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A CLASS OF STOCHASTIC DIFFERENTIAL EQUATIONS WITH PATHWISE UNIQUE SOLUTIONS[1]

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We propose a new method viz., using stochastic partial differential equations to study the pathwise uniqueness of stochastic (ordinary) differential equations. We prove the existence and pathwise uniqueness of a class of stochastic differential equations with coefficients in suitable Hermite-Sobolev class using our approach.

Key words : Stochastic differential equation; stochastic partial differential equation; pathwise uniqueness; Hermite-Sobolev functions.

1. Introduction

Results on existence and uniqueness of solutions to finite dimensional stochastic differential equations (SDEs) may be viewed as falling into two classes viz. strong solutions [10] under Lipschitz type condition on the coefficients and weak solutions for (say) bounded continuous coefficients but with a nondegeneracy condition on the diffusion coefficients [23]. Subsequent efforts have focused on existence and uniqueness results for equations with less regular coefficients. The main result available in this direction is the Yamada-Watanabe result on diffusions in $\mathbb{R}$ with holder continuous coefficients with exponent more than half and various variants of this result by many others. The existence of a unique strong solution of the SDE (2.1) is well known under the local Lipschitz continuity and linear growth assumptions. Also under the nondegeneracy condition, i.e. the infimum of the eigen values of $\bar{\sigma}\bar{\sigma}^t$ is strictly positive, where $\bar{\sigma}$ denotes the diffusion coefficient of the SDE and $\bar{\sigma}^t$ its transpose, one

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has the existence of unique strong solutions with bounded measurable drift and Lipschitz continuous diffusion coefficients. These results are due to Zvonkin in [28] for the one dimensional case and Veretennikov in [27] for the multidimensional case. Also an interesting special class of nondegenerate diffusions is considered in [2]. Here authors consider piecewise constant diffusion coefficient and bounded measurable drift and obtain pathwise uniqueness. For the one dimensional case, stochastic differential equation with irregular case is extensively studied in the monograph, see [4]. But to the best of our knowledge, in the case of irregular coefficients, only limited results are available without a nondegeneracy condition on the diffusion coefficient. For example in [25], Swart considers a class of SDEs which evolve in the unit ball in $\mathbb{R}^d$ with the diffusion coefficient being locally Lipschitz in the interior and Hölder continuous with exponent $\frac{1}{2}$ on the boundary of the ball. In [6], Fang and Zhang proved the existence of pathwise unique solutions by relaxing the Lipschitz condition on the coefficients by a logarithmic factor, i.e., Osgood type condition. In an alternate approach to study the well posedness of the SDEs under non smooth coefficients, one studies the Kolmogorov forward equation, i.e. the stochastic continuity equation and weak uniqueness of the SDE is obtained using the forward Kolmogorov's equation; see for example [3, 7, 8, 16]. We refer to [16] for issues connecting the well-posedness of the SDE (2.1) with the well-posedness of the associated forward Kolmogorov's equation. In [24], the existence of pathwise unique solution is shown for a class of SDEs evolving in the positive orthant without the Lipschitz condition on the coefficients. There a set of sufficient conditions on the coefficients is given which prevents the process from hitting the boundary of the orthant.

