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NON-TRIVIALITY OF A PRODUCT IN THE ADAMS E_2-TERM

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Let p be a prime greater than five and A the mod p Steenrod algebra. In this paper, we prove that $h_n h_m \tilde{\delta}_{s+4} \in \mathrm{Ext}_A^{s+6,t(s,n,m)+s}(Z/p, Z/p)$ is nontrivial in the Adams E_2-term when $m \geq n+2 \geq 7$ and $0 \leq s < p-4$, and trivial in the Adams E_2-term when $m \geq n+2 = 6$ and $0 \leq s < p-4$, where $\tilde{\delta}_{s+4}$ stands for the fourth Greek letter element and $t(s,n,m) = 2(p-1)[(s+1)+(s+2)p+(s+3)p^2+(s+4)p^3+p^n+p^m]$.

Key words : Stable homotopy groups of spheres; Adams spectral sequence; Smith-Toda spectrum; May spectral sequence.

1. Introduction and Statement of Results

For a prime p, there is a spectral sequence, called the Adams spectral sequence, converging to the stable homotopy groups of spheres localized at p, $\pi_*^s(S)$, whose E_2-term is isomorphic to $\mathrm{Ext}_A^{*,*}(Z/p, Z/p)$, where A denotes the mod p Steenrod algebra (see [1]).

The known results on $\mathrm{Ext}_A^{*,*}(Z/p, Z/p)$ are as follows. $\mathrm{Ext}_A^{0,*}(Z/p, Z/p) = Z/p$ by its definition. From [4], we have $\mathrm{Ext}_A^{1,*}(Z/p, Z/p)$ has Z/p-basis consisting of $a_0 \in \mathrm{Ext}_A^{1,1}(Z/p, Z/p)$ and $h_i \in \mathrm{Ext}_A^{1,p^iq}(Z/p, Z/p)$ for all $i \geq 0$ and $\mathrm{Ext}_A^{2,*}(Z/p, Z/p)$ has Z/p-basis consisting of α_2, a_0^2, $a_0 h_i$ $(i > 0)$, g_i $(i \geq 0)$, k_i $(i \geq 0)$, b_i $(i \geq 0)$, and $h_i h_j$ $(j \geq i+2, i \geq 0)$ whose internal degrees are $2q+1$, 2, p^iq+1, $p^{i+1}q+2p^iq$, $2p^{i+1}q+p^iq$, $p^{i+1}q$ and p^iq+p^jq, respectively. In 1980, Aikawa [2] determined $\mathrm{Ext}_A^{3,*}(Z/p, Z/p)$ by the lambda algebra. Here, $q = 2(p-1)$ as usual.

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In [7], the elements $\tilde{\alpha}_s^{(n)}$ were defined in the Adams E_2-term $\mathrm{Ext}_A^{n,*}(Z/p, Z/p)$ for $n < p$ and $s \not\equiv 0, 1, \cdots, n-1 \bmod p$. Here $\tilde{\alpha}_s^{(n)}$ stands for the n-th letter of Greek alphabet and we call them Greek letter elements. When $n = 1$, 2, 3 and 4, $\tilde{\alpha}_s^{(n)}$ are denoted by $\tilde{\alpha}_s$, $\tilde{\beta}_s$, $\tilde{\gamma}_s$ and $\tilde{\delta}_s$, respectively.

For $n \leq 3$, it has already been proved that $\tilde{\alpha}_s$, $\tilde{\beta}_s$, $\tilde{\gamma}_s$ detect nontrivial elements α_s, β_s and γ_s in $\pi_*^s(S)$ in the Adams spectral sequence for $s \not\equiv 0, 1, \cdots, n-1 \bmod p$, respectively ([7]). In the case of $n \geq 4$, we have had few information on them yet.

The purpose of this paper is to prove the non-triviality and triviality of the product element $h_n h_m \tilde{\delta}_{s+4}$ in $\mathrm{Ext}_A^{s+6, t(s,n,m)+s}(Z/p, Z/p)$ under suitable restricts on m and n.

In this paper, our main result can be stated as follows:

***Theorem* 1.1** — *Let $p \geq 7$ and $0 \leq s < p-4$. Then in the Adams E_2-term $\mathrm{Ext}_A^{s+6, t(s,n,m)+s}$ $(Z/p, Z/p)$, we have:*

(1) *the product $h_n h_m \tilde{\delta}_{s+4}$ is nontrivial for $m \geq n+2 \geq 7$.*

(2) *the product $h_4 h_m \tilde{\delta}_{s+4}$ is trivial for $m \geq 6$.*

Here $t(s,n,m) = q[(s+1) + (s+2)p + (s+4)p^2 + (s+4)p^3 + p^n + p^m]$.

The paper is arranged as follows: we recall some knowledge on the May spectral sequence in Section 2, then we will prove Theorem 1.1 in Section 3.

2. The May Spectral Sequence

For the sake of completeness, we first recall some basic knowledge on the May spectral sequence.

