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APPLICATION OF THE NOVEL (G'/G)-EXPANSION METHOD TO CONSTRUCT TRAVELING WAVE SOLUTIONS TO THE POSITIVE GARDNER-KP EQUATION

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The novel (G'/G)-expansion method is one of the powerful methods accredited at the present time for establishing exact traveling wave solutions to nonlinear evolution equations (NLEEs). In this article, the method has been implemented to find the traveling wave solutions to the positive Gardner-KP equation. The efficiency of this method for finding exact and traveling wave solutions has been demonstrated. The obtained solutions have been compared with the solution obtained by other methods. The solutions have also been demonstrated by figures. It has been shown that the method is straightforward and an effective tool for solving NLEES that occur in applied mathematics, mathematical physics, and engineering.

Key words : Novel (G'/G)-expansion method; positive Gardner-KP equation; nonlinear evolution equation; traveling wave solutions; solitary wave solutions.

1. INTRODUCTION

The mathematical modeling of complex phenomena that change over time depends closely on the study of a variety of systems of ordinary and partial differential equations. Similar models are developed in diverse fields of study, such as, physics, biology, fluid mechanics, solid-state physics, biophysics, chemical kinematics, geochemistry, electricity, propagation of shallow water waves, plasma physics, high-energy physics, condensed matter physics, quantum mechanics, optical fibers, elastic media and so on. Thus, the investigation of solutions to NLEEs plays a very important role to uncover the obscurity of many phenomena and processes throughout the natural sciences. Therefore, in order to find out exact solutions to NLEEs different groups of mathematicians, engineers, and physicist have been working industriously. Accordingly, in the recent years, they establish several methods to search exact solutions, for instance, the Miura transformation method [1], the Jacobi elliptic function method [2], the Adomian decomposition method [3, 4], the method of bifurcation of planar dynamical systems [5, 6], the wave translation method [7], the ansatz method [8, 9], the Darboux transformation method [10], the Cole-Hopf transformation method [11], the (G'/G)-expansion method [12-23], the improved (G'/G)-expansion method [24, 25], the modified simple equation method [26, 27], the Exp-function method [28-30], the inverse scattering transform method [31], the multiple-expansion method [32], the novel (G'/G)-expansion method [33] etc.

The objective of this article is to implement the novel (G'/G) expansion method to construct exact and traveling wave solutions to the positive Gardner-KP equation to exhibit the suitability and competence of the method.

The rest of the article is organized as follows: In Section 2, the method has been discussed. In Section 3, we apply this method to the positive Gardner-KP equation. In Section 4, the physical explanations of the obtained solutions have been provided. In Section 5, we compare the obtained solutions with already published solutions. Finally, in Section 6, we have drawn our conclusions.

2. Description of the Method

Suppose the nonlinear evolution equation is of the form

$$P(u,\, u_t,\, u_x,\, u_{tt},\, u_{t\,x},\, u_{xx}, \cdots) = 0, \tag{1}$$

where P is a polynomial in $u(x,\, t)$and its partial derivatives wherein the highest order partial derivatives and the nonlinear terms are concerned. In order to investigate traveling wave solutions through the novel (G'/G)-expansion method we have to execute the following steps:

Step 1 : Combining the real variables x and t by a compound variable ξ, we suppose that

$$u(x,\, t) = u(\xi), \xi = x \pm V\, t, \tag{2}$$

where V is the speed of the traveling wave. The wave variable (2), transforms equation (1) into an ODE for $u = u(\xi)$:

$$Q(u,\, u',\, u'',\, u''', \cdots) = 0, \tag{3}$$

where Q is a function of $u(\xi)$ and its derivatives wherein prime indicates the derivative with respect to ξ.