In this article, we propose a new method for proving the existence and pathwise uniqueness of strong solutions of stochastic differential equations with irregular diffusion and drift coefficients without the assumption of nondegeneracy. We use a technique based on [19, 20] and will prove the existence of pathwise unique solutions when the coefficients are in certain Hermite-Sobolev spaces. A brief description of the approach is as follows. We identify an associated stochastic partial differential equation (SPDE) such that if $X(\cdot)$ is a solution of the SDE (2.1), then the process $\{\tau_{X_t} y | t \geq 0\}$, where y is a tempered distribution in an appropriate Hermite-Sobolev space and $\tau_x y$ is the translation of y by $x \in \mathbb{R}^d$, is a solution of the associated SPDE (see eqns. (3.2), (3.3) below). Now by imposing appropriate conditions on the coefficients of the SDE (2.1) (see condition (A1) below in section 2), we show that the associated SPDE satisfies the so called 'monotonicity inequality'. That such an inequality leads to uniqueness results for SPDE's is well known [11, 13, 22]. For second order partial differential operators of diffusion type with constant coefficients this inequality was first proved in [5] and then derived using properties of differential operators on the Hermite-Sobolev spaces in [1]. In [20], an extension of this inequality to nonlinear operators is used to prove existence and uniqueness for a general class of SPDE's. Using this nonlinear extension, in this paper we establish the pathwise uniqueness property for the SPDE associated with our finite dimensional SDE. This in turn leads us to the existence of pathwise unique solution of our finite dimensional SDE (2.1). We remark here that the condition (A1) mentioned above is interesting in itself and raises the question of when a given function can be written as the convolution of two other functions with specified properties. The problem seems to be well known [21] and points to some connection with harmonic analysis.

The article is structured as follows. In Section 2, the problem and main result are described. In Section 3, we prove the general existence and pathwise unique solution result the SDE (2.1). In Section 4, we give give some specific classes of SDE's which comes under this general theorem.

2. Problem Description

We consider the SDE

$$dX(t) = \bar{b}(X_t)\,dt + \bar{\sigma}(X_t)\cdot dW(t),\ X(0) = x, \tag{2.1}$$

where $\bar{b} : \mathbb{R}^d \to \mathbb{R}^d,\ \bar{\sigma} : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ are bounded continuous. There are examples in this class where one can indeed get multiple solutions. For example, take $d = 1, \bar{\sigma} = 0$ and $\bar{b}(x) := \sqrt{|x|}, |x| \leq 1$, and $\bar{b}(x)$ has value 0 for $|x| > 2$ and is Hölder continuous with exponent $\frac{1}{2}$. Our aim in this article is to study the pathwise uniqueness of the SDE (2.1) using pathwise uniqueness of an associated SPDE. As observed in [19], one can associate with any solution $X(\cdot)$ of (2.1), with appropriate conditions on the coefficients, an SPDE of the form

$$dY_t \ = \ \hat{L}Y_t\,dt + \hat{A}Y_t\cdot dW_t,\ Y_0 = y,$$

where $\hat{A} = (\hat{A}_1, \cdots, \hat{A}_d), \hat{L}$ are appropriate nonlinear operators from the space of tempered distributions viz. $\mathcal{S}'(\mathbb{R}^d)$ to itself. The operators $\hat{A},\ \hat{L}$ can be identified using Itô's formula for $f(X_t) := \langle f, \delta_{X_t}\rangle$ for a class of test functions f. In fact Itô's formula implies that the $\mathcal{S}'(\mathbb{R}^d)$- valued process $\{\delta_{X_t} | t \geq 0\}$ is a solution to the SPDE given above with $y = \delta_x$. In this paper, we will extend this approach to prove pathwise uniqueness to the SDE (2.1) for irregular coefficients.

Now we will describe the framework for the SPDE's considered in this article. Let $\mathcal{S}_p(\mathbb{R}^d),\ p \in \mathbb{R}$ be the Hermite-Sobolev spaces which are (real) Hilbert spaces obtained as the completion of the

Schwartz space $\mathcal{S}(\mathbb{R}^d)$ with the inner product given by

$$< \varphi, \psi >_p := \sum_{m=0}^{\infty} \sum_{n \in \mathbb{Z}_+^d; |n|=m} (2m+d)^{2p} < \varphi, h_n >< \psi, h_n >,$$

