Indian J. Pure Appl. Math., **47**(1): 9-22, March 2016

DOI: 10.1007/s13226-015-0165-0

CUSPIDALITY AND THE GROWTH OF FOURIER COEFFICIENTS OF MODULAR FORMS : A SURVEY

Soumya Das

Department of Mathematics, Indian Institute of Science,
Bangalore 560 012, *India*
e-mails: somu@math.iisc.ernet.in, soumya.u2k@gmail.com

(*Received* 14 *January* 2015; *accepted* 9 *July* 2015)

In this paper, we present a survey of the recent results on the characterization of the cuspidality of classical modular forms on various groups by a suitable growth of their Fourier coefficients.

Key words : Hecke bound; cuspidality; growth of Fourier coefficients; Siegel modular forms.

1. Introduction

The role of automorphic functions in general and modular forms in particular has been of central importance in number theory, especially the Fourier coefficients of modular forms often encode many interesting number-theoretic information. It is well known that in the classical version of the theory, the space of modular forms of weight $k > 0$ on some congruence subgroup of the modular group $SL(2, \mathbf{Z})$ (which we will define soon, in more generality) possess an absolutely convergent Fourier expansion on the complex upper half space. Moreover, this space of modular forms is made up of the space of Eisenstein series, whose Fourier coefficients are well-known and computable, and by the space of cusp forms. However, the Fourier coefficients of the cusp forms are rather mysterious objects.

In this paper, we will be concerned with an analytic property of such Fourier coefficients by addressing the basic question of deciding whether a modular form is actually a cusp form by looking at the growth-rate of it's Fourier coefficients. We will work in the more general setting of the Siegel modular forms; which were introduced by Siegel for studying quadratic forms. Siegel modular forms of weight k and degree n are holomorphic functions on the Siegel upper half-space $\mathbf{H}_n$ with certain invariance property with respect to the congruence subgroups of the Siegel modular group.

1.1 *Some facts about Siegel modular forms*

The Siegel modular group, denoted by Γ_n is the symplectic group of order n over $\mathbf{Z}$:

$$\Gamma_n := Sp_n(\mathbf{Z}) = \left\{ M \in M_{2n\times 2n}(\mathbf{Z}) \mid M^t J M = J \right\},\ J := \left(\begin{smallmatrix} 0 & -I_{n\times n} \\ I_{n\times n} & 0 \end{smallmatrix} \right).$$

The Siegel upper half space, denoted as $\mathbf{H}_n$ is

$$\mathbf{H}_n = \{ Z \in M_{n\times n}(\mathbf{C}) \mid Z = Z^t,\ Im(Z) > 0 \}.$$

Γ_n acts on $\mathbf{H}_n$ by $Z \mapsto \gamma \circ Z = (AZ+B)(CZ+D)^{-1}$, where $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right)$.

Fix an integer $k \geq 0$. Γ_n acts on functions $F \colon \mathbf{H}_n \to \mathbf{C}$ by

$$F(Z) \mapsto F\mid_k \gamma(Z) := \det(CZ+D)^{-k} F(\gamma\langle Z \rangle).$$

$\Gamma \subset \Gamma_n$ is called a congruence subgroup of level N if N is the smallest positive integer such that it contains some principal congruence subgroup of level N, which is defined as

$$\Gamma^n(N) := \{ \gamma \in \Gamma_n \mid \gamma \equiv 1_{n\times n} \bmod N \}.$$

An important example of such a congruence subgroup is

$$\Gamma_0^n(N) := \{ \gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right) \mid C \equiv 0 \bmod N \}.$$

We say that F is a Siegel modular forms for the group Γ of weight k if

$$F\mid_k \gamma = F \quad \text{for all } \gamma \in \Gamma,$$

if $n = 1$, we also require that $f\mid_k \gamma$ should be bounded as $Im(z) \to \infty$ for all $\gamma \in \mathrm{SL}(2,\mathbf{Z})$.

It is a well known fact that the Fourier coefficients of Siegel modular forms have a certain polynomial growth. More precisely, let F be a Siegel modular form on the congruence subgroup Γ of the Siegel modular group Γ_n of integral weight k (denoted as $M_k^n(\Gamma)$) with the Fourier expansion (at the cusp "∞"):

$$F(Z) = \sum_{T\in\Lambda_n} a(F,T) \exp(2\pi i/M \cdot \mathrm{tr} TZ), \quad \text{where } Z \in \mathbf{H}_n,$$

where M is a positive integer and Λ_n (respectively Λ_n^+) is the set of all $n \times n$ symmetric, positive semi-definite (respectively positive definite), half-integral matrices over $\mathbf{Z}$, i.e., $T_{ii} \in \mathbf{Z}, T_{ij} \in \frac{1}{2}\mathbf{Z}$. For an arbitrary modular form $F \in M_k^n(\Gamma)$, one has the following estimate on its Fourier coefficients (see e.g., [1]):