From [5], there is the May spectral sequence $\{E_r^{s,t,*}, d_r\}$ which converges to $\mathrm{Ext}_A^{s,t}(Z/p, Z/p)$ with E_1-term

$$E_1^{*,*,*} = E(h_{m,i} | m > 0, i \geq 0) \otimes P(b_{m,i} | m > 0, i \geq 0) \otimes P(a_n | n \geq 0), \tag{2.1}$$

where $E(\ )$ is the exterior algebra, $P(\ )$ is the polynomial algebra, and

$$h_{m,i} \in E_1^{1, 2(p^m-1)p^i, 2m-1}, b_{m,i} \in E_1^{2, 2(p^m-1)p^{i+1}, p(2m-1)}, a_n \in E_1^{1, 2p^n-1, 2n+1}.$$

One has

$$d_r : E_r^{s,t,u} \to E_r^{s+1,t,u-r} \tag{2.2}$$

and if $x \in E_r^{s,t,*}$ and $y \in E_r^{s',t',*}$, then

$$d_r(x \cdot y) = d_r(x) \cdot y + (-1)^s x \cdot d_r(y). \tag{2.3}$$

In particular, the first May differential d_1 is given by

$$d_1(h_{i,j}) = \sum_{0<k<i} h_{i-k,k+j}h_{k,j},\ d_1(a_i) = \sum_{0\le k<i} h_{i-k,k}a_k,\ d_1(b_{i,j}) = 0. \tag{2.4}$$

There also exists a graded commutativity in the May spectral sequence: $x \cdot y = (-1)^{ss'+tt'} y \cdot x$ for $x, y = h_{m,i}, b_{m,i}$ or a_n.

For each element $x \in E_1^{s,t,u}$, we define dim $x = s$, deg $x = t$, $M(x) = u$. Then we have that

$$\begin{cases} \text{dim } h_{i,j} = \text{dim } a_i = 1, \\ \text{dim } b_{i,j} = 2, \\ \text{deg } h_{i,j} = q(p^{i+j-1} + \cdots + p^j), \\ \text{deg } b_{i,j} = q(p^{i+j} + \cdots + p^{j+1}), \\ \text{deg } a_i = q(p^{i-1} + \cdots + 1) + 1, \\ \text{deg } a_0 = 1, \\ M(h_{i,j}) = M(a_{i-1}) = 2i - 1, \\ M(b_{i,j}) = (2i-1)p, \end{cases} \tag{2.5}$$

where $i \ge 1$, $j \ge 0$.

For each integer $m \ge 0$, it can be expressed uniquely as

$$m = q(c_n p^n + c_{n-1}p^{n-1} + \cdots + c_1 p + c_0) + e, \tag{2.6}$$

where $0 \le c_i < p$ ($0 \le i < n$), $p > c_n > 0$, $0 \le e < q$.

In this paper, we mainly consider the May E_1-terms of the form $E_1^{s,tq+b,*}$, where s, t, b are three integers with $s > 0, t > 0$ and $b \ge 0$ and satisfy the following conditions:

(1) $0 \le b < q$;

(2) $s < q$.

We denote $a_i, h_{i,j}$ and $b_{i,j}$ by x, y and z, respectively. By the graded commutativity of $E_1^{*,*,*}$, a generator of $E_1^{s,tq+b,*}$ must be of the form

$$g = (x_1 \cdots x_b)(y_1 \cdots y_v)(z_1 \cdots z_l). \tag{2.7}$$

Moreover, by (2.5) and (2.6), the degrees of x_i, y_i and z_i can be expressed uniquely as

$$\begin{cases} \text{deg } x_i = (x_{i,0} + x_{i,1}p + \cdots + x_{i,n}p^m)q + 1, \\ \text{deg } y_i = (y_{i,0} + y_{i,1}p + \cdots + y_{i,n}p^m)q, \\ \text{deg } z_i = (0 + z_{i,1}p + \cdots + z_{i,n}p^m)q, \end{cases} \tag{2.8}$$

and we also have that

$$\begin{cases} (a)(x_{i,0},\dots,x_{i,k},x_{i,k+1},\dots,x_{i,m}) \text{ is of the form } (1,\dots,1,0,\cdots,0)(\text{in} \\ \text{particular, } (x_{i,0},\cdots,x_{i,k},x_{i,k+1},\dots,x_{i,m})=(0,\dots,0,0,\dots,0) \text{ if } x_i=a_0; \\ (b) \text{ if } \omega_{i,k}=1=\omega_{i,l} \text{ for } k<l, \text{ then } \omega_{i,t}=1 \text{ for all } t \text{ between } k \text{ and } l, \\ \text{where } \omega_{i,*} \text{ denotes } y_{i,*} \text{ or } z_{i,*}. \end{cases} \tag{2.9}$$

3. Proof of Theorem 1.1

In this section we first give three lemmas which are needed in the proof of Theorem 1.1. Then we will give the proof of Theorem 1.1.

First recall the following Representation Theorem, due to Liu and Zhao.

Lemma 3.1 [3, Lemma 3.1] — For $p\geq 7$ and $0\leq s<p-4$. Then the fourth Greek letter element $\tilde{\delta}_{s+4}\in \mathrm{Ext}_A^{s+4,t_1(s)+s}(Z/p,Z/p)$ is represented by

$$a_4^s h_{4,0}h_{3,1}h_{2,2}h_{1,3}\in E_1^{s+4,t_1(s)+s,*}$$

in the May E_1-term, where $t_1(s)=[(s+1)+(s+2)p+(s+3)p^2+(s+4)p^3]q$.

Now we first show an important lemma of this section. The proof of Theorem 1.1 depends on it.

