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HYPO-EP OPERATORS[1]

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In this paper we study closed range operators on a Hilbert space such that the range is contained in the range of its adjoint. Some results pertaining to these operators and operator matrices are discussed.

Key words : Hypo-EP operator; EP operator; operator matrices.

1. INTRODUCTION

Let H be a separable Hilbert space and $BL(H)$ be the space of all bounded linear operators from H to H. We say that $A \in BL(H)$ is hypo-EP if the range $\mathcal{R}(A)$ of A is closed, and $\mathcal{R}(A) \subset \mathcal{R}(A^*)$ equivalently $\mathcal{N}(A) \subset \mathcal{N}(A^*)$, where A^* denotes the adjoint of A.

For $A \in BL(H)$ with $\mathcal{R}(A)$ closed, there is a unique $A^\dagger \in BL(H)$ such that

$$AA^\dagger A = A,\ A^\dagger AA^\dagger = A^\dagger,\ (AA^\dagger)^* = AA^\dagger,\ (A^\dagger A)^* = A^\dagger A.$$

The operator $A^\dagger$ is known as the Moore-Penrose inverse of A [11]. Infact $AA^\dagger$ is the orthogonal projection onto $\mathcal{R}(A)$ and $A^\dagger A$ is the orthogonal projection onto $\mathcal{R}(A^*)$.

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An operator $A \in BL(H)$ is called an EP operator if $A^\dagger A = AA^\dagger$ [11]. In fact $A \in BL(H)$ is EP if and only if $\mathcal{R}(A)$ is closed and $\mathcal{R}(A) = \mathcal{R}(A^*)$. It is well known that EP stands for equiprojection. An operator $A \in BL(H)$ is normal if $A^*A = AA^*$ and hyponormal if $||A^*x|| \leq ||Ax||$ for each $x \in H$.

It is observed in [9], that A is hypo-EP if and only if $||A^\dagger Ax|| \geq ||AA^\dagger x||$ for each $x \in H$. It is also proved that A is hypo-EP if and only if $A^\dagger A^2 A^\dagger = AA^\dagger$. It is observed in ([10], Corollary 2.5), that for closed range operators A and B, AB is a closed ranged operator if and only if $\mathcal{N}(A) + \mathcal{R}(B)$ is closed. We note that the unilateral right shift operator on the Hilbert space l^2 is hypo-EP but it is not EP. For $A \in BL(H)$, it is easy to observe the following:

(1) Both A and A^* are hypo-EP if and only if A is EP.

(2) A hypo-EP operator on a finite dimensional Hilbert space is EP.

(3) A hyponormal operator with closed range is hypo-EP.

Infinite dimensional EP operators have been studied by several authors. Hypo-EP operators have been introduced by Itoh [9] presumably motivated by hyponormal operators. In the present paper, we discuss powers and product of hypo-EP operators. We also discuss hypo-EP operator matrices.

2. Powers and Product of Hypo-EP Operators

For a hyponormal $A \in BL(H)$, A^2 need not be hyponormal [6]; however for a hypo-EP operator, we have the following.

***Theorem* 2.1** — *If $A \in BL(H)$ is hypo-EP, then A^n is hypo-EP for each n.*

We shall need a couple of lemmas. It is well known that for a hyponormal operator $A \in BL(H)$, $\mathcal{N}(A) = \mathcal{N}(A^n)$ for each n, where $\mathcal{N}(A) = \{x \in H : Ax = 0\}$ is the null space of A. The following is a hypo-EP analogue of this.

Lemma 2.2 — If $A \in BL(H)$ is hypo-EP, then $\mathcal{N}(A) = \mathcal{N}(A^n)$ for each n.

Proof : We prove this result by induction on n. First we prove it for $n = 2$. Clearly, $\mathcal{N}(A) \subset \mathcal{N}(A^2)$. Let $x \in \mathcal{N}(A^2)$. Then $Ax \in \mathcal{N}(A)$. Since A is hypo-EP, $A^*Ax = 0$. So, $||Ax||^2 = \langle A^*Ax,\ x\rangle = 0$. Thus $x \in \mathcal{N}(A)$. Hence $\mathcal{N}(A) = \mathcal{N}(A^2)$. Now suppose that the result holds for $n = k$. Let $x \in \mathcal{N}(A^{k+1})$. Then $Ax \in \mathcal{N}(A^k) = \mathcal{N}(A) \subset \mathcal{N}(A^*)$. Thus $A^*Ax = 0$, which implies that $Ax = 0$. So, $x \in \mathcal{N}(A)$. Thus $\mathcal{N}(A^{k+1}) \subset \mathcal{N}(A)$. Hence $\mathcal{N}(A) = \mathcal{N}(A^{k+1})$. □

The converse of above lemma need not be true as is seen from the following example. Consider $A : \mathbb{C} \times \mathbb{C} \to \mathbb{C} \times \mathbb{C}$ define by $A(x,\ y) = (x,\ ix)$. Then $A^*(x,\ y) = (x - iy,\ 0)$. Since A is idempotent, $\mathcal{N}(A) = \mathcal{N}(A^n)$ for each n. Also $\mathcal{N}(A) = \{(0,\ y)/y \in \mathbb{C}\}$ and $\mathcal{N}(A^*) = \{(x,\ y)/x = iy,\ x,\ y \in \mathbb{C}\}$. Thus A is not hypo-EP.