$$a(F,T) \ll_F \det(T)^k \quad (T > 0). \tag{1.1}$$

1.1.1 *Cusps and cusp forms* : The set of (0-dimensional) cusps of Γ is

$$\mathcal{C}_\Gamma := \Gamma\backslash\Gamma_n/\Gamma_\infty,$$

where $\Gamma_\infty := \Gamma \cap \Gamma_{n,\infty}$. Here $\Gamma_{n,\infty}$ denotes the Siegel parabolic subgroup of Γ_n defined by

$$\Gamma_{n,\infty} := \{\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_n \mid C = 0\}.$$

It is known that $\mathcal{C}_\Gamma$ is an union of finitely many double cosets, and by abuse of notation, we call the set of the representatives of such double cosets as the (0-dimensional) cusps.

It is well known (see [1] for example) that $F \in M_k^n(\Gamma)$ is a cusp form if and only if its Fourier coefficients at all the cusps γ are supported on Λ_n^+:

$$F \mid_k \gamma(Z) = \sum_{T \in \Lambda_n^+} a_\gamma(F,T) \exp(2\pi i/M \cdot \mathrm{tr} TZ).$$

In this case one has the following better estimate, also known as the "Hecke bound" on its Fourier coefficients:

$$a(F,T) \ll_F \det(T)^{k/2} \quad \text{for all } T \in \Lambda_n^+. \tag{1.2}$$

Let us denote the space of cusp forms inside $M_k^n(\Gamma)$ by $S_k^n(\Gamma)$ and define $\mathcal{E}_k^n(\Gamma)$, the space of Eisenstein series, to be the orthogonal complement of $S_k^n(\Gamma)$ under the Petersson-inner product defined by

$$\langle F, G \rangle = \int_{\mathcal{F}_\Gamma} F(Z)\overline{G(Z)} \det(Y)^k \frac{dX\, dY}{\det(Y)^{n+1}};$$

where at least one of $F, G \in S_k^n(\Gamma)$.

2. The Converse Question

It is natural to ask whether the converse of the assertion in (1.2) above is true, i.e., if the Fourier coefficients of $F \in M_k^n(\Gamma)$ satisfy (1.2), is it necessarily cuspidal? Thus we seek to characterize Siegel modular forms (as being cuspidal or not) by the growth of their Fourier coefficients. This question has recently received the attention of many authors, following a proposal by Kohnen [20] who considered this question in the context of elliptic modular forms.

It is desirable to answer the above question in any degree. For cusp forms on congruence subgroups, it is evidently true that the Hecke bound (1.2) holds on the Fourier coefficients for the Fourier expansion at all the cusps. Moreover, it can be proved (see [17]) that (1.1) actually holds for the exponent $k - \frac{n+1}{2}$.

In view of this, in [4, 5] a more refined version of the converse question to the above was made (which we believe may have been known before as folkore):

Conjecture 2.1 — Let Γ be a congruence subgroup of Γ_n. Let $F \in M_k^n(\Gamma)$ be such that it's Fourier coefficients supported on Λ_n^+ satisfy

$$a_F(T) \ll_F \det(T)^\alpha$$

in at least one of the cusps of Γ. Then F is a cusp form if $\alpha < k - \frac{n+1}{2}$ and $k \geq n/2$.

Remark 2.2 : The above conjecture is the best possible. This can be seen from the Fourier coefficients $b_k^n(T)$ of the Siegel Eisenstein series of weight k and degree n defined for $k > n+1$ by

$$E_k^n(Z) = \sum_{\{C,D\}} \det(CZ+D)^{-k},$$

where $\{C, D\}$ denotes the equivalence classes of co-prime symmetric pairs (C, D) (i.e, those tuples which can be completed to an element of Γ_n) modulo the action of the unimodular group $\mathrm{GL}_n(\mathbf{Z})$: $U \cdot (C, D) = (UC, UD)$. One has

$$b_k^n(T) \asymp \det(T)^{k-(n+1)/2}, \quad (T > 0),$$

where $\asymp$ denotes "the same order of magnitude". This shows that the conjecture stated above is indeed the best possible, or in other words, there can not be any uniform cancellations in the (positive-definite) Fourier coefficients of any non-zero linear combination of Eisenstein series.