Lemma 3.2 — Let $p\geq 7$, $m\geq n+2\geq 6$, $0<s<p-4$. Then we have the May E_1-term

$$E_1^{s+5,t(s,n,m)+s,*}=\begin{cases} M & n=4, \\ K & n=5, \text{ and } s=p-5, \\ 0 & n>5, \text{ or } n=5 \text{ and } 0<s<p-5. \end{cases}$$

Here, $t(s,n,m)=[(s+1)+(s+2)p+(s+3)p^2+(s+4)p^3+p^n+p^m]q$, M is the Z/p-module generated by the following five elements:

$$\begin{cases} \mathbf{g}1=a_5a_4^{s-1}h_{4,0}h_{3,1}h_{2,2}h_{1,3}h_{1,m}, \\ \mathbf{g}2=a_4^s h_{5,0}h_{3,1}h_{2,2}h_{1,3}h_{1,m}, \\ \mathbf{g}3=a_4^s h_{4,0}h_{4,1}h_{2,2}h_{1,3}h_{1,m}, \\ \mathbf{g}4=a_4^s h_{4,0}h_{3,1}h_{3,2}h_{1,3}h_{1,m}, \\ \mathbf{g}5=a_4^s h_{4,0}h_{3,1}h_{2,2}h_{2,3}h_{1,m}, \end{cases}$$

and K is the Z/p-module generated by the following six elements:

$$\begin{cases} \mathbf{g}6 = a_m^{p-6} a_6 h_{m,0} h_{m-1,1} h_{m-2,2} h_{m-3,3} h_{m-4,4}, \\ \mathbf{g}7 = a_m^{p-5} h_{6,0} h_{m-1,1} h_{m-2,2} h_{m-3,3} h_{m-4,4}, \\ \mathbf{g}8 = a_m^{p-5} h_{m,0} h_{5,1} h_{m-2,2} h_{m-3,3} h_{m-4,4}, \\ \mathbf{g}9 = a_m^{p-5} h_{m,0} h_{m-1,1} h_{4,2} h_{m-3,3} h_{m-4,4}, \\ \mathbf{g}10 = a_m^{p-5} h_{m,0} h_{m-1,1} h_{m-2,2} h_{3,3} h_{m-4,4}, \\ \mathbf{g}11 = a_m^{p-5} h_{m,0} h_{m-1,1} h_{m-2,2} h_{m-3,3} h_{2,4}. \end{cases}$$

PROOF : Consider the generator $g = \omega_1 \omega_2 \cdots \omega_l \in E_1^{s+5, t(s,n,m)+s,*}$ in the May spectral sequence, where $t(s,n,m) = q[(s+1) + (s+2)p + (s+3)p^2 + (s+4)p^3 + p^n + p^m]$, ω_i is one of a_k, $h_{\mu,j}$ or $b_{u,z}$, $0 \le k \le m+1$, $1 \le \mu + j \le m+1$, $1 \le u+z \le m$, $\mu \ge 1$, $j \ge 0$, $u \ge 1$, $z \ge 0$. Assume that

$$\text{deg } \omega_i = q(c_{i,m} p^m + c_{i,m-1} p^{m-1} + \cdots + c_{i,1} p + c_{i,0}) + e_i,$$

where $c_{i,j} = 0$ or 1, $e_i = 1$ if $\omega_i = a_{k_i}$, or $e_i = 0$. Then we have

$$\text{dim } g = \sum_{i=1}^{l} \text{dim } \omega_i = s + 5 \tag{3.1}$$

and

$$\begin{aligned} & \text{deg } g \\ = \; & \sum_{i=1}^{l} \text{deg } \omega_i \\ = \; & q[(\sum_{i=1}^{l} c_{i,m}) p^m + \cdots + (\sum_{i=1}^{l} c_{i,n}) p^n + \cdots + (\sum_{i=1}^{l} c_{i,1}) p + (\sum_{i=1}^{l} c_{i,0})] + (\sum_{i=1}^{l} e_i) \\ = \; & q[(s+1) + (s+2)p + (s+3)p^2 + (s+4)p^3 + p^n + p^m] + s. \end{aligned} \tag{3.2}$$

From (3.1) we have $l \leq s+5 \leq p$. Then from (3.2), we have the following:

$$\begin{cases} \sum\limits_{i=1}^{l} e_i = s, & \sum\limits_{i=1}^{l} c_{i,0} = s+1; \\ \sum\limits_{i=1}^{l} c_{i,1} = s+2, & \sum\limits_{i=1}^{l} c_{i,2} = s+3; \\ \sum\limits_{i=1}^{l} c_{i,3} = s+4, & \sum\limits_{i=1}^{l} c_{i,4} = \lambda_4 p,\ \lambda_4 \geq 0; \\ \sum\limits_{i=1}^{l} c_{i,k} + \lambda_{k-1} = \lambda_k p, & \lambda_k \geq 0\ (5 \leq k \leq n-1); \\ \sum\limits_{i=1}^{l} c_{i,n} + \lambda_{n-1} = 1 + \lambda_n p, & \lambda_n \geq 0; \\ \sum\limits_{i=1}^{l} c_{i,k} + \lambda_{k-1} = \lambda_k p, & \lambda_k \geq 0\ (n+1 \leq k \leq m-1); \\ \sum\limits_{i=1}^{l} c_{i,m} + \lambda_{m-1} = 1. \end{cases} \tag{3.3}$$

Case $1 : n > 5$.

