\gamma\rho_2+\mu_0,
\end{aligned}
$$

i.e. $\gamma\rho_2 \ge \gamma\rho_2 + \mu_0$, which is a contradiction. Hence by (ii) of Theorem 2.2, it follows that $i_K(T, \Omega_{\rho_2}) = 0$.

Since $\rho_1 < \gamma\rho_2$ and Lemma 3.4 (b), we have $\overline{K}_{\rho_1} \subset K_{\gamma\rho_2} \subset \Omega_{\rho_2}$. Therefore (iii) of Theorem 2.2 implies that BVP (1.6), (1.4) and (1.5) has at least one positive solution u with $u \in \Omega_{\rho_2} \setminus \overline{K}_{\rho_1}$. $\square$

4. EXAMPLE

Example 4.1 : Let $\mathbb{T} = \{0\} \cup \left\{\left(\frac{1}{2}\right)^{\mathbb{N}}\right\} \cup \left(\frac{1}{2}, 1\right]$, where $\mathbb{N}$ denotes the set of all natural numbers. Consider the following problem

$$-[\phi_3(u^{\triangle}(t))]^{\nabla} = f(t, u(t)),\ t \in [0,1]_{\mathbb{T}} \setminus \{\frac{1}{2}\}, \tag{4.1}$$

$$-\left(u(\frac{1}{2}^{+}) - u(\frac{1}{2}^{-})\right) = 0.49, \tag{4.2}$$

$$u^{\triangle}(0) = 0,\ \frac{1}{3}u(1) + \frac{2}{5}u^{\triangle}(1) = \frac{1}{5}u(\frac{1}{4}). \tag{4.3}$$

In the BVP (4.1)-(4.3), $p = 3,\ n = 1,\ m = 3,\ a_1 = \frac{1}{5},\ \xi_1 = \frac{1}{4},\ t_1 = \frac{1}{2},\ I_1\left(u(\frac{1}{2})\right) = 0.49$, $\alpha = \frac{1}{3}$ and $\beta = \frac{2}{5}$. Let

$$f(t,u) = \begin{cases} \dfrac{549}{21420}u + \dfrac{1}{35},\ (t,u) \in [0,1]_{\mathbb{T}} \times \left[0, \dfrac{68}{9}\right], \\ \dfrac{5191}{13}(u-9) + 577,\ (t,u) \in (t,u) \in [0,1]_{\mathbb{T}} \times \left(\dfrac{68}{9}, \infty\right). \end{cases}$$

Take $a = \frac{1}{16},\ b = 4,\ c = 9$. Clearly, $0 < a < b < c$ and $\gamma = \frac{9}{17}$. It is easy to see that assumptions $(H1) - (H3)$ are satisfied. Next, we show that $(H4) - (H6)$ are also satisfied. Through some simple calculations, we obtain

$$[0,\xi_1]_{\mathbb{T}} \times \left[[c, \frac{1}{\gamma}c\right] = [0, \frac{1}{4}]_{\mathbb{T}} \times [9,17], \quad [0,1]_{\mathbb{T}} \times \left[0, \frac{1}{\gamma}b\right] = [0,1]_{\mathbb{T}} \times \left[0, \frac{68}{9}\right],$$

$$[0,1]_{\mathbb{T}} \times [0,a] = [0,1]_{\mathbb{T}} \times \left[0, \frac{1}{16}\right], \quad A = \frac{3}{8}, \quad B = 8,$$

$$\max_{(t,u)\in[0,1]_{\mathbb{T}}\times\left[0,\frac{1}{\gamma}b\right]} f(t,u) < 0.23, \quad \min_{(t,u)\in[0,1]_{\mathbb{T}}\times[0,a]} f(t,u) \geq \frac{1}{35}, \quad \min_{(t,u)\in[0,\xi_1]_{\mathbb{T}}\times\left[[c,\frac{1}{\gamma}c\right]} f(t,u) \geq 577,$$

$$0 < a = \frac{1}{16} < \frac{A}{B}b = \frac{3}{16} < \frac{A\gamma}{B}c = \frac{243}{1088}.$$

For $(t,u) \in [0, \frac{1}{4}]_{\mathbb{T}} \times [9,17]$, since $f(t,u) \geq 577$ and $\phi_p\left(\frac{c}{A}\right) = \phi_3(24) = 576$, we know that

$$f(t,u) > \phi_p\left(\frac{c}{A}\right), \quad \text{for all} \quad (t,u) \in [0,\xi_1]_{\mathbb{T}} \times [c, \frac{1}{\gamma}c].$$

Then, $(H4)$ is satisfied.