We note that the right unilateral shift operator A on l^2 is hypo-EP but $\mathcal{R}(A) \neq \mathcal{R}(A^n)$ for any $n > 1$.

Corollary 2.3 — If $A \in BL(H)$ is hypo-EP and nilpotent, then $A = 0$.

Lemma 2.4 — If $A \in BL(H)$ is hypo-EP, then $\mathcal{R}(A^n)$ is closed for each n.

PROOF : First we prove that $\mathcal{N}(A)^{\perp}$ is invariant under A^j for each j. Let $x \in \mathcal{N}(A)$. Since A is hypo-EP, $x \in \mathcal{N}(A^*)$ and so $A^*x \in \mathcal{N}(A)$. Thus $A^*(\mathcal{N}(A)) \subset \mathcal{N}(A)$. Hence $A(\mathcal{N}(A)^{\perp}) \subset \mathcal{N}(A)^{\perp}$. Thus $A^j(\mathcal{N}(A)^{\perp}) \subset \mathcal{N}(A)^{\perp}$. To see that $\mathcal{R}(A^n)$ is closed, let $x \in \mathcal{N}(A^n)^{\perp}$. Since A is hypo-EP, by Lemma 2.2, $\mathcal{N}(A) = \mathcal{N}(A^j)$ for each j. Thus $x \in \mathcal{N}(A)^{\perp}$. So $A^jx \in \mathcal{N}(A)^{\perp}$ for each j. In particular $A^{n-1}x \in \mathcal{N}(A)^{\perp}$. Since $\mathcal{R}(A)$ is closed, there exists $\alpha \geq 0$ such that $||Ay|| \geq \alpha||y||$ for each $y \in \mathcal{N}(A)^{\perp}$. So $||A(A^{n-1}x)|| \geq \alpha||A^{n-1}x||$. Now again using the fact that $A^{n-2}x \in \mathcal{N}(A)^{\perp}$, we get $||A^nx|| \geq \alpha^2||A^{n-2}x||$. Continuing this process we get $||A^nx|| \geq \alpha^n||x||$ for $x \in \mathcal{N}(A^n)^{\perp}$. Hence $\mathcal{R}(A^n)$ is closed. □

PROOF OF THEOREM 2.1 : By Lemma 2.4, $\mathcal{R}(A^n)$ is closed for each n. So it is enough to prove that $\mathcal{N}(A^n) \subset \mathcal{N}(A^{*n})$. Now by Lemma 2.2, $\mathcal{N}(A) = \mathcal{N}(A^n)$ for each n. Thus $\mathcal{N}(A^n) = \mathcal{N}(A) \subset \mathcal{N}(A^*) \subset \mathcal{N}(A^{*n})$. Hence $\mathcal{N}(A^n) \subset \mathcal{N}(A^{*n})$. Thus A^n is hypo-EP. □

Hartwig [8] and Basket and Katz [1], discussed the product of two EP operators on finite dimensional spaces. Necessary and sufficient conditions for the product of two EP operators to be an EP operator have been discussed by Djordjevic [4]. It is shown that the product of two commuting EP operators is EP. We discuss the case of hypo-EP operators.

***Theorem* 2.5** — *Let $A,\ B \in BL(H)$ be hypo-EP operators.*

(a) If $\mathcal{R}(AB)$ is closed, $\mathcal{R}(AB) \subset \mathcal{R}(A) \cap \mathcal{R}(B^*)$ and $\mathcal{N}(AB) \subset \overline{\mathcal{N}(A) + \mathcal{N}(B)}$, then AB is hypo-EP.

(b) If A is injective and $\mathcal{R}(AB) \subset \mathcal{R}(A) \cap \mathcal{R}(B^*)$, then AB is hypo-EP.

(c) If AB is hypo-EP, then $\mathcal{R}(AB) \subset \mathcal{R}(A) \cap \mathcal{R}(B^*)$.

PROOF : (a) Since $\mathcal{R}(AB)$ is closed, it is enough to prove that $\mathcal{R}(AB) \subset \mathcal{R}((AB)^*)$.

$$\begin{aligned} \text{Now, } \mathcal{R}(AB) &\subset \mathcal{R}(A) \cap \mathcal{R}(B^*) = (\mathcal{R}(A) \cap \mathcal{R}(B^*))^{\perp\perp} \\ &= (\overline{\mathcal{R}(A)^{\perp} + \mathcal{R}(B^*)^{\perp}})^{\perp} = (\overline{\mathcal{N}(A^*) + \mathcal{N}(B)})^{\perp} \\ &\subset (\overline{\mathcal{N}(A) + \mathcal{N}(B)})^{\perp} \subset \overline{\mathcal{R}((AB)^*)} = \mathcal{R}((AB)^*). \end{aligned}$$ Thus AB is hypo-EP.