(a) If $l < p$, it follows that $(\lambda_4, \lambda_5, \cdots, \lambda_{m-1}) = (0, 0, \cdots, 0)$, which implies $c_{k,n} = c_{t,m} = 1$ and $c_{k,3} = c_{t,3} = 0$ for some k, t with $k \neq t$ by (2.9). However, on the other hand we have $\sum\limits_{i=1}^{l} c_{i,3} = s+4$ from (3.3). These together imply

$$l - 2 \geq \sum_{\substack{i=1 \\ i \neq k,t}}^{l} c_{i,3} = \sum_{i=1}^{l} c_{i,3} = s+4,$$

i.e.,

$$l \geq s+6,$$

which contradicts $l \leq s+5$.

(b) If $l \geq p$, we deduce $l = s+5 = p$. Then g only consists of $h_{\mu,j}$'s. A similar argument shows $\lambda_4 = 1$. Then by (2.9), we see that $\lambda_i = 1$ for $i \geq 4$. Then we have $\sum\limits_{i=1}^{l} c_{i,4} = \sum\limits_{i=1}^{l} c_{i,n} = p$ and $\sum\limits_{i=1}^{l} c_{i,t} = p-1$ $(5 \leq t \leq n-1)$, which contradicts (2.9). So this case is also impossible.

It follows that in this case

$$E_1^{s+5, t(s,n,m)+s,*} = 0.$$

Case $2 : n = 5$. We can deduce $l \geq p$ by an argument similar to that used in Case 1. In fact, we also have $(\lambda_4, \lambda_5, \cdots, \lambda_{m-1}) = (1, 1, \cdots, 1)$. Then (3.3) turns into

$$\begin{cases} \sum_{i=1}^{l} e_i = s, \quad \sum_{i=1}^{l} c_{i,0} = s+1, \quad \sum_{i=1}^{l} c_{i,1} = s+2, \\ \sum_{i=1}^{l} c_{i,2} = s+3, \quad \sum_{i=1}^{l} c_{i,3} = s+4, \\ \sum_{i=1}^{l} c_{i,4} = \sum_{i=1}^{l} c_{i,5} = p = l = s+5, \\ \sum_{i=1}^{l} c_{i,6} = \cdots = \sum_{i=1}^{l} c_{i,m-1} = p-1, \\ \sum_{i=1}^{l} c_{i,m} = 0. \end{cases} \tag{3.4}$$

Then up to sign the generator g must be of the form $(x_1 x_2 \cdots x_{p-5})(y_1 y_2 y_3 y_4 y_5)$ as in (2.7). A direct computation gives six non-trivial generators as follows:

$$\begin{cases} \mathbf{g}6 = a_m^{p-6} a_6 h_{m,0} h_{m-1,1} h_{m-2,2} h_{m-3,3} h_{m-4,4}, \\ \mathbf{g}7 = a_m^{p-5} h_{6,0} h_{m-1,1} h_{m-2,2} h_{m-3,3} h_{m-4,4}, \\ \mathbf{g}8 = a_m^{p-5} h_{m,0} h_{5,1} h_{m-2,2} h_{m-3,3} h_{m-4,4}, \\ \mathbf{g}9 = a_m^{p-5} h_{m,0} h_{m-1,1} h_{4,2} h_{m-3,3} h_{m-4,4}, \\ \mathbf{g}10 = a_m^{p-5} h_{m,0} h_{m-1,1} h_{m-2,2} h_{3,3} h_{m-4,4}, \\ \mathbf{g}11 = a_m^{p-5} h_{m,0} h_{m-1,1} h_{m-2,2} h_{m-3,3} h_{2,4}, \end{cases}$$

where $M(\mathbf{g}j) = (2m+1)(p-5) + 4m - 13$ $(6 \le j \le 11)$.

Case 3 $n = 4$. Now (3.3) turns into

$$\begin{cases} \sum_{i=1}^{l} e_i = s, \quad \sum_{i=1}^{l} c_{i,0} = s+1, \quad \sum_{i=1}^{l} c_{i,1} = s+2, \\ \sum_{i=1}^{l} c_{i,2} = s+3, \quad \sum_{i=1}^{l} c_{i,3} = s+4, \quad \sum_{i=1}^{l} c_{i,4} = 1, \\ \sum_{i=1}^{l} c_{i,5} = \lambda_5 p \ (\lambda_5 \ge 0); \\ \cdots \\ \sum_{i=1}^{l} c_{i,m-1} + \lambda_{m-2} = 0 + \lambda_{m-1} p \ (\lambda_{m-1} \ge 0); \\ \sum_{i=1}^{l} c_{i,m} + \lambda_{m-1} = 1. \end{cases} \tag{3.5}$$