For $(t,u) \in [0,1]_{\mathbb{T}} \times \left[0, \frac{68}{9}\right]$, since $f(t,u) < 0.23,\ I_1(u) = 0.49$, and $\phi_p\left(\frac{b}{B}\right) = \phi_3\left(\frac{1}{2}\right) = 0.25$, we have

$$f(t,u) < \phi_p\left(\frac{b}{B}\right),\ \sup\{I_1(u)\} < \frac{b}{B}, \quad \text{for all} \quad (t,u) \in [0,1]_{\mathbb{T}} \times [0, \frac{1}{\gamma}b].$$

Then, $(H5)$ holds.

For $(t,u) \in [0,1]_{\mathbb{T}} \times \left[0, \frac{1}{16}\right]$, since $f(t,u) \geq \frac{1}{35}$ and $\phi_p\left(\frac{a}{A}\right) = \phi_3\left(\frac{1}{6}\right) = \frac{1}{36}$, we have

$$f(t,u) > \phi_p\left(\frac{a}{A}\right), \quad \text{for all} \quad (t,u) \in [0,1]_{\mathbb{T}} \times [0,a].$$

So, $(H6)$ is satisfied. Then, all conditions of Theorem 3.1 hold. Hence, we get the BVP (4.1)-(4.3) has at least two positive solutions.

Example 4.2 : In BVP (1.6), (1.4) and (1.5), suppose that $p = 3,\ n = 1,\ m = 3,\ \lambda = \frac{1}{3}$, $\mathbb{T} = [0,1]_{\mathbb{T}},\ t_1 = \frac{1}{2},\ \xi_1 = \frac{1}{4},\ a_1 = \frac{1}{10},\ \alpha = \frac{1}{5}$ and $\beta = \frac{2}{5}$, i.e.,

$$-[\phi_3(u^{\triangle}(t))]^{\nabla} = \frac{1}{3}f(t,u(t)),\ t \in [0,1]_{\mathbb{T}} \setminus \{\frac{1}{2}\},\ \mathbb{T} = [0,1]_{\mathbb{T}}, \tag{4.4}$$

$$-\left(u(\frac{1}{2}^{+}) - u(\frac{1}{2}^{-})\right) = I_1(u(\frac{1}{2})), \tag{4.5}$$

$$u^{\triangle}(0) = 0,\ \frac{1}{5}u(1) + \frac{2}{5}u^{\triangle}(1) = \frac{1}{10}u(\frac{1}{4}), \tag{4.6}$$

where

$$f(t,u) = \begin{cases} 0.9u,\ (t,u) \in [0,1]_{\mathbb{T}} \times [0,20]; \\ 90.2(u-20) + 18,\ (t,u) \in [0,1]_{\mathbb{T}} \times (20,\infty); \end{cases}$$

$$I_1(u) = \frac{u-1}{8}.$$

Taking $\rho_1 = 20,\ \rho_2 = 70$, we have $l = \frac{1}{8},\ L = \frac{1}{4},\ \gamma = \frac{3}{7}$. We can obtain that

$$\rho_1 = 20 < \gamma\rho_2 = 30.$$

Now, we show that $(H7)$ is satisfied:

$$f_{30}^{70} = \frac{46}{245} \geq \frac{\phi_p(L)}{\lambda} = \frac{3}{16},$$
$$f_0^{20} = \frac{9}{200} \leq \frac{\phi_p(l)}{\lambda} = \frac{3}{64},$$
$$I_0^{20} = 2.375 < l\rho_1 = 2.5.$$

Then, condition $(H7)$ of Theorem 3.2 holds. Thus, the BVP (4.4)-(4.6) has at least one positive solution.

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