Step 2 : Let us assume that the solution of Eq. (3) can be expressed in powers $\psi(\xi)$:

$$u(\xi) = \sum_{j=-N}^{N} \alpha_j \left(\psi(\xi)\right)^j , \tag{4}$$

where

$$\psi(\xi) = \left(d + \frac{G'(\xi)}{G(\xi)}\right). \tag{5}$$

Herein α_{-N} or α_N might be zero, but both of them could not be zero simultaneously. α_j $(j = 0, \pm 1, \pm 2, \cdots, \pm N)$ and d are constants to be determined later and $G = G(\xi)$ satisfies the second order nonlinear ODE:

$$G\,G'' = \lambda\,G\,G' + \mu\,G^2 + \nu\,(G')^2 \tag{6}$$

where prime denotes the derivative with respect to ξ and λ, μ, ν are real parameters.

The Cole-Hopf transformation $\Phi(\xi) = \ln\left(G(\xi)\right)_\xi = \frac{G'(\xi)}{G(\xi)}$ reduces Eq. (6) into Riccati equation:

$$\Phi'(\xi) = \mu + \lambda\ \Phi(\xi) + (\nu - 1)\ \Phi^2(\xi). \tag{7}$$

Eq. (7) has individual twenty five solutions (see Zhu, [34] for details).

Step 3 : The value of the positive integer N can be determined by balancing the highest order linear terms with the nonlinear terms of the highest order come out in Eq. (3). If the degree of $u(\xi)$ is $D\left[u\left(\xi\right)\right] = n$, then the degree of the other expressions will be as follows:

$$D\,[\frac{d^p u\left(\xi\right)}{d\xi^p}] = n + p, \quad D\,[u^p \left(\frac{d^q u\left(\xi\right)}{d\xi^q}\right)^s] = n\,p + s\,(n + q).$$

Step 4 : Substituting Eq. (4) including Eqs. (5) and (6) into Eq. (3), we obtain polynomials in $\left(d + \frac{G'(\xi)}{G(\xi)}\right)^j$ and $\left(d + \frac{G'(\xi)}{G(\xi)}\right)^{-j}$, $(j = 0,\ 1,\ 2,\ \cdots, N)$. Collecting all coefficients of identical power of the resulted polynomials to zero, yields an over-determined set of algebraic equations for α_j $(j = 0, \pm 1, \pm 2, \cdots, \pm N)$, d and V.

Step 5 : Suppose the value of the constants can be obtained by solving the algebraic equations obtained in Step 4. Substituting the values of the constants together with the solutions of Eq. (6), we will obtain some new and comprehensive exact traveling wave solutions to the nonlinear evolution equation (1).

Remark 1 : It is noteworthy to observe that if we replace λ by $-\lambda$ and μ by $-\mu$ and put $\nu = 0$ in Eq. (6), then the novel (G'/G)-expansion method coincides with the generalized and improved (G'/G)-expansion method [16]. On the other hand, if we put $d = 0$ in Eq. (5) and $\nu = 0$ in Eq. (6) then the method is identical to the improved (G'/G)-expansion method presented by Zhang *et al*. [24]. Again if we set $d = 0$, $\nu = 0$ and negative the exponents of (G'/G) are zero in Eq. (4), then the method turns out into the basic (G'/G)-expansion method introduced by Wang *et al*. [12]. Finally, if we put $\nu = 0$ in Eq. (6) and α_j $(j = 1,\ 2,\ 3, \cdots, N)$ are functions of x and t instead of constants then the method transforms into the generalized (G'/G)-expansion method developed by Zhang *et al*. [19]. Thus, the methods presented in the Ref. [12, 16, 19, 24] are only particular cases of the novel (G'/G)-expansion method.

3. Application of the Method

In this section, we will implement the novel (G'/G)-expansion method to obtain some new and more general exact traveling wave solutions to the celebrated positive Gardner-KP equation.

Let us consider the positive Gardner-KP equation,

$$(u_t + 6uu_x + 6u^2u_x + u_{xxx})_x + u_{yy} = 0. \tag{8}$$

This equation was first obtained rigorously within the asymptotic theory for long internal waves in a two-layer fluid with a density jump at the interface. The competition among dispersion, quadratic, and cubic nonlinearities constitutes the main interest of this equation [4].