(b) Suppose A is injective and $\mathcal{R}(AB) \subset \mathcal{R}(A) \cap \mathcal{R}(B^*)$. As A is injective and B is hypo-EP, $\mathcal{N}(A) + \mathcal{R}(B) = \mathcal{R}(B)$ is closed. Hence $\mathcal{R}(AB)$ is closed. Since $\mathcal{N}(AB) = \mathcal{N}(B)$, by part (a), AB is hypo-EP.

(c) Suppose AB is a hypo-EP operator. Since A is hypo-EP, $\mathcal{R}(A^*)$ is closed. Thus $H = \mathcal{R}(A^*) \oplus \mathcal{N}(A)$. Consider the following decomposition of A:

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{pmatrix} \to \begin{pmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{pmatrix}$$

Since B is hypo-EP, $\mathcal{R}(B^*)$ is closed. Thus $H = \mathcal{R}(B^*) \oplus \mathcal{N}(B)$. Let $x \in H$. Then $x = x_1 + x_2$, for some $x_1 \in \mathcal{R}(B^*),\ x_2 \in \mathcal{N}(B)$. Since $Bx \in H = \mathcal{R}(A^*) \oplus \mathcal{N}(A)$, $Bx = y_1 + y_2$, for some $y_1 \in \mathcal{R}(A^*),\ y_2 \in \mathcal{N}(A)$. Define $B_1 : \mathcal{R}(B^*) \to \mathcal{R}(A^*)$ and $B_2 : \mathcal{R}(B^*) \to \mathcal{N}(A)$ by $B_1(x_1) = y_1$ and $B_2(x_1) = y_2$. Clearly $B_1,\ B_2$ are well defined. Thus we have following decomposition for B

$$B = \begin{pmatrix} B_1 & 0 \\ B_2 & 0 \end{pmatrix} : \begin{pmatrix} \mathcal{R}(B^*) \\ \mathcal{N}(B) \end{pmatrix} \to \begin{pmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{pmatrix}.$$

Since $B_1 : \mathcal{R}(B^*) \to \mathcal{R}(A^*)$, $\mathcal{R}(B^*) = \mathcal{N}(B_1) \oplus (\mathcal{N}(B_1)^{\perp} \cap \mathcal{R}(B^*))$.

Now $AB = \begin{pmatrix} A_1B_1 & 0 \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} \mathcal{R}(B^*) \\ \mathcal{N}(B) \end{pmatrix} \to \begin{pmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{pmatrix}$.

Clearlly $\mathcal{N}(B_1) \oplus \mathcal{N}(B) \subset \mathcal{N}(AB)$. Since A_1 is injective, $\mathcal{N}(AB) \subset \mathcal{N}(B_1) \oplus \mathcal{N}(B)$. Thus $\mathcal{N}(AB) = \mathcal{N}(B) \oplus \mathcal{N}(B_1)$. Therefore $\mathcal{N}(AB)^{\perp} = (\mathcal{N}(B_1) \oplus \mathcal{N}(B))^{\perp} = \mathcal{N}(B_1)^{\perp} \cap \mathcal{N}(B)^{\perp} = \mathcal{N}(B_1)^{\perp} \cap \mathcal{R}(B^*)$. Since AB is hypo-EP, $\mathcal{R}(AB) \subset \mathcal{R}(AB)^* = \mathcal{N}(AB)^{\perp} \subset \mathcal{R}(B^*)$. Therefore $\mathcal{R}(AB) \subset \mathcal{R}(A) \cap \mathcal{R}(B^*)$. □

Corollary 2.6 — Let $A,\ B \in BL(H)$ be hypo-EP with A injective and $AB = BA$, then AB is hypo-EP.

The following example shows that we cannot drop the condition $AB = BA$ in Corollary 2.6. We do not know whether the injectivity assumption can be omitted.

Example : Let $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Then A is invertible and B is EP. Then $AB = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$ with $\mathcal{N}(AB) = \{(x,\ -x)\ :\ x \in \mathbb{R}\}$ and $\mathcal{N}((AB)^*) = \{(-2x,\ x)\ :\ x \in \mathbb{R}\}$. Thus $\mathcal{N}(AB)$ is not contained in $\mathcal{N}((AB)^*)$. Hence AB is not hypo-EP.