where $h_n : \mathbb{R}^d \to \mathbb{R}$ denotes the Hermite function corresponding to the multi-index $n = (n_1, \cdots, n_d) \in \mathbb{Z}_+^d$ and $< .,. >_0 \quad := \quad < .,. >$ is the $L^2(\mathbb{R}^d)$ inner product. The family $\{h_n | n \in \mathbb{Z}_+^d\}$ form an orthonormal basis for $L^2(\mathbb{R}^d)$. The norm corresponding to the inner product $< .,. >_p$ will be denoted by $\|.\|_p$. We then have $\mathcal{S}_p \subseteq \mathcal{S}_q$ if $q < p$. Further if $\mathcal{S}'(\mathbb{R}^d)$ denotes the continuous dual of $\mathcal{S}(\mathbb{R}^d)$ - the tempered distributions on $\mathbb{R}^d$ - then note that $\mathcal{S}'(\mathbb{R}^d) = \bigcup_{p \in \mathbb{R}} \mathcal{S}_p$ and $\mathcal{S}(\mathbb{R}^d) = \bigcap_{p \in \mathbb{R}} \mathcal{S}_p$. Also the derivatives $\partial_i : \mathcal{S}_p(\mathbb{R}^d) \to \mathcal{S}_{p-\frac{1}{2}}(\mathbb{R}^d),\ i = 1, \cdots, d$ are bounded linear maps. We refer to [10] for more details on the properties of the spaces $\mathcal{S}_p$. In section 4, we will require the Hermite-Sobolev spaces over the complex field $\mathbb{C}$, denoted by $\mathcal{S}_p(\mathbb{R}^d; \mathbb{C})$ or simply $\mathcal{S}_p(\mathbb{C})$, when the dimension is understood. These are defined in a manner similar to the real case, with the inner product $< \varphi, \psi >_p$ defined as above except that in the RHS above the factor $< \psi, h_n >$ is replaced by its complex conjugate $< \psi, \bar{h}_n >$.

We make the following assumptions on the coefficients $\bar{b}$, $\bar{\sigma}$ of eqn.(2.1) :

(A1) The functions $\bar{b} = (\bar{b}_1, \cdots, \bar{b}_d) : \mathbb{R}^d \to \mathbb{R}^d$ and $\bar{\sigma} = (\bar{\sigma}_{ij}) : \mathbb{R}^d \to \mathbb{R}^{d \times d}$ are such that for some $\alpha \in \mathbb{R}$, there exists a $y \in \mathcal{S}_{-\alpha+1}(\mathbb{R}^d)$ and $b_i, \sigma_{ij} \in \mathcal{S}_\alpha(\mathbb{R}^d), i, j = 1, \cdots, d$ satisfying

$$\bar{b}_i(x) \ = \ \langle b_i, \tau_x y \rangle, \ \bar{\sigma}_{ij}(x) \ = \ \langle \sigma_{ij}, \tau_x y \rangle, \ x \in \mathbb{R}^d,$$

where $\langle , \rangle$ denote the dual bracket associated with the dual pair $(\mathcal{S}_\alpha(\mathbb{R}^d),\ \mathcal{S}_{-\alpha}(\mathbb{R}^d))$ and the translation operator $\tau_x : \mathcal{S}_\alpha(\mathbb{R}^d) \to \mathcal{S}_\alpha(\mathbb{R}^d),\ x \in \mathbb{R}^d$ is given by

$$\tau_x y(u) \ = \ y(u - x), u \in \mathbb{R}^d, \ y \in \mathcal{S}_\alpha(\mathbb{R}^d).$$