The wave transformation $\xi = x + y - Vt$ transforms the equation (8) into an ordinary differential equation and integrating we obtain

$$(1 - V)\, u + 3\, u^2 + 2\, u^3 + u'' + C = 0, \tag{9}$$

where C is a constant of integration. Now, balancing the highest order derivative u'' with the nonlinear term of the highest order u^3, yields $N = 1$.

Therefore, the solution of Eq. (9) takes the form

$$u(\xi) = \alpha_{-1}\ (\psi(\xi))^{-1} + \alpha_0 + \alpha_1\ (\psi(\xi))\,. \tag{10}$$

Substituting Eq. (10) into Eq. (9), the left hand side is transformed into polynomials of $\left(d + \frac{G'(\xi)}{G(\xi)}\right)^j$ and $\left(d + \frac{G'(\xi)}{G(\xi)}\right)^{-j}$, $(j = 0,\ 1,\ \ 2,\ \cdots, N)$. Equating each coefficient of these polynomials to zero, we obtain an over-determined set of algebraic equations (for simplicity we leave out to display the equations) for α_0, α_1, α_{-1}, d, C and V. Solving the over-determined set of algebraic equations by using the symbolic computation software, such as Maple, we obtain

Set 1 : $\alpha_{-1} = 0$, $\alpha_0 = \pm I\,\{-d\,(\nu-1) + \frac{\lambda}{2}\} - \frac{1}{2}$, $\alpha_1 = \pm I\,(\nu-1)$, $C = -\mu\,\nu + \frac{1}{4} + \mu + \frac{1}{4}\lambda^2$,

$$V = 2\mu\,\nu - \frac{\lambda^2}{2} - 2\mu - \frac{1}{2}. \tag{11}$$

where d, λ, μ and ν are arbitrary constants and $I = \sqrt{-1}$.

Set 2 : $\alpha_1 = 0$, $\alpha_{-1} = \pm I\,\{d^2(\nu-1) + \mu - d\lambda\}$,$\alpha_0 = \mp\,\frac{\lambda - 2\,d\,\nu \pm I + 2\,d}{2i}$, $C = -\mu\;\nu + \frac{\lambda^2}{4} + \mu + \frac{1}{4}$,

$$V = 2\mu\,\nu - \frac{\lambda^2}{2} - 2\mu - \frac{1}{2}. \tag{12}$$

Substituting (11) and (12) into Eq. (10), we respectively obtain

$$u_1(x,\,t) = \pm\,I\,\{-d\,(\nu-1) + \frac{\lambda}{2}\} - \frac{1}{2} \pm I\,(\nu-1) \times \left(d + (G'/G)\right), \tag{13}$$

where $\xi = x - \left\{2\mu\,\nu - \frac{\lambda^2}{2} - 2\mu - \frac{1}{2}\right\}\,t$, and d, λ and ν are arbitrary constants.

$$u_2(x,\,t) = \mp\,\frac{\lambda - 2\,d\,\nu \pm I + 2\,d}{2i} \pm I\,\{d^2(\nu-1) + \mu - d\lambda\} \times \left(d + (G'/G)\right)^{-1}, \tag{14}$$

where $\xi = x - \left\{2\mu\,\nu - \frac{\lambda^2}{2} - 2\mu - \frac{1}{2}\right\}\,t$, and d, λ and ν are arbitrary constants.

By using the solutions for $G(\xi)$ obtained from Eq. (6) into Eq. (13) and simplifying, we obtain the following solutions:

When $\Omega = \lambda^2 - 4\,\mu\,\nu + 4\,\mu > 0$ and $\lambda\,(\nu-1) \neq 0$ (or $\mu\;(\nu-1) \neq 0$),

$$u_{1\,1}(x,\,t) = \pm\frac{I\,\sqrt{\Omega}}{2}\;\tanh(\frac{1}{2}\sqrt{\Omega}\,\xi) - \frac{1}{2}, \tag{15}$$

$$u_{1\,2}(x,\,t) = \pm\frac{I\,\sqrt{\Omega}}{2}\;\coth(\frac{1}{2}\sqrt{\Omega}\,\xi) - \frac{1}{2}, \tag{16}$$

$$u_{1\,3}(x,\,t) = \pm\frac{\sqrt{\Omega}}{2}\;\left\{I\;\tanh(\sqrt{\Omega}\,\xi) - \sec h(\sqrt{\Omega}\,\xi)\right\} - \frac{1}{2}, \tag{17}$$

$$u_{1\,4}(x,\,t) = \pm\frac{I\,\sqrt{\Omega}}{2}\;\left\{\coth(\sqrt{\Omega}\,\xi) + \csc h(\sqrt{\Omega}\,\xi)\right\} - \frac{1}{2}, \tag{18}$$

$$u_{1\,5}(x,\,t) = \pm\frac{I\,\sqrt{\Omega}}{2}\left\{\frac{A\;\cosh(\sqrt{\Omega}\,\xi) \mp \sqrt{A^2+B^2}}{A\;\sinh(\sqrt{\Omega}\,\xi) + B}\right\} - \frac{1}{2}, \tag{19}$$

where A and B are real constants.

$$u_{1\,6}(x,\,t)=\pm\frac{I\,\lambda}{2}\pm\frac{2\,I\,(\nu-1)\,\mu\,f_{i,j}}{\sqrt{\Omega}\,\sinh(\frac{1}{2}\sqrt{\Omega}\,\xi)-\lambda\,\cosh(\frac{1}{2}\sqrt{\Omega}\,\xi)}-\frac{1}{2},\tag{20}$$

where $i,\ j=0,\ 1$; $i\neq j$. If $i>j$, $f_{i\,j}=\cosh(\frac{1}{2}\sqrt{\Omega}\,\xi)$ and $i<j$, $f_{i\,j}=\sinh(\frac{1}{2}\sqrt{\Omega}\,\xi)$.

$$u_{1\,7}(x,\,t)=\pm\frac{I\,\lambda}{2}\pm\frac{2\,I\,(\nu-1)\,\mu\,\cosh(\sqrt{\Omega}\,\xi)}{\sqrt{\Omega}\,\sinh(\sqrt{\Omega}\,\xi)-\lambda\,\cosh(\sqrt{\Omega}\,\xi)\pm I\,\sqrt{\Omega}}-\frac{1}{2},\tag{21}$$

$$u_{1\,8}(x,\,t)=\pm\frac{I\,\lambda}{2}\pm\frac{2\,I\,(\nu-1)\,\mu\,\sinh(\sqrt{\Omega}\,\xi)}{\sqrt{\Omega}\,\cosh(\sqrt{\Omega}\,\xi)-\lambda\,\sinh(\sqrt{\Omega}\,\xi)\pm\sqrt{\Omega}}.\tag{22}$$

When $\Omega=\lambda^2-4\,\mu\,\nu+4\,\mu<0$ and $\lambda\,(\nu-1)\neq 0$ (or $\mu\,(\nu-1)\neq 0$),

$$u_{1\,9}(x,\,t)=\pm\frac{I\,\sqrt{-\Omega}}{2}\,\tan(\frac{1}{2}\sqrt{-\Omega}\,\xi)-\frac{1}{2},\tag{23}$$

$$u_{1\,10}(x,\,t)=\pm\frac{I\,\sqrt{-\Omega}}{2}\,\cot(\frac{1}{2}\sqrt{-\Omega}\,\xi)-\frac{1}{2},\tag{24}$$

$$u_{1_{11}}(x,\,t)=\pm\frac{I\,\sqrt{-\Omega}}{2}\,\Big(\tan(\sqrt{-\Omega}\,\xi)\pm\sec(\sqrt{-\Omega}\,\xi)\Big)-\frac{1}{2},\tag{25}$$