2.1 *Some old results*

We briefly recall the results obtained so far in favour of the above conjecture. Namely in [20, 21], it was proved for elliptic and resp. Siegel modular forms of weights 2 and k on $\Gamma_0(N)$ (resp. Γ_n) at the cusp ∞ by using an explicit description of the space of elliptic (resp. Jacobi) Eisenstein series. In the context of Jacobi, Siegel, Hilbert or half-integral weight modular forms, see the works [4], [21], [8], [22], [23]; all of which answer this converse question assuming mostly the Hecke bound in either the case of level one or for the congruence subgroup $\Gamma_0^n(N)$ (N square-free) at the cusp ∞ or for forms of small degrees.

One of the basic tools in these papers (in the case of the Hecke bound) is to write $F = E + g$, where E is in $\mathcal{E}_k^n(\Gamma)$, and g is a cusp form. Now since g already satisfies the Hecke bound, it is enough to assume $F = E$. Now one argues by writing down an explicit basis for $\mathcal{E}_k^n(\Gamma)$, whose elements are known to have explicitly determined Fourier coefficients (in particular, their growth-rate is known), but only for small degrees, say for $n = 1, 2$ or for Jacobi forms of degree 1. In [23], the approach was different from the rest, and used Imai's converse theorem for the Koecher-Maass series attached to degree 2 Siegel cusp forms. Weissauer has a generalization of this type of converse theorem for level one, but nothing is known for higher levels, where the situation is much more delicate. So, even this method has its limitations at the moment.

As was observed in [4], in higher degrees it is difficult to argue using explicit formulas for the Fourier coefficients of the elements $\mathcal{E}_k^n(\Gamma)$ of degree n. First of all, such formulas are not explicit enough (in the case of higher levels) and not known in most cases. Furthermore, one has to consider at the same time several Klingen type Eisenstein series; all of these have the same kind of growth properties (with respect to $\det(T)$). Thus in [4], we took a route different from the previous approaches to prove the following.

***Theorem* 2.3** — *Let $n \geq 1$ and N be either 1 or square–free. Suppose that $a(T) \ll_F \det(T)^\alpha$ for some $F \in M_k^n(\Gamma_0^n(N))$ at the cusp ∞. Then f is a cusp form provided that $\alpha < k - n$ and $k \geq 2n + 1$.*

In the above result, we answered the question by transferring it into a problem about characterizing cuspidal Hecke eigenforms by the growth of their eigenvalues. Essential tools in [4] were the Andrianov-identity (describing the standard L-function in terms of a Dirichlet series involving Fourier coefficients) and the Zharkovskaya-relation (describing the standard L-function of a noncuspial Hecke eigenform in terms of the standard L-function of a cusp form of lower degree). The result then followed from *analytic* considerations about the possible poles of the standard L-function. We now see more details.

2.1.1 *Brief sketch of proof*

(a) Reduce to Hecke eigenforms by diagonalizing the Hecke module generated by F.

(b) Use Andrianov's identity, which relates the Fourier coefficients with Hecke eigenvalues (via the Standard zeta function), to get good convergence properties of the Standard zeta function $D^n(F, s)$ attached to F. We briefly recall them below.

Definition 2.4 [The Andrianov identity] — Let $F = \sum_T a(F, T) \exp(\operatorname{tr} TZ) \in M_k^n(\Gamma_0(N))$ be a Hecke eigenform. We fix $T_0 \in \Lambda_n^+$ with $a(F, T_0) \neq 0$ and put $M = N \cdot \det(2T_0)$. Then

$$\sum_X a(F, T_0[X]) \det(X)^{-s-k+1} = a(F, T_0) \Xi_M(s) \cdot D_M^n(F, s) \tag{2.1}$$

where

$$\Xi_M(s) = \begin{cases} L_M(s+m, \chi_{T_0})^{-1} \cdot \prod_{i=0}^{m-1} \zeta_M(2s-2i)^{-1} & \text{if } n = 2m, \\ \prod_{i=0}^{m} \zeta_M(2s+2i)^{-1} & \text{if } n = 2m+1. \end{cases}$$

Here X runs over all nonsingular integral matrices of size n with $(\det X, M) = 1$, modulo the action of $\mathrm{GL}(n, \mathbf{Z})$ from the right. Moreover $L(s, \chi_{T_0})$ is the Dirichlet L-series attached to the quadratic character $\chi_{T_0} := \left(\frac{(-1)^m \det(2T_0)}{\cdot}\right)$ and $\zeta_M(\cdot)$ (resp. $L_M(\cdot)$) means that the Euler factors corresponding to $p \mid M$ are omitted.

Definition 2.5 [Standard zeta function] —

$$D_N^n(F, s) = \prod_{p \nmid N} D_{p,F}(p^{-s}), \quad (\mathrm{Re(s)} \gg 1)$$

where the p-Euler factor is given by

$$D_{p,F}(X) = (1-X) \prod_{i=1}^{n} (1-\alpha_{i,p}X)(1-\alpha_{i,p}^{-1}X),$$

where $(\alpha_{0,p}, \ldots, \alpha_{n,p})$ are the Satake p-parameters of F.