***Theorem* 2.7** — *If every operator on a Hilbert space H of rank n is hypo-EP, then $dim\ H = n$.*

PROOF : Suppose that $dim\ H > n$. Then there exist orthonormal vectors $u_1,\ u_2,\ ....\ u_n$ and orthonormal vectors $w_1,\ w_2,\ ....\ w_n$ such that $Span\{u_1,\ u_2,\ .....\ u_n\} \neq Span\{w_1,\ w_2,\ ....\ w_n\}$. Define $T\ :\ H \rightarrow H$ by $Tx = \sum_{i=1}^{n} \langle x,\ u_i\rangle w_i$, $x \in H$. Thus for $x,\ y \in H$, $\langle Tx,\ y\rangle = \sum_{i=1}^{n} \langle x,\ u_i\rangle\langle w_i,\ y\rangle = \sum_{i=1}^{n} \langle x,\ \langle y,\ w_i\rangle u_i\rangle = \langle x,\ \sum_{i=1}^{n} \langle y,\ w_i\rangle u_i\rangle = \langle x,\ T^* y\rangle$. Therefore $T^*y = \sum_{i=1}^{n} \langle y,\ w_i\rangle u_i$. Thus we have $\mathcal{N}(T) = \{u_1,\ u_2,\ ...\ u_n\}^{\perp}$ and $\mathcal{N}(T^*) = \{w_1,\ w_2,\ ...\ w_n\}^{\perp}$. Since T is hypo-EP, $\{u_1,\ u_2,\ ...\ u_n\}^{\perp} \subset \{w_1,\ w_2,\ ...\ w_n\}^{\perp}$. Therefore $Span\{w_1,\ w_2,\ ...\ w_n\} \subset Span\{u_1,\ u_2,\ ...\ u_n\}$. Thus $Span\{w_1,\ w_2,\ ....\ w_n\} = Span\{u_1,\ u_2,\ ....\ u_n\}$. This contradiction gives $dim\ H = n$. □

***Theorem* 2.8** — *Let $A \in BL(H)$ be hypo-EP and $B \in BL(H)$ be unitarily equivalent to A. Then B is hypo-EP.*

PROOF : Since $\mathcal{R}(A)$ is closed and B is unitarily equivalent to A, $\mathcal{R}(B)$ is closed. Since B is unitarily equivalent to A, $B = U^*AU$ for some unitary operator U. Let $x \in \mathcal{N}(B)$. Then $U^*AUx = 0$, so that $AUx = 0$. Since A is hypo-EP, $A^*Ux = 0$. Thus $B^*x = 0$. Hence $\mathcal{N}(B) \subset \mathcal{N}(B^*)$. Therefore B is hypo-EP. □

The following example shows that in above theorem unitary equivalence cannot be replaced by similarity.

Example : Let $A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$. Take $B = C^{-1}AC$. Then A is EP and $B = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$ with $\mathcal{N}(B) = \{(0,\ 0,\ \alpha)\ :\ \alpha \in \mathbb{R}\}$ and $\mathcal{N}(B^*) = \{(0,\ \alpha,\ \alpha)\ :\ \alpha \in \mathbb{R}\}$. Thus B is not hypo-EP.

3. Hypo-EP Operator Matrices

Hartwig [7] has discussed when a block matrix $\begin{pmatrix} A & C \\ B & D \end{pmatrix}$ is EP. In this section, we discuss hypo-EP operator matrices. We generalize some results of Hartwig to infinite dimensional spaces.

***Theorem* 3.1** — *Let* $A,\ X \in BL(H)$ *and* $M = \begin{pmatrix} A & AX \\ X^*A & X^*AX \end{pmatrix}$. *Then* M *is hypo-EP if and only if* A *is hypo-EP.*

Proof : Suppose M is a hypo-EP operator. Let $x \in \mathcal{N}(A)$. Then $\begin{pmatrix} A & AX \\ X^*A & X^*AX \end{pmatrix} \begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Since M is a hypo-EP operator, $\begin{pmatrix} A^* & A^*X \\ X^*A^* & X^*A^*X \end{pmatrix} \begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Therefore $A^*x = 0$. Thus $\mathcal{N}(A) \subset \mathcal{N}(A^*)$. Next we prove that $\mathcal{R}(A)$ is closed. Suppose $A(x_n) \to u$, for some $u \in H$. Then $\begin{pmatrix} A & AX \\ X^*A & X^*AX \end{pmatrix} \begin{pmatrix} x_n \\ 0 \end{pmatrix} \to \begin{pmatrix} u \\ X^*u \end{pmatrix}$. Since $\mathcal{R}(M)$ is closed, there exist $x,\ y \in H$ such that $\begin{pmatrix} u \\ X^*u \end{pmatrix} = \begin{pmatrix} A & AX \\ X^*A & X^*AX \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} A(x+Xy) \\ X^*A(x+Xy) \end{pmatrix}$. Thus $u = A(x+Xy)$. Therefore $\mathcal{R}(A)$ is closed. Hence A is hypo-EP.