We easily get $(\lambda_5, \lambda_6, \cdots, \lambda_{m-1}) = (0, 0, \cdots, 0)$, which is equivalent to $\sum_{i=1}^{l} c_{i,t} = 0$ $(5 \leq t \leq m-1)$ and $\sum_{i=1}^{l} c_{i,m} = 1$. Thus, there must exist an element of the form $\omega_l = h_{1,m}$ or $b_{1,m-1}$ in g. If $\omega_l = b_{1,m-1}$, we have $\sum_{i=1}^{l-1} c_{i,3} = s+4 \leq l-1$ which implies that $l = s+5$. So we get

$$\dim g = \sum_{i=1}^{l-1} \dim \omega_i + \dim b_{1,m-1} \geq l-1+2 = s+6,$$

which contradicts (3.1). Then up to sign the generator g must be of the form $(x_1x_2\cdots x_{p-5})(y_1y_2y_3y_4h_{1,m})$ as in (2.7). A direct computation gives five non-trivial generators as follows:

$$\begin{cases} \mathbf{g}1 = a_5 a_4^{s-1} h_{4,0} h_{3,1} h_{2,2} h_{1,3} h_{1,m}, \\ \mathbf{g}2 = a_4^s h_{5,0} h_{3,1} h_{2,2} h_{1,3} h_{1,m}, \\ \mathbf{g}3 = a_4^s h_{4,0} h_{4,1} h_{2,2} h_{1,3} h_{1,m}, \\ \mathbf{g}4 = a_4^s h_{4,0} h_{3,1} h_{3,2} h_{1,3} h_{1,m}, \\ \mathbf{g}5 = a_4^s h_{4,0} h_{3,1} h_{2,2} h_{2,3} h_{1,m}, \end{cases}$$

where $M(\mathbf{g}i) = 9s+19$ for $1 \leq i \leq 5$.

Combining Cases 1-3 gives the desired result. □

The same method leads us to the case when $s = 0$.

Lemma 3.3 — Let $p \geq 7$, $m \geq n+2 \geq 6$. Then the May E_1-term satisfies

$$E_1^{5,t(0,n,m),*} = \begin{cases} \bar{M} & n = 4, \\ 0 & n \geq 5. \end{cases}$$

Here, $t(0,n,m) = (1+2p+3p^2+4p^3+p^n+p^m)q$, $\bar{M}$ is the Z/p-module generated by the following four elements:

$$\begin{cases} \mathbf{g}\bar{2} = h_{5,0}h_{3,1}h_{2,2}h_{1,3}h_{1,m}, \\ \mathbf{g}\bar{3} = h_{4,0}h_{4,1}h_{2,2}h_{1,3}h_{1,m}, \\ \mathbf{g}\bar{4} = h_{4,0}h_{3,1}h_{3,2}h_{1,3}h_{1,m}, \\ \mathbf{g}\bar{5} = h_{4,0}h_{3,1}h_{2,2}h_{2,3}h_{1,m}, \end{cases}$$

where $M(\mathbf{g}\bar{i}) = 19$ $(2 \le i \le 5)$.

Now we are in a position to show Theorem 1.1.

PROOF [Proof of Theorem 1.1] : The proof of Theorem 1.1 is divided into the following two cases.

Case 1 : $s > 0$.

(1) It is known that $h_{1,n} \in E_1^{1,p^n q,*}$ is a permanent cocycle and represents $h_n \in \mathrm{Ext}_A^{1,p^n q}(Z/p, Z/p)$ in the May spectral sequence for $n \ge 0$. From Lemma 3.1, $\tilde{\delta}_{s+4}$ is represented by $a_4^s h_{4,0}h_{3,1}h_{2,2}h_{1,3} \in E_1^{s+4,t_1(s)+s,*}$ in the May spectral sequence. So, we get that $h_{1,n}h_{1,m}a_4^s h_{4,0}h_{3,1}h_{2,2}h_{1,3} \in E_1^{s+6,t(s,n,m)+s,9s+18}$ is a permanent cocycle in the May spectral sequence and represents $h_n h_m \tilde{\delta}_{s+4} \in \mathrm{Ext}_A^{s+6,t(s,n,m)+s}(Z/p, Z/p)$.

(i) $0 < s < p-5$ or $n > 5$. From Lemma 3.2, we have that the May E_1-term

$$E_1^{s+5,t(s,n,m)+s,*} = 0,$$

which implies

$$E_r^{s+5,t(s,n,m)+s,*} = 0$$

for $r \ge 1$. Consequently, the permanent cocycle $h_{1,n}h_{1,m}a_4^s h_{4,0}h_{3,1}h_{2,2}h_{1,3}$ cannot be hit by any May differential in the May spectral sequence. Thus in this case, we have that

$$h_n h_m \tilde{\delta}_{s+4} \ne 0$$

in the Adams spectral sequence.

(ii) $s = p-5$ and $n = 5$. By Lemma 3.2, we have

$$E_1^{s+5,t(s,n,m)+s,*} = E_1^{p,t(p-5,n,m),*} = Z/p\{\mathbf{g6}, \cdots, \mathbf{g11}\}.$$

Note that

$$M(\mathbf{g}i) = (2m+1)(p-5) + 4m - 13$$

and

$$M(h_{1,n}h_{1,m}a_4^{p-5}h_{4,0}h_{3,1}h_{2,2}h_{1,3}) = 9(p-5)+18.$$

By the reason of May filtration, we have that $h_{1,n}h_{1,m}a_4^{p-5}h_{4,0}h_{3,1}h_{2,2}h_{1,3}$ is not in $d_1(E_1^{p,t(p-5,n,m)+p-5,(2m+1)(p-5)+4m-13})$. At the same time, one can have the first May differentials of the generators as follows:

$$\begin{cases}
d_1(\mathbf{g6}) & = d_1(a_m^{p-6}a_6h_{m,0}h_{m-1,1}h_{m-2,2}h_{m-3,3}h_{m-4,4}) \\
& = a_m^{p-6}a_6h_{m,0}h_{m-1,1}h_{m-2,2}h_{m-3,3}d_1(h_{m-4,4}) + \cdots \\
& = -a_m^{p-6}a_6h_{m,0}h_{m-1,1}h_{m-2,2}h_{m-3,3}h_{1,4}h_{m-5,5} + \cdots \\
& \neq 0, \\
d_1(\mathbf{g7}) & = d_1(a_m^{p-5}h_{6,0}h_{m-1,1}h_{m-2,2}h_{m-3,3}h_{m-4,4}) \\
& = a_m^{p-5}h_{6,0}h_{m-1,1}h_{m-2,2}h_{m-3,3}d_1(h_{m-4,4}) + \cdots \\
& = -a_m^{p-5}h_{6,0}h_{m-1,1}h_{m-2,2}h_{m-3,3}h_{1,4}h_{m-5,5} + \cdots \\
& \neq 0, \\
d_1(\mathbf{g8}) & = d_1(a_m^{p-5}h_{m,0}h_{5,1}h_{m-2,2}h_{m-3,3}h_{m-4,4}) \\
& = a_m^{p-5}h_{m,0}h_{5,1}h_{m-2,2}h_{m-3,3}d_1(h_{m-4,4}) + \cdots \\
& = -a_m^{p-5}h_{m,0}h_{5,1}h_{m-2,2}h_{m-3,3}h_{1,4}h_{m-5,5} + \cdots \\
& \neq 0, \\
d_1(\mathbf{g9}) & = d_1(a_m^{p-5}h_{m,0}h_{m-1,1}h_{4,2}h_{m-3,3}h_{m-4,4}) \\
& = a_m^{p-5}h_{m,0}h_{m-1,1}h_{4,2}h_{m-3,3}d_1(h_{m-4,4}) + \cdots \\
& = -a_m^{p-5}h_{m,0}h_{m-1,1}h_{4,2}h_{m-3,3}h_{1,4}h_{m-5,5} + \cdots \\
& \neq 0, \\
d_1(\mathbf{g10}) & = d_1(a_m^{p-5}h_{m,0}h_{m-1,1}h_{m-2,2}h_{3,3}h_{m-4,4}) \\
& = a_m^{p-5}h_{m,0}h_{m-1,1}h_{m-2,2}h_{3,3}d_1(h_{m-4,4}) + \cdots \\
& = -a_m^{p-5}h_{m,0}h_{m-1,1}h_{m-2,2}h_{3,3}h_{1,4}h_{m-5,5} + \cdots \\
& \neq 0, \\
d_1(\mathbf{g11}) & = d_1(a_m^{p-5}h_{m,0}h_{m-1,1}h_{m-2,2}h_{m-3,3}h_{2,4}) \\
& = a_m^{p-5}h_{m,0}h_{m-1,1}h_{m-2,2}h_{m-3,3}d_1(h_{2,4}) + \cdots \\
& = -a_m^{p-5}h_{m,0}h_{m-1,1}h_{m-2,2}h_{m-3,3}h_{1,4}h_{1,5} + \cdots \\
& \neq 0.
\end{cases}$$

One can check that the first May differential of each of the six generators contains at least a term which is not in the first May differential of the other generators. Thus, the six May differentials above are linearly independent, showing

$$E_2^{p,t(p-5,n,m),(2m+1)(p-5)+4m-13} = 0.$$

Then we have that for $r > 2$,

$$E_r^{p,t(p-5,n,m),(2m+1)(p-5)+4m-13} = 0.$$

Thus, $h_{1,n}h_{1,m}a_4^{p-5}h_{4,0}h_{3,1}h_{2,2}h_{1,3}$ is not in $d_r(E_r^{p,t(p-5,n,m)+p-5,(2m+1)(p-5)+4m-13})$ for $r \geq 1$. Thus $h_{1,n}h_{1,m}a_4^{p-5}h_{4,0}h_{3,1}h_{2,2}h_{1,3}$ cannot be hit by any May differential, showing that

$$h_n h_m \tilde{\delta}_{p-1} \neq 0$$

in the Adams spectral sequence.

This completes the proof of Theorem 1.1 (1).

(2) Now we prove $h_4 h_m \tilde{\delta}_{s+4} = 0$. It suffices to prove $h_{1,4}h_{1,m}a_4^s h_{4,0}h_{3,1}h_{2,2}h_{1,3} \in E_1^{s+6,t(s,4,m)+s,9s+18}$ which represents $h_4 h_m \tilde{\delta}_{s+4} \in \mathrm{Ext}_A^{s+6,t(s,4,m)+s}(Z/p, Z/p)$ in the May spectral sequence is in $d_1(E_1^{s+5,t(s,4,m)+s,9s+19})$.