Definition 2.6 [The Zharkovskaya relation] —

$$D_M^n(F, s) = D_M^{n-1}(\Phi F, s)\zeta_M(s-k+n)\zeta_M(s+k-n), \quad (\Phi F \neq 0)$$

where Φ is the Siegel's Φ operator : $M_k^n(\Gamma_0^n(N)) \to M_k^{n-1}(\Gamma_0^{n-1}(N))$ defined by

$$\Phi F(Z) := \lim_{\lambda \to \infty} F\left(\begin{smallmatrix} Z & 0 \\ 0 & i\lambda \end{smallmatrix}\right).$$

We now continue with the sketch of the proof.

(c) We use the good convergence properties of $D_M^n(F, s)$ thus obtained to compare the location of poles on the two sides of the Zharkovskaya relation. We find that the LHS of (3) is holomorphic in $Re(s) > n+1$, whereas we have a pole on the RHS at $s = k-n+1$.

(d) In order to get better range of weights k, we have to iterate the Zharkovskaya relation enough number of times to hit a cusp form of lower degree.

(e) Show that these convergence properties contradict the location of poles of the standard zeta function, if F was not a cusp form.

(f) When $N > 1$, one can not directly carry out (c), since it involves the standard zeta function attached to $\Phi(F)$, where Φ is the Siegel's Φ operator. $\Phi(F)$ could be zero even if F is not a cusp form!

(g) A way to get around this is to resort to the 'solution of the basis problem' due to Böcherer, Katsurada, Sculze-Pillot when $k \geq 2n+1$ and N is square-free. This says that any modular form in $M_n^k(N)$ can be written as a linear combination of theta series, i.e.,

$$F = \sum_{i=1}^{r} c_i \vartheta^n(S_i) = \underbrace{\sum_{i_1} c_{i_1} \vartheta^n(S_{i_1})}_{Genus\ \Upsilon_1} + \cdots + \underbrace{\sum_{i_t} c_{i_t} \vartheta^n(S_{i_t})}_{Genus\ \Upsilon_t},$$

where $\vartheta^n(S_i)$ are the usual theta functions attached to symmetric, positive-definite, even matrices S_i of size $2k$ and level N

$$\vartheta^n(S_i)(Z) = \sum_{X \in \mathbf{Z}^{(2k,n)}} \exp\left(\pi i \text{ tr } X^t S_i X \cdot Z\right),$$

and we have grouped together the theta series according to the genera Υ_j in which they lie.

By a result of Andrianov [2], note that each theta block consisting of theta series corresponding to the quadratic forms defined by S_i in the same genus can be diagonalized under the Hecke operators. This provides us with a $G = \sum_j c_j \vartheta^n(S_j)$ with the same eigenvalues as that of F (by linear algebra) with $\Phi G \neq 0$, since it is known by the work of Böcherer and Schulze-Pillot that

$$\underbrace{\sum_j c_j \vartheta^n(S_j)}_{Genus\ \Upsilon_j} \quad \text{is cuspidal} \quad \iff \Phi\left(\sum c_j \vartheta^n(S_j)\right) = 0 \quad \text{if } N \text{ is square-free.}$$

Thus we can now replace F by G and use the Zharkovskaya relation to complete the proof.

2.1.2 *Some variants* : We now discuss some interesting variants of the question that we are concerned with.

Degenerate indices : A cusp form F of degree n has all its degenerate Fourier coefficients $a(F,T)$ (the ones with the rank of T less than n) equal to zero. Conversely, it is natural to ask if one can characterize cuspidality by information only on the degenerate coefficients? We have the following result in this regard.

Theorem 2.7 — *Suppose that for $F \in M_k^n$, α real, we have*

$$a(F,T) \ll_F \delta(T)^{\alpha} \quad \textit{if } \operatorname{rank}(T) < n.$$

$$\textit{If} \quad k > \max\left\{n,\ \frac{n}{2} + 2\alpha\right\},$$

then F is a cusp form, i.e., the degenerate coefficients vanish.

Here $\delta(T) = \det(T_1)$, where we write $U'TU = \left(\begin{smallmatrix} T_1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, with T_1 invertible $r \times r$ matrix with $r =$ the rank of T.

Remark 2.8 : This is a counter-part to the previous theorems and has a completely different proof since we can not apply Andrianov's identity. Here we use the Rankin-Selberg method, [4] for details.