$$u_{1\,12}(x,\,t)=\pm\frac{I\,\sqrt{-\Omega}}{2}\,\Big(\cot(\sqrt{-\Omega}\,\xi)\mp\csc(\sqrt{-\Omega}\,\xi)\Big)-\frac{1}{2},\tag{26}$$

$$u_{1\,13}(x,\,t)=\pm\frac{I\,\sqrt{-\Omega}}{2}\,\frac{\Big\{A\,\cos(\sqrt{-\Omega}\,\xi)\pm\sqrt{(A^2-B^2)}\Big\}}{A\,\sin(\sqrt{-\Omega}\,\xi)+B}-\frac{1}{2},\tag{27}$$

where A and B are arbitrary constants such that $A^2-B^2>0$.

$$u_{1\,14}(x,\,t)=\pm\frac{I\,\lambda}{2}\pm\frac{2\,I\,(\nu-1)\,\mu\,\cos(\frac{1}{2}\sqrt{-\Omega}\,\xi)}{\sqrt{-\Omega}\,\sin(\frac{1}{2}\sqrt{-\Omega}\,\xi)+\lambda\,\cos(\frac{1}{2}\sqrt{-\Omega}\,\xi)}-\frac{1}{2},\tag{28}$$

$$u_{1\,15}(x,\,t)=\pm\frac{I\,\lambda}{2}\pm\frac{2\,I\,(\nu-1)\,\mu\,\sin(\frac{1}{2}\sqrt{-\Omega}\,\xi)}{\sqrt{-\Omega}\,\cos(\frac{1}{2}\sqrt{-\Omega}\,\xi)-\lambda\,\sin(\frac{1}{2}\sqrt{-\Omega}\,\xi)}-\frac{1}{2},\tag{29}$$

$$u_{1\,16}(x,\,t)=\pm\frac{I\,\lambda}{2}\mp\frac{2\,I\,(\nu-1)\,\mu\,\cos(\sqrt{-\Omega}\,\xi)}{\sqrt{-\Omega}\,\sin(\sqrt{-\Omega}\,\xi)+\lambda\,\cos(\sqrt{-\Omega}\,\xi)\pm\sqrt{-\Omega}}-\frac{1}{2},\tag{30}$$

$$u_{1\,17}(x,\,t)=\pm\frac{I\,\lambda}{2}\mp\frac{2\,I\,(\nu-1)\,\mu\,\sin(\frac{1}{2}\sqrt{-\Omega}\,\xi)}{\sqrt{-\Omega}\,\cos(\sqrt{-\Omega}\,\xi)-\lambda\,\sin(\sqrt{-\Omega}\,\xi)\pm\sqrt{-\Omega}}-\frac{1}{2}.\tag{31}$$

When $\mu=0$ and $\lambda\,(\nu-1)\neq 0$,

$$u_{1\,18}(x,\,t)=\pm\frac{I\,\lambda}{2}\mp\frac{I\,\lambda\,k}{k+\cosh(\lambda\,\xi)-\sinh(\lambda\,\xi)}-\frac{1}{2},\tag{32}$$

$$u_{1\,19}(x,\,t) = \pm\frac{I\,\lambda}{2} \mp \frac{I\,\lambda\,\{\cosh(\lambda\,\xi)+\sinh(\lambda\,\xi)\}}{k+\cosh(\lambda\,\xi)+\sinh(\lambda\,\xi)} - \frac{1}{2}, \tag{33}$$

where k is an arbitrary constant.

When $(\nu-1)\neq 0$ and $\lambda=\mu=0$, the solution of Eq. (8) is

$$u_{1\,20}(x,\,t) = \pm\frac{I\,\lambda}{2} \mp \frac{I\,(\nu-1)}{(\nu-1)\;\xi+c_1} - \frac{1}{2}, \tag{34}$$

where c_1 is an arbitrary constant.