Conversely suppose A is a hypo-EP operator. Let $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{N}(M)$. Then $\begin{pmatrix} A(x+Xy) \\ X^*A(x+Xy) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Thus $x + Xy \in \mathcal{N}(A)$. Since A is a hypo-EP operator, $x + Xy \in \mathcal{N}(A^*)$. Therefore $M^* \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} A^*(x+Xy) \\ X^*A^*(x+Xy) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Thus $\mathcal{N}(M) \subset \mathcal{N}(M^*)$. To see that $\mathcal{R}(M)$ is closed, let $M \begin{pmatrix} x_n \\ y_n \end{pmatrix} \to \begin{pmatrix} u \\ v \end{pmatrix}$. Then $\begin{pmatrix} A(x_n+Xy_n) \\ X^*A(x_n+Xy_n) \end{pmatrix} \to \begin{pmatrix} u \\ v \end{pmatrix}$. Thus $A(x_n + Xy_n) \to u$ and $X^*A(x_n + Xy_n) \to v = X^*u$. Since $\mathcal{R}(A)$ is closed, $u = Ax$ and $v = X^*u = X^*Ax$. Thus $\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} A & AX \\ X^*A & X^*AX \end{pmatrix} \begin{pmatrix} x \\ 0 \end{pmatrix}$. Therefore $\mathcal{R}(M)$ is closed. Hence M is hypo-EP. □

Corollary 3.2 — Let $A,\ X \in BL(H)$ and $M = \begin{pmatrix} A & AX \\ X^*A & X^*AX \end{pmatrix}$. Then M is EP if and only if A is EP.

PROOF : Suppose M is EP. Then M^* is also EP. Therefore M and M^* are hypo-EP. Thus by Theorem 3.1, A and A^* are hypo-EP. Therefore A is EP.

Conversely suppose A is EP. Therefore A and A^* are hypo-EP. Thus by Theorem 3.1, $M = \begin{pmatrix} A & AX \\ X^*A & X^*AX \end{pmatrix}$ and $M^* = \begin{pmatrix} A^* & A^*X \\ X^*A^* & X^*A^*X \end{pmatrix}$ are hypo-EP. Hence M is EP. □

The following results are observed by Hartwig [7], for EP matrices. Next we discuss similar results for hypo-EP and EP operator matrices.

***Theorem* 3.3** — *An operator $A \in BL(H)$ is hypo-EP if and only if $M = \begin{pmatrix} A & 0 \\ A & A \end{pmatrix}$ is hypo-EP.*

PROOF : Suppose A is a hypo-EP operator. Let $\begin{pmatrix} A & 0 \\ A & A \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix} \to \begin{pmatrix} u \\ v \end{pmatrix}$. Then $Ax_n \to u$ and $Ay_n \to v - u$. Since $\mathcal{R}(A)$ is closed, there exist $x,\ y \in H$ such that $u = Ax$ and $v - u = Ay$. Thus $v = A(x + y)$. Therefore $\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} A & 0 \\ A & A \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$. Thus $\mathcal{R}(M)$ is closed. Now let $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{N}(M)$. Then $Ax = 0 = Ay$. Since A is hypo-EP, $A^*x = 0 = A^*y$. Therefore $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{N}(M^*)$. Hence M is hypo-EP.

Conversely suppose M is a hypo-EP operator. Let $Ax_n \to u$. Then $\begin{pmatrix} A & 0 \\ A & A \end{pmatrix} \begin{pmatrix} x_n \\ 0 \end{pmatrix} \to \begin{pmatrix} u \\ u \end{pmatrix}$. Since $\mathcal{R}(M)$ is closed, there exist $x,\ y \in H$ such that $\begin{pmatrix} A & 0 \\ A & A \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} u \\ u \end{pmatrix}$. Thus $Ax = u$ and $Ay = 0$. Thus $\mathcal{R}(A)$ is closed. Now let $x \in \mathcal{N}(A)$. Then $\begin{pmatrix} x \\ 0 \end{pmatrix} \in \mathcal{N}(M) \subset \mathcal{N}(M^*)$. Thus $\begin{pmatrix} A^* & A^* \\ 0 & A^* \end{pmatrix} \begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Therefore $A^*x = 0$. Thus $\mathcal{N}(A) \subset \mathcal{N}(A^*)$. Hence A is hypo-EP. □

For $A,\ C,\ D \in BL(H)$ with $\mathcal{R}(A)$ and $\mathcal{R}(D)$ closed, $\begin{pmatrix} A & C \\ 0 & D \end{pmatrix}^{\dagger} = \begin{pmatrix} A^{\dagger} & -A^{\dagger}CD^{\dagger} \\ 0 & D^{\dagger} \end{pmatrix}$ if and only if $\mathcal{N}(D) \subset \mathcal{N}(C)$ and $\mathcal{R}(C) \subset \mathcal{R}(A)$ [3].