Fourier-Jacobi coefficients : A Siegel cusp form has all its Fourier-Jacobi coefficients cuspidal. Suppose that $n \geq 2$ and

$$F(Z) = \sum_{m \in \Lambda_q} \phi_{F,m}(\tau, z) \exp(2\pi i \operatorname{tr} m\tau')$$

be the Fourier-Jacobi expansion of F, where we have decomposed $Z = \left(\begin{smallmatrix} \tau & z^t \\ z & \tau' \end{smallmatrix}\right)$, ($\tau \in \mathbf{H}_{n-q}, z \in \mathbf{C}^{(q,n-q)}, \tau' \in \mathbf{H}_q, z^t$ being the transpose of z). Here $\phi_{F,m}$ are the Fourier-Jacobi coefficients of F (Jacobi forms of weight k and index m) corresponding to this decomposition.

Note that cuspidality of $\phi_{F,m}$ ($m > 0$) implies the same for F (use Jacobi decomposition, Hecke bound for $\phi_{F,m}$ and Theorem 2.3). Also note that $\phi_{F,m}(\tau, z) = \phi_{F,m[u]}(\tau, u^t \cdot z)$, u in $GL_q(\mathbf{Z})$, so cuspidality is preserved under the action $GL_q(\mathbf{Z})$ on the index m.

What if we knew cuspidality of $\phi_{F,m}$ only on some smaller subsets? A partial answer is provided by the next theorem.

Theorem 2.9 — *Let $F \in M_k^n$ be such that for some $1 \leq q \leq n-1$ its Fourier-Jacobi coefficients $\phi_{F,m}$ are cusp forms for all $m \in \Lambda_q \setminus \mathcal{S}$, where $\mathcal{S}$ consists of finitely many $GL_q(\mathbf{Z})$- equivalence classes. Then F is a cusp form if $k \geq \frac{n-1}{2}$.*

Question : How big the set $\mathcal{S}$ can be in the above theorem?

Recall that the Andrianov-identity involves the $GL_n(\mathbf{Q})$ equivalence class of any *one* $T_0 > 0$ such that $a(F, T_0)$ is non-zero. Therefore we could rephrase all our main results by requesting only a growth property within one such class of Fourier coefficients.

2.2 *Difficulties in higher degrees*

It also is desirable to have such a result for modular forms for arbitrary congruence subgroups. Here we meet several new obstacles, which do not allow us to proceed directly along the same lines as sketched above. Namely, there are several cusps, and several Siegel's Φ-operators (F is a cusp form if and only if it is in the kernel of all such operators) which are used to lower the degree of a form. Both the Andrianov-identity and the Zharkovskaya-relation do not generalize in a smooth way to principal congruence subgroups; in particular, the cases, where the cusps under consideration are not equal to the one usually called ∞, are quite delicate.

Further, the reduction of the question to an eigenform for the full Hecke algebra is also not clear. One can assume that the growth property is given at the cusp ∞ for some principal congruence subgroup. Even if one can reduce the question to some eigenfunction of a 'nice' Hecke subalgebra, and hoped to carry through the *analytic* arguments via L-functions, one should require that $\Phi F \neq 0$, in order to be able to apply Zharkovskaya relaton. This was averted in [4, Thm. 4.4] by using theta series attached to quadratic forms in a fixed genus for *square-free levels*. These had rather special properties; but in the general case such a trick seems not available. These considerations lead us to look for a more flexible method, avoiding the plausible difficulties discussed above.

2.3 *A new 'local' approach*

In this approach we avoid such problems by using a 'local' version of the identities above. Clearly, it suffices to prove Conj. 2.1 for all Siegel-principal congruence subgroups. Thus we consider modular forms for the principal congruence subgroup of level N and use the above mentioned identities only for special Hecke operators for primes $p \equiv 1 \bmod N$. Then we note that only *algebraic* considerations (as opposed to analytic ones) play a role. *We emphasize that we need such a growth condition only in one cusp*! Our main result can then be stated as follows.

***Theorem* 2.10** *(Local method) — Let Γ be a congruence subgroup of Γ_n such that the Fourier coefficients $a(F,T)$ of $F \in M_k^n(\Gamma)$ satisfy*

$$a(F,T) \ll_F (\det T)^\alpha \quad (T \in \Lambda_n^+),$$

for some $\alpha \in \mathbf{R}$ in some cusp of Γ. Suppose that $k > 2n$ and $\alpha < k - n$. Then F is a cusp form.

Remark 2.11 : (i) Note that the condition $k > 2n$ in the Theorem comes from the proof and thus we can not say anything for the weights in the range $[n/2, 2n]$.

(ii) It would be interesting to find a proof for small weights. Later we will see an improvement when $\Gamma = \Gamma_0^n(N)$.