Note that for $\alpha \in \mathbb{R}$ and $y \in \mathcal{S}_\alpha(\mathbb{R}^d)$, the map $x \to \tau_x y$ from $\mathbb{R}^d \to \mathcal{S}_\alpha(\mathbb{R}^d)$ is a continuous map, hence the condition (A1) implies that $\bar{b}_i$, $\bar{\sigma}_{ij}$ are continuous. We refer to [18] for some properties of the operators τ_x acting on the spaces $\mathcal{S}_\alpha$. We shall refer to σ_{ij} and y as the factors of $\bar{\sigma}_{ij}$ and similarly for the b_i's. We note that under (A1), $\bar{b}_i(x) = y^- * b_i(x)$ where $y^-(x) := y(-x)$ for functions, extended by duality to distributions and the $*$ denotes convolution. In particular if one of the factors of $\bar{b}_i$ is smooth then so is $\bar{b}_i$. It is known that $\mathcal{S}_{|\alpha|} \subset C^k$ if $|\alpha| > d + \frac{k}{2}$ where C^k is the space of k times continuously differentiable functions on $\mathbb{R}^d$. Hence the range of α for which the condition (A1) could possibly yield irregular coefficients is $|\alpha| \leq d + \frac{1}{2}$.

(A2) The functions $\bar{b} = (\bar{b}_1, \cdots, \bar{b}_d) : \mathbb{R}^d \to \mathbb{R}^d$ and $\bar{\sigma} = (\bar{\sigma}_{ij}) : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ are bounded.

Our main result is the following general theorem regarding the existence of unique pathwise solution to the SDE (2.1).

Theorem — *Assume (A1) and (A2). Then the SDE (2.1) has a pathwise unique strong solution.*

3. A GENERAL EXISTENCE UNIQUENESS RESULT

In this section, we prove our general theorem on the existence of pathwise unique solution to the SDE (2.1).

For $\alpha \in \mathbb{R}$, consider $b = (b_1, \cdots, b_d),\ b_i \in \mathcal{S}_\alpha(\mathbb{R}^d), \sigma = (\sigma_{ij}), \sigma_{ij} \in \mathcal{S}_\alpha(\mathbb{R}^d)$. Since $\mathcal{S}_\alpha(\mathbb{R}^d) \equiv (\mathcal{S}_{-\alpha}(\mathbb{R}^d))^*$, we have (after identification) $b_i, \sigma_{ij} : \mathcal{S}_{-\alpha}(\mathbb{R}^d) \to \mathbb{R}$ satisfying

$$\begin{aligned} |\langle b_i, \varphi\rangle| &\leq \|b_i\|_\alpha \|\varphi\|_{-\alpha}, \\ |\langle \sigma_{ij}, \varphi\rangle| &\leq \|\sigma_{ij}\|_\alpha \|\varphi\|_{-\alpha},\ \varphi \in \mathcal{S}_{-\alpha}(\mathbb{R}^d). \end{aligned}$$

Since $\mathcal{S}_{-\alpha+1}(\mathbb{R}^d) \subseteq \mathcal{S}_{-\alpha}(\mathbb{R}^d)$, if $\hat{b}_i, \hat{\sigma}_{ij}$ denote the restriction of b_i, σ_{ij} to $\mathcal{S}_{-\alpha+1}(\mathbb{R}^d)$, it follows that $\hat{b}_i : \mathcal{S}_{-\alpha+1}(\mathbb{R}^d) \to \mathbb{R},\ \hat{\sigma}_{ij} : \mathcal{S}_{-\alpha+1}(\mathbb{R}^d) \to \mathbb{R}$ satisfies

$$\begin{aligned} |\langle \hat{b}, \varphi\rangle_i| := |\langle \hat{b}_i, \varphi\rangle| &\leq \|b_i\|_\alpha \|\varphi\|_{-\alpha}, \\ |\langle \hat{\sigma}, \varphi\rangle_{ij}| := |\langle \hat{\sigma}_{ij}, \varphi\rangle| &\leq \|\sigma_{ij}\|_\alpha \|\varphi\|_{-\alpha},\ \varphi \in \mathcal{S}_{-\alpha+1}(\mathbb{R}^d). \end{aligned} \tag{3.1}$$