Finally, by using the solutions for $G(\xi)$ obtained from Eq. (6) into Eq. (14), and simplifying we obtain the following solutions:

When $\Omega=\lambda^2-4\,\mu\,\nu+4\,\mu>0$ and $\lambda\,(\nu-1)\neq 0$ (or $\mu\;(\nu-1)\neq 0$),

$$\begin{aligned} u_{2\,1}(x,\,t) &= \mp\,\tfrac{\lambda-2\,d\,\nu\pm I+2\,d}{2i} \pm I\,\{d^2(\nu-1)+\mu-d\lambda\} \\ &\quad \times\left\{d-\tfrac{1}{2\,(\nu-1)}\left(\lambda+\sqrt{\Omega}\,\tanh\left(\tfrac{1}{2}\sqrt{\Omega}\,\xi\right)\right)\right\}^{-1}, \end{aligned} \tag{35}$$

$$\begin{aligned} u_{2\,2}(x,\,t) &= \mp\,\tfrac{\lambda-2\,d\,\nu\pm I+2\,d}{2i} \pm I\,\{d^2(\nu-1)+\mu-d\lambda\} \\ &\quad \times\left\{d-\tfrac{1}{2\,(\nu-1)}\left(\lambda+\sqrt{\Omega}\,\coth\left(\tfrac{1}{2}\sqrt{\Omega}\,\xi\right)\right)\right\}^{-1}, \end{aligned} \tag{36}$$

$$\begin{aligned} u_{2\,3}(x,\,t) &= \mp\,\tfrac{\lambda-2\,d\,\nu\pm I+2\,d}{2i} \pm I\,\{d^2(\nu-1)+\mu-d\lambda\} \\ &\quad \times\left[d-\tfrac{1}{2\,(\nu-1)}\,\left\{\lambda+\sqrt{\Omega}\,\left(\tanh(\sqrt{\Omega}\,\xi)\pm i\,\sec h(\sqrt{\Omega}\,\xi)\right)\right\}\right]^{-1} \end{aligned} \tag{37}$$

Similarly, we can write down the other families of exact solutions which are omitted for convenience.

When $\Omega=\lambda^2-4\,\mu\,\nu+4\,\mu<0$ and $\lambda\,(\nu-1)\neq 0$ (or $\mu\;(\nu-1)\neq 0$),

$$\begin{aligned} u_{2\,4}(x,\,t) &= \mp\,\tfrac{\lambda-2\,d\,\nu\pm I+2\,d}{2I} \pm I\,\{d^2(\nu-1)+\mu-d\lambda\} \\ &\quad \times\left\{d+\tfrac{1}{2\,(\nu-1)}\left(-\lambda+\sqrt{-\Omega}\,\tan(\tfrac{1}{2}\sqrt{-\Omega}\,\xi)\right)\right\}^{-1}, \end{aligned} \tag{38}$$

$$\begin{aligned} u_{2\,5}(x,\,t) &= \mp\,\tfrac{\lambda-2\,d\,\nu\pm I+2\,d}{2I} \pm I\,\{d^2(\nu-1)+\mu-d\lambda\} \\ &\quad \times\left\{d-\tfrac{1}{2\,(\nu-1)}\left(\lambda+\sqrt{-\Omega}\,\cot(\tfrac{1}{2}\sqrt{-\Omega}\,\xi)\right)\right\}^{-1}, \end{aligned} \tag{39}$$

$$\begin{aligned} u_{2\,6}(x,\,t) &= \mp\,\tfrac{\lambda-2\,d\,\nu\pm I+2\,d}{2\,I} \pm I\,\{d^2(\nu-1)+\mu-d\lambda\} \\ &\quad \times\left\{d+\tfrac{1}{2\,(\nu-1)}\,\left\{-\lambda+\sqrt{-\Omega}\,\left(\tan(\sqrt{-\Omega}\,\xi)\pm\sec(\sqrt{-\Omega}\,\xi)\right)\right\}\right\}^{-1}. \end{aligned} \tag{40}$$

For simplicity the other families of exact solutions of are omitted.