2.3.1. *Brief idea of proof* : A filtration on $M_k^n(\Gamma)$. For a congruence subgroup Γ, we define a filtration on $M_k^n(\Gamma)$ as follows:

$$S_k^n(\Gamma) = \mathscr{N}_k^{n,0}(\Gamma) \subset \mathscr{N}_k^{n,1}(\Gamma) \subset \cdots \subset \mathscr{N}_k^{n,n}(\Gamma) = M_k^n(\Gamma), \tag{2.2}$$

where the subspaces $\mathscr{N}_k^{n,r}(\Gamma)$ are defined by

$$\mathscr{N}_k^{n,r}(\Gamma) := \{f \in M_k^n(\Gamma) \mid \forall \gamma \in \mathrm{Sp}(n, \mathbf{Z}) : (f \mid_k \gamma) \mid \Phi^{r+1} = 0\}. \tag{2.3}$$

Here Φ is the Siegel's Φ operator.

One shows that these filtration is Hecke equivariant and stable under inclusion of subgroups, these allow us to assume $\Gamma = \Gamma^n(N)$ for some N.

For $1 \leq j \leq n$ we denote by α_j a positive real number satisfying

$$\alpha_j < k - n + \frac{j-1}{2}.$$

***Theorem* 2.12** — *Assume that $f \in \mathscr{N}_k^{n,j}(\Gamma)$ satisfies the condition $\mathcal{K}(\alpha_j)$ in some cusp. Then f belongs to $\mathscr{N}_k^{n,j-1}$. For $j = n$ this holds for all weights $k \geq \frac{n+1}{2}$, whereas for $1 \leq j \leq n-1$ we have to impose the conditions $k \geq \max(n, 2n - 2j + 2)$.*

Theorem 2.10 follows by passing succesively from $j = n$ in the Theorem 2.12 to $j = 1$ and choosing $\alpha_1 := \alpha$ and $\alpha_j := \alpha + \frac{j-1}{2}$. To see this, recall that our assumption on α is $\alpha < k - n$. Then f also has the $\mathcal{K}(\alpha_j)$ property for all $j \geq 1$ and we can apply the above theorem to pass through the filtration

$$S_k^n(\Gamma) = \mathscr{N}_k^{n,0}(\Gamma) \subset \mathscr{N}_k^{n,1}(\Gamma) \subset \cdots \subset \mathscr{N}_k^{n,n}(\Gamma) = M_k^n(\Gamma), \tag{2.4}$$

to reach the space of cusp forms. In some more details,

(a) Choose a Hecke subalgebra generated by the double cosets

$$T(D_{n,p}) := \Gamma^n(N) \begin{pmatrix} D_{n,p} & 0 \\ 0 & D_{n,p}^{-1} \end{pmatrix} \Gamma^n(N)$$

where the diagonal matrices $D_{n,p}$ are given by

$$D_{n,p} := \mathrm{diag}(1, 1, \ldots, 1, \mathrm{p}),$$

with $p \equiv 1 \mod N$. Upto a normalisation, $T(D_{n,p})$ is the Hecke operator $T_{1,n-1}(p^2)$.

(b) Reduce to an eigenform for this subalgebra (using the results of Andrianov - Evdokimov) and use a local version of the Andrianov's identity to transfer the bound on Fourier coefficients to an upper bound for the eigenvalues.

(c) For $j = n$ we essentially encounter forms which behave like the Siegel Eisenstein series under the Φ operator; for $1 \leq j \leq n-1$, we can reduce to cusp form of degree $n - j$ by (essentially) considering $f \mid \Phi^j$. We look at the second case, its more interesting.

(d) The estimate that we have from the local version of the Andrianov-identity is (setting $\lambda_n := \lambda(D_{n,p})$)

$$|\lambda_n| \ll_f p^{t(\alpha_j)}$$

with a certain explicit function $t(\cdot)$.

(e) On the other hand, λ_n computed recursively by using a local version of the so-called Zarkovskaya relation due to Krieg, Evdokimov, and we get under the condition of the weight in Theorem 2.12 that

$$|\lambda_n| \sim p^{k+j-1}.$$

This bound p^{k+j-1} is then shown to be incompatible with the upper bound for λ_n obtained before under the hypothesis on α_j in Theorem 2.12.

2.4 *Small weights*

It is quite interesting to obtain these results for all admissible weights, i.e., for all weights $k \geq n/2$. When $k < n/2$, all modular forms are singular, i.e, their Fourier coefficients are supported only over degenerate indices, so that they can not be cuspidal, unless zero.

The local method and all the previous results on this topic were for the so-called large weights $k > 2n$, a common feature in the theory of Siegel modular forms. To get these results for the smaller weights in the local method, we would need non-trivial estimates for the eigenvalues of cusp forms (while carrying out (e) above), which is perhaps not available for arbitrary levels and degrees at the moment. So one has to argue differently, and we now state our result in this direction.