***Theorem* 3.4** — *Let $A,\ C,\ D \in BL(H)$ with $\mathcal{N}(D) \subset \mathcal{N}(C)$ and $\mathcal{R}(C) \subset \mathcal{R}(A)$. Then $M = \begin{pmatrix} A & C \\ 0 & D \end{pmatrix}$ is hypo-EP if and only if A and D are hypo-EP.*

PROOF : Suppose M is hypo-EP. Let $x \in \mathcal{N}(A)$, then $\begin{pmatrix} x \\ 0 \end{pmatrix} \in \mathcal{N}(M)$. Thus $\begin{pmatrix} x \\ 0 \end{pmatrix} \in \mathcal{N}(M^*)$. Therefore $x \in \mathcal{N}(A^*)$. Hence $\mathcal{N}(A) \subset \mathcal{N}(A^*)$. Suppose $Ax_n \to u$, for some $u \in H$. Then $\begin{pmatrix} A & C \\ 0 & D \end{pmatrix} \begin{pmatrix} x_n \\ 0 \end{pmatrix} \to \begin{pmatrix} u \\ 0 \end{pmatrix}$. Since $\mathcal{R}(M)$ is closed, there exists $x,\ y \in H$ such that $\begin{pmatrix} u \\ 0 \end{pmatrix} = \begin{pmatrix} A & C \\ 0 & D \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$. Thus $Dy = 0$ and $Ax + Cy = u$. Since $\mathcal{N}(D) \subset \mathcal{N}(C)$, $Ax = u$. Thus $\mathcal{R}(A)$ is closed. Hence A is hypo-EP. Let $y \in \mathcal{N}(D)$. Then $y \in \mathcal{N}(C)$. Thus $\begin{pmatrix} A & C \\ 0 & D \end{pmatrix} \begin{pmatrix} 0 \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Since M is hypo-EP, $\begin{pmatrix} A^* & 0 \\ C^* & D^* \end{pmatrix} \begin{pmatrix} 0 \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Thus $y \in \mathcal{N}(D^*)$. Hence $\mathcal{N}(D) \subset \mathcal{N}(D^*)$. To Prove $\mathcal{R}(D)$ is closed, it is enough to prove that $\mathcal{R}(D^*)$ is closed. Let $D^*y_n \to v$. Then $\begin{pmatrix} A^* & 0 \\ C^* & D^* \end{pmatrix} \begin{pmatrix} 0 \\ y_n \end{pmatrix} \to \begin{pmatrix} 0 \\ v \end{pmatrix}$. Since $\mathcal{R}(M^*)$ is closed, there exist $x,\ y \in H$ such that $\begin{pmatrix} 0 \\ v \end{pmatrix} = \begin{pmatrix} A^* & 0 \\ C^* & D^* \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$. Therefore $A^*x = 0$ and $C^*x + D^*y = v$. Since $\mathcal{R}(C) \subset \mathcal{R}(A)$, $D^*y = v$. Therefore $\mathcal{R}(D^*)$ is closed.

Conversely suppose that A and D are hypo-EP. Thus $\mathcal{R}(A)$ and $\mathcal{R}(D)$ are closed. Since $\mathcal{N}(D) \subset \mathcal{N}(C)$ and $\mathcal{R}(C) \subset \mathcal{R}(A)$, $M^{\dagger} = \begin{pmatrix} A^{\dagger} & -A^{\dagger}CD^{\dagger} \\ 0 & D^{\dagger} \end{pmatrix}$. Thus $M^{\dagger}M^2M^{\dagger}$ $= \begin{pmatrix} AA^{\dagger} & -AA^{\dagger}CD^{\dagger} + A^{\dagger}ACD^{\dagger} \\ 0 & DD^{\dagger} \end{pmatrix} = \begin{pmatrix} AA^{\dagger} & 0 \\ 0 & DD^{\dagger} \end{pmatrix} = MM^{\dagger}$. Therefore M is hypo-EP. □

Corollary 3.5 — Let $A,\ D,\ X \in BL(H)$. If A and D are hypo-EP, then $\begin{pmatrix} A & AXD \\ 0 & D \end{pmatrix}$ is hypo-EP.

Corollary 3.6 — Let $A \in BL(H)$. Then A is hypo-EP if and only if $M = \begin{pmatrix} A & A \\ 0 & A \end{pmatrix}$ is hypo-EP.

***Theorem* 3.7** — *Let $A,\ C,\ D \in BL(H)$ with $\mathcal{N}(D) \subset \mathcal{N}(C)$ and $\mathcal{R}(C) \subset \mathcal{R}(A)$. Then $M = \begin{pmatrix} A & C \\ 0 & D \end{pmatrix}$ is EP if and only if A and D are EP.*

PROOF : Suppose M is EP. In view of Theorem 3.4, to prove A and D are EP, it is enough to prove $\mathcal{N}(A^*) \subset \mathcal{N}(A)$ and $\mathcal{N}(D^*) \subset \mathcal{N}(D)$. Let $x \in \mathcal{N}(A^*)$. Since $\mathcal{R}(C) \subset \mathcal{R}(A)$, $x \in \mathcal{N}(C^*)$. Therefore $\begin{pmatrix} A^* & 0 \\ C^* & D^* \end{pmatrix} \begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Since M is EP, $\begin{pmatrix} A & C \\ 0 & D \end{pmatrix} \begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Thus $Ax = 0$. Therefore $\mathcal{N}(A^*) \subset \mathcal{N}(A)$. Let $y \in \mathcal{N}(D^*)$. Then $\begin{pmatrix} A^* & 0 \\ C^* & D^* \end{pmatrix} \begin{pmatrix} 0 \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Since M is EP, $\begin{pmatrix} A & C \\ 0 & D \end{pmatrix} \begin{pmatrix} 0 \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Thus $Dy = 0$. Therefore $\mathcal{N}(D^*) \subset \mathcal{N}(D)$.