By Lemma 3.2,

$$E_1^{s+5,t(s,4,m)+s,9s+19} = Z/p\{\mathbf{g1}, \cdots, \mathbf{g5}\}.$$

By (2.3), we compute the first May differential of **gi** $(1 \leq i \leq 5)$ as follows:

$$\begin{cases}
d_1(\mathbf{g1}) = & (-1)^s(\underline{a_4^{s-1}a_0h_{5,0}h_{4,0}h_{3,1}h_{2,2}h_{1,3}h_{1,m}}_1 - \underline{a_4^{s-1}a_1h_{4,0}h_{4,1}h_{3,1}h_{2,2}h_{1,3}h_{1,m}}_2 \\
& +\underline{a_4^{s-1}a_2h_{4,0}h_{3,1}h_{3,2}h_{2,2}h_{1,3}h_{1,m}}_3 - \underline{a_4^{s-1}a_3h_{4,0}h_{3,1}h_{2,2}h_{2,3}h_{1,3}h_{1,m}}_4 \\
& +\underline{a_4^{s}h_{4,0}h_{3,1}h_{2,2}h_{1,3}h_{1,4}h_{1,m}}_5), \\
d_1(\mathbf{g2}) = & (-1)^s(-s\underline{a_4^{s-1}a_0h_{5,0}h_{4,0}h_{3,1}h_{2,2}h_{1,3}h_{1,m}}_1 - \underline{a_4^{s}h_{1,0}h_{4,1}h_{3,1}h_{2,2}h_{1,3}h_{1,m}}_6 \\
& +\underline{a_4^{s}h_{2,0}h_{3,1}h_{3,2}h_{2,2}h_{1,3}h_{1,m}}_7 - \underline{a_4^{s}h_{3,0}h_{3,1}h_{2,2}h_{2,3}h_{1,3}h_{1,m}}_8 \\
& +\underline{a_4^{s}h_{4,0}h_{3,1}h_{2,2}h_{1,3}h_{1,4}h_{1,m}}_5), \\
d_1(\mathbf{g3}) = & (-1)^s(s\underline{a_4^{s-1}a_1h_{4,0}h_{4,1}h_{3,1}h_{2,2}h_{1,3}h_{1,m}}_2 + \underline{a_4^{s}h_{1,0}h_{4,1}h_{3,1}h_{2,22}h_{1,3}h_{1,m}}_6 \\
& +\underline{a_4^{s}h_{4,0}h_{1,1}h_{3,2}h_{2,2}h_{1,3}h_{1,m}}_9 - \underline{a_4^{s}h_{4,0}h_{2,1}h_{2,2}h_{2,3}h_{1,3}h_{1,m}}_{10} \\
& +\underline{a_4^{s}h_{4,0}h_{3,1}h_{2,2}h_{1,3}h_{1,4}h_{1,m}}_5), \\
d_1(\mathbf{g4}) = & (-1)^s(-s\underline{a_4^{s-1}a_2h_{4,0}h_{3,1}h_{3,2}h_{2,2}h_{1,3}h_{1,m}}_3 - \underline{a_4^{s}h_{2,0}h_{3,1}h_{3,2}h_{2,2}h_{1,3}h_{1,m}}_7 \\
& -\underline{a_4^{s}h_{4,0}h_{1,1}h_{3,2}h_{2,2}h_{1,3}h_{1,m}}_9 - \underline{a_4^{s}h_{4,0}h_{3,1}h_{1,2}h_{2,3}h_{1,3}h_{1,m}}_{11} \\
& +\underline{a_4^{s}h_{4,0}h_{3,1}h_{2,2}h_{1,3}h_{1,4}h_{1,m}}_5), \\
d_1(\mathbf{g5}) = & (-1)^s(s\underline{a_4^{s-1}a_3h_{4,0}h_{3,1}h_{2,2}h_{2,3}h_{1,3}h_{1,m}}_4 + \underline{a_4^{s}h_{3,0}h_{3,1}h_{2,2}h_{2,3}h_{1,3}h_{1,m}}_8 \\
& +\underline{a_4^{s}h_{4,0}h_{2,1}h_{2,2}h_{2,3}h_{1,3}h_{1,m}}_{10} + \underline{a_4^{s}h_{4,0}h_{3,1}h_{1,2}h_{2,3}h_{1,3}h_{1,m}}_{11} \\
& +\underline{a_4^{s}h_{4,0}h_{3,1}h_{2,2}h_{1,3}h_{1,4}h_{1,m}}_5).
\end{cases}$$

Without generality, we let s be even. Then we easily get

$$\begin{pmatrix} d_1(\mathbf{g1}) \\ d_1(\mathbf{g2}) \\ d_1(\mathbf{g3}) \\ d_1(\mathbf{g4}) \\ d_1(\mathbf{g5}) \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -s & 0 & 0 & 0 & 1 & -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & s & 0 & 0 & 1 & 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & -s & 0 & 1 & 0 & -1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & s & 1 & 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \text{—1} \\ \text{—2} \\ \text{—3} \\ \text{—4} \\ \text{—5} \\ \text{—6} \\ \text{—7} \\ \text{—8} \\ \text{—9} \\ \text{—10} \\ \text{—11} \end{pmatrix}.$$

By direct computation, we can get

$$\underline{a_4^s h_{4,0}h_{3,1}h_{2,2}h_{1,3}h_{1,4}h_{1,m}}_5 = (s+4)^{-1}(sd_1(\mathbf{g}1) + d_1(\mathbf{g}2) + d_1(\mathbf{g}3) + d_1(\mathbf{g}4) + d_1(\mathbf{g}5)).$$

So $h_{1,4}h_{1,m}a_4^s h_{4,0}h_{3,1}h_{2,2}h_{1,3}$ is in $d_1(E_1^{s+5,t(s,4,m)+s,9s+19})$, showing that

$$h_4 h_m \tilde{\delta}_{s+4} = 0$$

in the Adams spectral sequence.

This finishes the proof of Theorem 1.1 when $s > 0$.