***Theorem* 2.13** ([5, 6]) — *Let the Fourier coefficients $a(F,T)$ of $F \in M_k^n(N)$ satisfy*

$$a(F,T) \ll_F (\det T)^{\alpha} \quad (T \in \Lambda_n^+),$$

for some $\alpha \in \mathbf{R}$ in some cusp. Suppose that

(i) $\alpha < k - \frac{n+3}{2}$, $k \geq \frac{n}{2} + 1$ and $N \geq 1$, or

(ii) $\alpha < k - \frac{n+1}{2}$, $k \geq \max\{2, \frac{n}{2}\}$ and N is square-free or equals 1.

Then F is cuspidal.

2.4.1 *Brief idea of the proof*

(a) For (i), we first write down explicit representatives for both the (finitely many) zero and one-dimensional cusps. Fourier expansions are described by the zero dimensional cusps, and to check cuspidality, one needs the one-dimensional cusps defined as: (put $\Gamma_n = Sp(n, \mathbf{Z})$)

$$\mathcal{C}_0 := \Gamma\backslash\Gamma_n/\Gamma_{n,\infty}, \quad \mathcal{C}_1 := \Gamma\backslash\Gamma_n/C_{n,n-1},$$

where

$$\Gamma_{n,\infty} := \{\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_n \mid C = 0_{n,n}\},$$
$$C_{n,n-1} := \{g \in \Gamma_n \mid g = \left(\begin{smallmatrix} * & * \\ 0_{1,2n-1} & * \end{smallmatrix}\right)\},$$

where $0_{r,s}$ denotes the $r \times s$ zero-matrix.

(b) We need (and prove) such representatives to be "diagonal", i.e., should be of the form $\omega \times \delta$, with $\omega \in \Gamma^{n-2}, \delta \in \Gamma^2$ etc., and

$$\omega \times \delta = \begin{pmatrix} A & 0 & B & 0 \\ 0 & a & 0 & b \\ C & 0 & D & 0 \\ 0 & c & 0 & d \end{pmatrix}, \omega = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right), \delta = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right).$$

Once this is done, there can be two ways of proceeding, each having its own merit (and limitation), which we briefly discuss below.

– *Fourier-Jacobi expansion:* We then use a Fourier-Jacobi expansion of type $(n-2, 2)$, i.e., we decompose $Z \in \mathbf{H}_n$ in the form $Z = \begin{pmatrix} \tau'^{(n-2)} & z \\ z' & \tau^{(2)} \end{pmatrix}$. We reduce the problem to degree 2 by considering the theta-components of the Fourier Jacobi coefficients.

(i) Implicit in the proof is a quite general method to answer the growth question for Jacobi forms of higher degree at least for level one, by looking at the theta components, thus avoiding complicated (and in most cases unknown) formulas for Fourier coefficients of Jacobi Eisenstein series.

(ii) We can not take the decomposition $(n-1, 1)$ as in this case we do not always get diagonal representatives, unless N is square–free.

– *The Witt operator:* For (ii) the level is one or square–free. We use the Witt-operator on modular forms defined by

$$(WF)(\tau, z) := F(\left(\begin{smallmatrix} \tau & 0 \\ 0 & z \end{smallmatrix}\right)), \quad \tau \in \mathbf{H}_1, z \in \mathbf{H}_{n-1}.$$

W has the following equivariant property. Writing $Z = \left(\begin{smallmatrix} \tau & x \\ x' & z \end{smallmatrix}\right) \in \mathbf{H}_n$, where $\tau \in \mathbf{H}$ and $z \in \mathbf{H}_{n-1}$:

$$W\left(F\mid_k \omega \times \delta\right)(Z) = j(\omega,\tau)^{-k} j(\delta,z)^{-k} WF\left(\omega\langle\tau\rangle, \delta\langle z\rangle\right) \tag{2.5}$$

where $\omega = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \in \Gamma_1, \delta = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \in \Gamma_{n-1}$ and $\omega \times \delta$ denotes the diagonal emedding

$$\omega \times \delta = \begin{pmatrix} a_1 & 0 & b_1 & 0 \\ 0 & a_2 & 0 & b_2 \\ c_1 & 0 & d_1 & 0 \\ 0 & c_2 & 0 & d_2 \end{pmatrix}.$$

(i) This implies that if $F \in M_k^n(\Gamma_0(N))$ then $(WF)(\tau, z)$ is a modular form in each of the variables τ, z of degree 1 and $n-1$ respectively.

(ii) We use this fact and the nice representatives to reduce the problem to degree 1, where we have a complete answer by Theorem 2.10.

Note added in Proof: Recently yet another approach to questions of this type was found [7], which uses the asymptotic behaviour of the modular form for $Y \to 0$ (requiring growth conditions for *all* Fourier coefficients).