When $(\nu-1)\neq 0$ and $\lambda=\mu=0$, the solution of Eq. (8) is

$$u_{2\,7}(x,\,t)=\mp\frac{\lambda-2\,d\,\nu\pm I+2\,d}{2\,I}\pm I\left\{d^2(\nu-1)+\mu-d\lambda\right\}\times\left\{d-\frac{1}{(\nu-1)\;\xi+c_1}\right\}^{-1},\quad (41)$$

where c_1 is an arbitrary constant.

The other families of exact solutions of Eq. (8) are displayed here for minimalism.

4. Physical Explanations

The modulus of solutions (21), (22) and (41) describe the solitons. Solitons are special kinds of solitary waves. The soliton solutions are localized traveling wave solutions, hence $u'(\xi),\ u''(\xi),\ u'''(\xi)\to 0$ as $\xi\to\pm\infty$, $\xi=x-V\,t$. Solitons have a remarkable property that it keeps its identity upon interacting with other solitons. Fig. 1 shows the soliton obtained from solution (21). Solutions (15)-(20) and (34)-(37) are the multiple soliton solution. Fig. 2 shows the shape of the exact multiple soliton solution drawn from solution (15) of the positive Gardner-KP equation. The shape of figure of solutions (15)-(20) and (34)-(37) are similar to the figure of the solution (15). Solutions (23), (25), (27)-(31), (38)-(40) represent the exact periodic traveling wave solutions. Periodic solutions are traveling wave solutions that are periodic such as $\cos(x-t)$. Fig. 3, shows the periodic solution of Eq. (27). For convenience the other figures are omitted. Solutions (24) and (26) are the exact singular periodic traveling wave solutions. Fig. 4 shows the singular periodic solutions (24). For convenience other figures are omitted. Solution (32) and (33) represent compacton. Compacton is a new class of solitons with compact spatial support such that each compacton is a soliton confined to a finite core. Compactons are defined by solitary waves with the remarkable soliton property that after colliding with other compactons, they reemerge with the same coherent shape. This particle like wave exhibits the elastic collision which is similar to the soliton collision. The Fig. 5 shows the shape of the exact compacton solution (32) of the positive Gardner-KP equation.

Remark 2 : We have checked the obtained solutions by putting them back into the original equation and found correct.

5. Comparison

From the above solutions we observe that if we put $\nu=0$, $d=0$ and λ and μ are replaced by $-\lambda$ and $-\mu$ respectively in the obtained solution, then the hyperbolic function solution obtained by Shafiof *et al*. [35] is identical to our solution $u_{1\,1}$ when $C_2=0$, and when $C_1=0$ it is identical to our solution $u_{1\,2}$. Similarly, the trigonometric function solutions are identical to our solutions $u_{1\,9}$ and $u_{1\,10}$ when $C_2=0$ and $C_1=0$ respectively. On the other hand, the rational function solution obtained by Shafiof

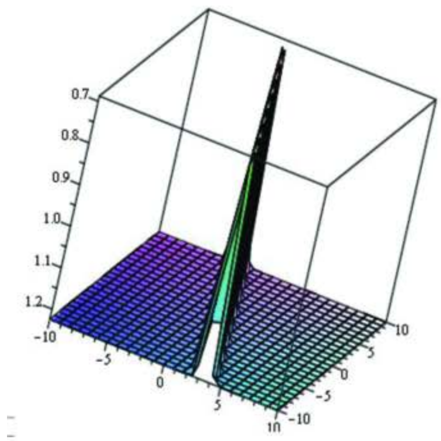

Fig. 1. Graph of the soliton solution of Eq. (21) for l = 1, m= -1, n = 2, *d* = 1 with -10 £ *x*, *t* £ 10

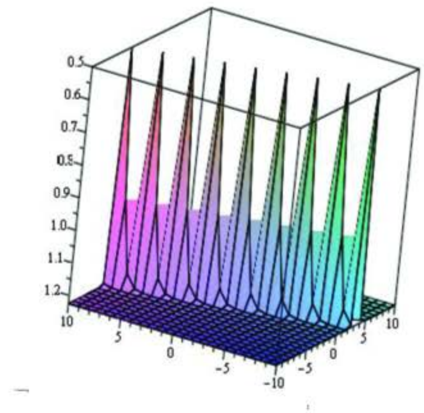

Fig. 2. Graph of the singular soliton solution (15) for l = 1, m= -1, n = 2, *d* = 1 with -10 £ *x*, *t* £ 10