Conversely suppose that A and D are EP operators. Thus $\mathcal{R}(A)$ and $\mathcal{R}(D)$ are closed. Therefore under the assumption of hypothesis we have $M^\dagger = \begin{pmatrix} A^\dagger & -A^\dagger C D^\dagger \\ 0 & D^\dagger \end{pmatrix}$. Thus $M^\dagger M - M M^\dagger = \begin{pmatrix} A^\dagger A - A A^\dagger & 0 \\ 0 & D^\dagger D - D D^\dagger \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Therefore M is EP. □

We study similar results to above for another type of operator matrices in the following.

***Theorem* 3.8** — *Let $A,\ B,\ C \in BL(H)$ with $\mathcal{N}(B) = \mathcal{N}(C) \subset \mathcal{N}(A)$ and $\mathcal{R}(A) \subset \mathcal{R}(C)$. Then $M = \begin{pmatrix} A & C \\ B & 0 \end{pmatrix}$ is hypo-EP if and only if C and B are hypo-EP.*

PROOF : Suppose M is hypo-EP. Let $x \in \mathcal{N}(C)$. Since $\mathcal{N}(B) = \mathcal{N}(C) \subset \mathcal{N}(A)$, $x \in \mathcal{N}(A)$ and $x \in \mathcal{N}(B)$. Therefore $\begin{pmatrix} A & C \\ B & 0 \end{pmatrix} \begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Since M is hypo-EP, $\begin{pmatrix} A^* & B^* \\ C^* & 0 \end{pmatrix} \begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Thus $x \in \mathcal{N}(C^*)$. Hence $\mathcal{N}(C) \subset \mathcal{N}(C^*)$. Suppose $Cx_n \to u$, for some $u \in H$.

Thus $\begin{pmatrix} A & C \\ B & 0 \end{pmatrix}\begin{pmatrix} 0 \\ x_n \end{pmatrix} \to \begin{pmatrix} u \\ 0 \end{pmatrix}$. Since $\mathcal{R}(M)$ is closed, there exist $x,\ y \in H$ such that $\begin{pmatrix} u \\ 0 \end{pmatrix} = \begin{pmatrix} A & C \\ B & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$. Thus $Bx = 0$ and $Ax + Cy = u$. Since $\mathcal{N}(B) \subset \mathcal{N}(A)$, $Cy = u$. Thus $\mathcal{R}(C)$ is closed. Hence C is hypo-EP. Next we prove that B is a hypo-EP operator. Let $y \in \mathcal{N}(B)$. Since $\mathcal{N}(B) = \mathcal{N}(C)$, $y \in \mathcal{N}(C)$. Thus $\begin{pmatrix} A & C \\ B & 0 \end{pmatrix}\begin{pmatrix} 0 \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Since M is hypo-EP, $\begin{pmatrix} A^* & B^* \\ C^* & 0 \end{pmatrix}\begin{pmatrix} 0 \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Thus $y \in \mathcal{N}(B^*)$. Hence $\mathcal{N}(B) \subset \mathcal{N}(B^*)$. To prove $\mathcal{R}(B)$ is closed, it is enough to prove that $\mathcal{R}(B^*)$ is closed. Let $B^*y_n \to u$. Therefore $\begin{pmatrix} A^* & B^* \\ C^* & 0 \end{pmatrix}\begin{pmatrix} 0 \\ y_n \end{pmatrix} \to \begin{pmatrix} u \\ 0 \end{pmatrix}$. Since $\mathcal{R}(M^*)$ is closed, there exist $x,\ y \in H$ such that $\begin{pmatrix} u \\ 0 \end{pmatrix} = \begin{pmatrix} A^* & B^* \\ C^* & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$. Thus $C^*x = 0$ and $A^*x + B^*y = u$. Since $\mathcal{R}(A) \subset \mathcal{R}(C)$, $B^*y = u$. Therefore $\mathcal{R}(B^*)$ is closed. Hence B is hypo-EP.

Conversely suppose that B and C are hypo-EP. Let $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{N}(M)$. Then $\begin{pmatrix} A & C \\ B & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Thus $Bx = 0$ and $Ax + Cy = 0$. Since $\mathcal{N}(B) \subset \mathcal{N}(A)$, $Cy = 0$. Since $\mathcal{N}(B) = \mathcal{N}(C)$, $Cx = 0 = By$. As C and B are hypo-EP, $C^*x = 0 = B^*y$. Since $\mathcal{R}(A) \subset \mathcal{R}(C)$, $A^*x = 0$. Thus $\begin{pmatrix} A^* & B^* \\ C^* & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Thus $\mathcal{N}(M) \subset \mathcal{N}(M^*)$.