ACKNOWLEDGEMENT

The author thanks the Indian National Science Academy (INSA) for invitation to write this article. He also acknowledges financial support, in parts, from the UGC Center for Advanced Studies, DST India and IISc Bangalore.

REFERENCES

1. A. N. Andrianov and V. G. Zuravlev, *Modular Forms and Hecke operators*, Translations of Mathematical Monographs, **145**, American Mathematical Society, Providence, RI, 1995.
2. A. N. Andrianov, The multiplicative arithmetic of Siegel modular forms, *Russian Math. Surveys*, **34** (1979), 75-148.
3. S. Böcherer, Ein Rationalitätssatz für formale Heckereihen zur Siegelschen Modulgruppe, *Abh. Math. Sem. Univ. Hamburg*, **56** (1986), 35-47.
4. S. Böcherer and S. Das, Characterization of Siegel cusp forms by the growth of their Fourier coefficients, *Math. Ann.*, **359**(1-2) (2014), 169-188.
5. S. Böcherer and S. Das, Cuspidality and the growth of Fourier coefficients of modular forms, *J. Reine Angew. Math.*, (to appear), DOI 10.1515/crelle-2015-0075.
6. S. Böcherer and S. Das, Cuspidality and the growth of Fourier coefficients: small weights, *Math. Z.*, (to appear) DOI: 10.1007/s00209-015-1609-2.

7. S. Böcherer and W. Kohnen, *On the Fourier coefficients of Siegel modular forms*, preprint, 2015.
8. Y. Choie and W. Kohnen, On the Fourier coefficients of modular forms of half-integral weight, *Int. J. Number Theory*, **9**(8) (2013), 1879-1883.
9. W. Duke, R. Howe and J.-S. Li, Estimating Hecke eigenvalues of Siegel modular forms, *Duke Math. J.*, **67** (1992), 499-517.
10. S. A. Evdokimov, Euler products for congruence subgroups of the Siegel group of genus 2, *Math. USSR Sbornik*, **28** (1976), 431-458.
11. S. A. Evdokimov, A basis of eigenfunctions of Hecke operators in the theory of Siegel modular forms of genus n, *Math. USSR Sbornik*, **43** (1982), 299-321.
12. E. Freitag, Die Wirkung von Heckeoperatoren auf Thetarihen mit harmonischen Koeffizienten, *Math. Annalen*, **258**(4) (1982), 419-440.
13. E. Freitag, Siegelsche Modulfunktionen, *Grundl. Math. Wiss.*, **254** Springer–Verlag, (1983).
14. E. Freitag, Siegel Eisenstein series of arbitrary level and theta series, *Abh. Math. Sem. Univ. Hamburg*, **66** (1996), 229-247.
15. R. Godement, *Généralités sur les formes modulaires, I*. Séminaire Henri Cartan, tome **10**, n°1, (1957-1958), exp. n°**7**, p. 1-18.
16. Y. Kitaoka, Dirichlet series in the theory of Siegel modular forms, *Nagoya Math. J.*, **95** (1984), 73-84.
17. Y. Kitaoka, *Siegel Modular Forms and Representations by Quadratic Forms*, Lecture Notes, Tata Institute of Fundamental Research, (1986).
18. A. Krieg, Das Vertauschungsgesetz zwischen Hecke Operatoren und dem Siegelschen ϕ-Operator, *Arch. Math.*, **46** (1986), 323-329.
19. G. Shimura, Convergence of zeta functions on symplectic and metaplectic groups, *Duke Math. J.*, **82** (1996), 327-347.
20. W. Kohnen, On certain generalized modular forms, *Funct. Approx. Comment. Math.*, **43** (2010), 23-29.
21. W. Kohnen and Y. Martin, A characterization of degree two cusp forms by the growth of their Fourier coefficients, *Forum Math.*, DOI 10.1515/forum-2011-0142.
22. B. Linowitz, Characterizing Hilbert modular cusp forms by coefficients size, *Kyushu Math. J.*, **68**(1) (2014), 105-111.
23. Y. Mizuno, On characterisation of Siegel cusp forms by the Hecke bound, *Mathematika*, **61**(1) (2015), 89-100.
24. G. Shimura, *Introduction to the arithmetic theory of automorphic functions*, Kano Memorial Lectures, No. 1. Publications of the Mathematical Society of Japan, No. 11. Iwanami Shoten, Publishers, Tokyo; Princeton University Press, Princeton, N.J., 1971. xiv+267 pp.
25. R. Weissauer, Vektorwertige Siegeische Modulformen kleinen Gewichtes, *Journal für die reine und angewandte Mathematik*, **343** (1983), 184-202.
26. R. Weissauer, *Stabile Modulformen und Eisensteinreihen*, Lecture Notes in Mathematics, **1219** (1986), Springer-Verlag.