Case 2 : $s = 0$. In this case, $h_{1,n}h_{1,m}h_{4,0}h_{3,1}h_{2,2}h_{1,3} \in E_1^{6,t(0,n,m),18}$ is a permanent cocycle in the May spectral sequence and represents $h_n h_m \tilde{\delta}_4 \in \mathrm{Ext}_A^{6,t(0,n,m)}(Z/p, Z/p)$.

(1) The May E_1-term $E_1^{5,t(0,n,m),*} = 0$ from Lemma 3.3, which implies that

$$E_r^{5,t(0,n,m),*} = 0$$

for $r \geq 1$. Consequently, the permanent cocycle $h_{1,n}h_{1,m}h_{4,0}h_{3,1}h_{2,2}h_{1,3}$ cannot be hit by any May differential in the May spectral sequence. Thus in this case,

$$h_n h_m \tilde{\delta}_4 \neq 0$$

in the Adams spectral sequence.

(2) Now we prove that $h_4 h_m \tilde{\delta}_4 = 0$. It suffices to prove $h_{1,4}h_{1,m}h_{4,0}h_{3,1}h_{2,2}h_{1,3} \in E_1^{6,t(0,4,m),18}$ which represents $h_4 h_m \tilde{\delta}_4 \in \mathrm{Ext}_A^{6,t(0,4,m)}(Z/p, Z/p)$ is in $d_1(E_1^{5,t(0,4,m),19})$. By Lemma 3.3 we get that

$$E_1^{5,t(0,4,m),19} = Z/p\{\mathbf{g}\bar{i} \mid 2 \le i \le 5\}.$$

By (2.3), we compute the first May differential of $\mathbf{g}\bar{i}$ $(2 \le i \le 5)$ by using the same method as above, and similarly obtain the following equalities:

$$\begin{cases} d_1(\mathbf{g}\bar{2}) = & -\underline{h_{1,0}h_{4,1}h_{3,1}h_{2,2}h_{1,3}h_{1,m}}_6 + \underline{h_{2,0}h_{3,1}h_{3,2}h_{2,2}h_{1,3}h_{1,m}}_7 \\ & -\underline{h_{3,0}h_{3,1}h_{2,2}h_{2,3}h_{1,3}h_{1,m}}_8 + \underline{h_{4,0}h_{3,1}h_{2,2}h_{1,3}h_{1,4}h_{1,m}}_5, \\ d_1(\mathbf{g}\bar{3}) = & \underline{h_{1,0}h_{4,1}h_{3,1}h_{2,22}h_{1,3}h_{1,m}}_6 + \underline{h_{4,0}h_{1,1}h_{3,2}h_{2,2}h_{1,3}h_{1,m}}_9 \\ & -\underline{h_{4,0}h_{2,1}h_{2,2}h_{2,3}h_{1,3}h_{1,m}}_{10} + \underline{h_{4,0}h_{3,1}h_{2,2}h_{1,3}h_{1,4}h_{1,m}}_5, \\ d_1(\mathbf{g}\bar{4}) = & -\underline{h_{2,0}h_{3,1}h_{3,2}h_{2,2}h_{1,3}h_{1,m}}_7 - \underline{h_{4,0}h_{1,1}h_{3,2}h_{2,2}h_{1,3}h_{1,m}}_9 \\ & -\underline{h_{4,0}h_{3,1}h_{1,2}h_{2,3}h_{1,3}h_{1,m}}_{11} + \underline{h_{4,0}h_{3,1}h_{2,2}h_{1,3}h_{1,4}h_{1,m}}_5, \\ d_1(\mathbf{g}\bar{5}) = & \underline{h_{3,0}h_{3,1}h_{2,2}h_{2,3}h_{1,3}h_{1,m}}_8 + \underline{h_{4,0}h_{2,1}h_{2,2}h_{2,3}h_{1,3}h_{1,m}}_{10} \\ & +\underline{h_{4,0}h_{3,1}h_{1,2}h_{2,3}h_{1,3}h_{1,m}}_{11} + \underline{h_{4,0}h_{3,1}h_{2,2}h_{1,3}h_{1,4}h_{1,m}}_5. \end{cases}$$

Then we easily get

$$\begin{pmatrix} d_1(\mathbf{g}\bar{2}) \\ d_1(\mathbf{g}\bar{3}) \\ d_1(\mathbf{g}\bar{4}) \\ d_1(\mathbf{g}\bar{5}) \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 & -1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & -1 & 0 \\ 1 & 0 & -1 & 0 & -1 & 0 & -1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \text{—}_5 \\ \text{—}_6 \\ \text{—}_7 \\ \text{—}_8 \\ \text{—}_9 \\ \text{—}_{10} \\ \text{—}_{11} \end{pmatrix}.$$

By a direct computation, we can get

$$\underline{h_{4,0}h_{3,1}h_{2,2}h_{1,3}h_{1,4}h_{1,m}}_5 = -4^{-1}(d_1(\mathbf{g}\bar{2}) + d_1(\mathbf{g}\bar{3}) + d_1(\mathbf{g}\bar{4}) + d_1(\mathbf{g}\bar{5})).$$

So $h_{1,4}h_{1,m}h_{4,0}h_{3,1}h_{2,2}h_{1,3}$ is in $d_1(E_1^{5,t(0,4,m),19})$, showing that

$$h_4 h_m \tilde{\delta}_4 = 0$$

in the Adams spectral sequence.

From Cases 1 and 2, Theorem 1.1 follows.

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