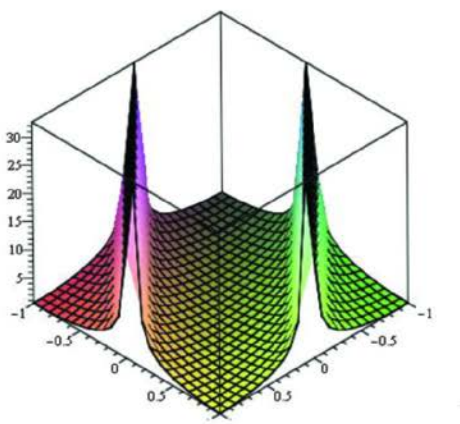

Fig. 3. Graph of the periodic solution of Eq. (27) for l = 1, m= -1, n = 2, *d* = 1 with -1 £ *x*, *t* £ 1

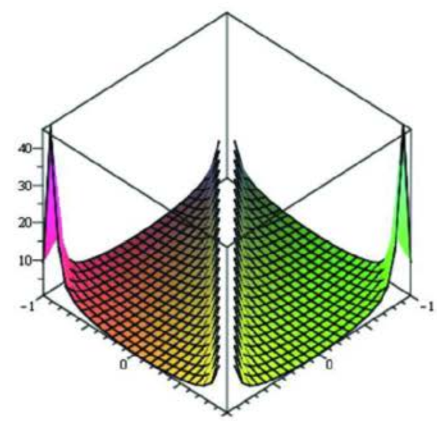

Fig. 4. Graph of the singular periodic solutions (24) for l = 1, m= -1, n = 2, *d* = 1 with -1 £ *x*, *t* £ 1

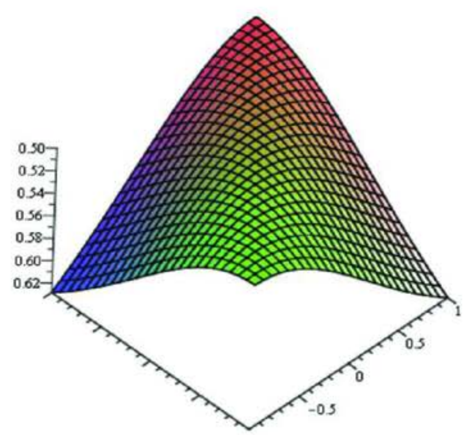

Fig. 5. Graph of the compacton solution (32) for l = 1, m= 0, n = 2, *d* = 1, a = 1, *k* = 1 with -10 £ *x*, *t* £ 10

et al. is identical to our solutions $u_{1\,20}$. Shafiof *et al*. [35] did not find any more solution, but by using the novel (G'/G)-expansion method, for Set 1, apart from these solutions, we obtain fifteen more new solutions. The solutions obtained in this article for Set 2 are not obtained by Shafiof *et al*. It can be shown that solutions obtained by the improved (G'/G)-expansion method [24] and the basic (G'/G)-expansion method [12] are only particular cases of the novel (G'/G)-expansion method.

6. CONCLUSION

The novel (G'/G)-expansion method has successfully been implemented to establish travelling wave solutions to the positive Gardner-KP equation. The performance of this method is reliable and effective to be used in finding exact solutions to NLEEs and it provides more general solutions which contain further arbitrary constants and the arbitrary constants imply that these solutions have rich local structures. It is important to notice that the basic (G'/G)-expansion method, the improved (G'/G)-expansion and the generalized and improved (G'/G)-expansion method are only special cases of the novel (G'/G)-expansion method, and thus the novel (G'/G)-expansion method could be a powerful mathematical tool for solving NLEEs.

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