Consider the SPDE

$$dY_t = L(Y_t)\, dt + A(Y_t) \cdot dW_t,\ Y(0) = y, \tag{3.2}$$

where $y \in \mathcal{S}_{-\alpha+1}(\mathbb{R}^d)$, $L : \mathcal{S}_{-\alpha+1}(\mathbb{R}^d) \to \mathcal{S}_{-\alpha}(\mathbb{R}^d),\ A = (A_1, \cdots, A_d) : \mathcal{S}_{-\alpha+1}(\mathbb{R}^d) \to \mathcal{S}_{-\alpha}(\mathbb{R}^d) \times \cdots \times \mathcal{S}_{-\alpha}(\mathbb{R}^d)$ are nonlinear operators given by

$$\begin{aligned} A_j(\varphi) &= -\sum_{i=1}^d \langle \hat{\sigma}, \varphi\rangle_{ij} \partial_i \varphi,\ \ j = 1, \cdots d, \\ L(\varphi) &= -\sum_{i=1}^d \langle \hat{b}, \varphi\rangle_i \partial_i \varphi + \frac{1}{2} \sum_{i,j=1}^d (\langle \hat{\sigma}, \varphi\rangle \langle \hat{\sigma}, \varphi\rangle^t)_{ij} \partial_{ij}^2 \varphi. \end{aligned} \tag{3.3}$$

Here the derivatives are defined in the distributional sense and $\langle ., . \rangle$ denote the dual bracket associated with the dual pair $(\mathcal{S}_\alpha(\mathbb{R}^d),\ \mathcal{S}_{-\alpha}(\mathbb{R}^d))$. That the operators L and A_i satisfy $L : \mathcal{S}_{-\alpha+1}(\mathbb{R}^d) \to \mathcal{S}_{-\alpha}(\mathbb{R}^d)$, and $A_i : \mathcal{S}_{-\alpha+1}(\mathbb{R}^d) \to \mathcal{S}_{-\alpha}(\mathbb{R}^d), i = 1, \cdots d$, follows from the properties of the derivative operators ∂_i mentioned above. Let $\{\mathcal{F}_t^W\}$ denote the augmented filtration of the filtration generated by $W(\cdot)$.

Definition 3.1 — For $y \in \mathcal{S}_{-\alpha+1}(\mathbb{R}^d)$, and a Wiener process $W(\cdot)$ on $(\Omega, \mathcal{F}, P)$, we say that an $\mathcal{S}_{-\alpha+1}(\mathbb{R}^d)$-valued continuous $\{\mathcal{F}_t^W\}$-adapted process $Y(\cdot)$ is a (strong) solution to the SPDE (3.2) on $[0, \infty)$ if

(1) $Y(0) = y$ a.s.

(2) The following equation holds a.s. in $\mathcal{S}_{-\alpha}(\mathbb{R}^d)$, for every $t \geq 0$:

$$Y(t) \ = \ y + \int_0^t L(Y_s)ds + \sum_{i=1}^d \int_0^t A_i(Y_s)dW_s^i.$$

Here note that since $\{A_i(Y_t), t \geq 0\}$, $\{L(Y_t), t \geq 0\}$ are $\{\mathcal{F}_t^W\}$-adapted locally bounded $\mathcal{S}_{-\alpha}(\mathbb{R}^d)$ processes, the stochastic integrals $\int_0^t L(Y_s)ds, \int_0^t A_i(Y_s)dW_s^i$ are well defined $\mathcal{S}_{-\alpha}(\mathbb{R}^d)$ valued continuous $\{\mathcal{F}_t^W\}$-adapted processes.

First we prove the following pathwise uniqueness result for the SPDE (3.2).

***Theorem* 3.1** — *Assume that $b_i, \sigma_{ij} \in \mathcal{S}_\alpha(\mathbb{R}^d), i, j = 1, \cdots, d$. Then the SPDE (3.2) has the pathwise uniqueness property.*

PROOF : From (3.1), we have for $\varphi, \psi \in \mathcal{S}_{-\alpha+1}(\mathbb{R}^d)$,

$$|\langle \hat{b}_i, \ \varphi - \psi \rangle| \ \ \leq \ \ K\|\varphi - \psi\|_{-\alpha} \tag{3.4}$$

$$|\langle \hat{\sigma}_{ij}, \varphi - \psi \rangle| \ \ \leq \ \ K\|\varphi - \psi\|_{-\alpha}, \tag{3.5}$$

where $K = \max_{i,j=1,\cdots,d}\{\|b_i\|_\alpha, \|\sigma_{ij}\|_\alpha\}$.