Let $\begin{pmatrix} A & C \\ B & 0 \end{pmatrix}\begin{pmatrix} x_n \\ y_n \end{pmatrix} \to \begin{pmatrix} u \\ v \end{pmatrix}$. Thus $Ax_n + Cy_n \to u$ and $Bx_n \to v$. Since $\mathcal{R}(B)$ is closed, there exists $x \in H$ such that $Bx = v$. As $\mathcal{R}(A) \subset \mathcal{R}(C)$, for each n there exists $z_n \in H$ such that $Ax_n = Cz_n$. Thus $C(z_n + y_n) \to u$. Since $\mathcal{R}(C)$ is closed, there exists $z \in H$ such that $u = Cz$. As $\mathcal{R}(A) \subset \mathcal{R}(C)$, $Cz - Ax \in \mathcal{R}(C)$. Thus there exists $y \in H$ such that $Cz - Ax = Cy$. Thus $\begin{pmatrix} A & C \\ B & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} Ax + Cy \\ Bx \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}$. Thus $\mathcal{R}(M)$ is closed. Hence M is hypo-EP.

***Theorem* 3.9** — *Let $A,\ B,\ C \in BL(H)$ with $\mathcal{N}(B) = \mathcal{N}(C) \subset \mathcal{N}(A)$ and $\mathcal{R}(A) \subset \mathcal{R}(C)$. Then $M = \begin{pmatrix} A & C \\ B & 0 \end{pmatrix}$ is EP if and only if C and B are EP.*

PROOF : Suppose M is EP. In view of Theorem 3.8, to prove C and B are EP, it is enough to prove that $\mathcal{N}(C^*) \subset \mathcal{N}(C)$ and $\mathcal{N}(B^*) \subset \mathcal{N}(B)$. Let $x \in \mathcal{N}(C^*)$. Since $\mathcal{R}(A) \subset \mathcal{R}(C)$, $x \in \mathcal{N}(A^*)$. Thus $\begin{pmatrix} A^* & B^* \\ C^* & 0 \end{pmatrix} \begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Since M is EP, $\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} A & C \\ B & 0 \end{pmatrix} \begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} Ax \\ Bx \end{pmatrix}$. Thus $Bx = 0$. Since $\mathcal{N}(B) = \mathcal{N}(C)$, $x \in \mathcal{N}(C)$. Therefore $\mathcal{N}(C^*) \subset \mathcal{N}(C)$. Let $y \in \mathcal{N}(B^*)$. Then $\begin{pmatrix} A^* & B^* \\ C^* & 0 \end{pmatrix} \begin{pmatrix} 0 \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Since M is EP, $\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} A & C \\ B & 0 \end{pmatrix} \begin{pmatrix} 0 \\ y \end{pmatrix} = \begin{pmatrix} Cy \\ 0 \end{pmatrix}$. Thus $Cy = 0$. Since $\mathcal{N}(B) = \mathcal{N}(C)$, $y \in \mathcal{N}(B)$. Therefore $\mathcal{N}(B^*) \subset \mathcal{N}(B)$.

Conversely suppose C and B are EP. In view of the Theorem 3.8, to prove M is EP, it is enough to prove that $\mathcal{N}(M^*) \subset \mathcal{N}(M)$. Let $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{N}(M^*)$. Thus $\begin{pmatrix} A^* & B^* \\ C^* & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Therefore $C^*x = 0$ and $A^*x + B^*y = 0$. Since $\mathcal{R}(A) \subset \mathcal{R}(C)$, $A^*x = 0$. Therefore $C^*x = 0 = B^*y$. Since C and B are EP, $Cx = 0 = By$. Since $\mathcal{N}(B) = \mathcal{N}(C) \subset \mathcal{N}(A)$, $Ax = 0 = Bx = Cy$. Therefore $\begin{pmatrix} A & C \\ B & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Thus $\mathcal{N}(M^*) \subset \mathcal{N}(M)$. □

***Theorem* 3.10** — *Let $A \in BL(H)$ be hypo-EP and $B \in BL(H)$ be invertible. If $M = \begin{pmatrix} A & 0 \\ B & D \end{pmatrix}$ is hypo-EP, then M is injective.*

PROOF : Suppose M is hypo-EP. Let $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{N}(M)$. Then $\begin{pmatrix} Ax \\ Bx + Dy \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Therefore $x \in \mathcal{N}(A)$ and $Bx + Dy = 0$. Since A is hypo-EP, $A^*x = 0$. Also, since M is hypo-EP, $\begin{pmatrix} A^* & B^* \\ 0 & D^* \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Thus $A^*x + B^*y = 0$. So $B^*y = 0$. Since B^* is injective, $y = 0$. As $Bx + Dy = 0$, $Bx = 0$. So $x = 0$. Hence $\mathcal{N}(M) = \{0\}$. □

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