Set

$$\langle \hat{a}_{ij}, \varphi \rangle \ := \ (\langle \hat{\sigma}, \varphi \rangle \langle \hat{\sigma}, \varphi \rangle^t)_{ij}, \ \varphi \in \mathcal{S}_{-\alpha+1}(\mathbb{R}^d).$$

Then

$$\langle \hat{a}_{ij}, \varphi \rangle \ = \ \sum_{k=1}^d \langle \hat{\sigma}_{ik}, \varphi \rangle \langle \hat{\sigma}_{kj}, \varphi \rangle, \ \varphi \in \mathcal{S}_{-\alpha+1}(\mathbb{R}^d).$$

Therefore, it follows that for $\varphi, \psi \in B(0, \lambda, -\alpha + 1) := \{\eta \in \mathcal{S}_{-\alpha+1}(\mathbb{R}^d); \ \|\eta\|_{-\alpha+1} \leq \lambda\}$, with $\lambda > 0$ we have

$$\begin{aligned} |\langle \hat{a}_{ij}, \varphi\rangle - \langle \hat{a}_{ij}, \psi\rangle| &\leq \sum_{k=1}^{d} K(|\langle \hat{\sigma}_{ik}, \varphi\rangle| + |\langle \hat{\sigma}_{kj}, \psi\rangle|)\|\varphi - \psi\|_{-\alpha} \\ &\leq \sum_{k=1}^{d} K(\|\sigma_{ik}\|_{\alpha}\|\varphi\|_{-\alpha} + \|\sigma_{kj}\|_{\alpha}\|\psi\|_{-\alpha})\|\varphi - \psi\|_{-\alpha} \\ &\leq 2K^2 d\lambda\|\varphi - \psi\|_{-\alpha}. \end{aligned}$$

where in the second last inequality, we have used the fact that $\|\varphi\|_{-\alpha} \leq \|\varphi\|_{-\alpha+1}$ for $\varphi \in \mathcal{S}_{-\alpha+1}(\mathbb{R}^d)$. Hence using [20, Theorem 2.3] we can show that $\exists$ a constant $C = C(d, \alpha, \lambda)$ such that

$$\langle \varphi - \psi, L(\varphi) - L(\psi)\rangle_{-\alpha} + \sum_{j=1}^{d} \|A_j(\varphi) - A_j(\psi)\|_{-\alpha}^2 \leq C\ \|\varphi - \psi\|_{-\alpha}^2 \tag{3.6}$$

for all $\varphi, \psi \in B(0, \lambda, -\alpha + 1)$. Hence if (Y_t^1) and (Y_t^2) are two solutions of the SPDE (3.2) on the same probability space, on $[0, \infty)$, and if we define $\eta \equiv \eta(\lambda) := \inf\{t : Y_t^1 \ or\ Y_t^2 \notin B(0, \lambda, -\alpha+1)\}$ then we have in the interval $0 \leq t < \eta$,

$$\begin{aligned} \|Y_t^1 - Y_t^2\|_{-\alpha}^2 &= \int_0^t \{2\langle Y_s^1 - Y_s^2, L(Y_s^1) - L(Y_s^2)\rangle_{-\alpha} + \sum_{j=1}^{d} \|A_j(Y_s^1) - A_j(Y_s^2)\|_{-\alpha}^2\}\, ds \\ &\quad + M_t \end{aligned}$$

where (M_t) is a continuous local martingale. Now uniqueness follows using inequality (3.6), the Gronwall inequality and a localization argument (see Proof of Lemma 3.6, [19]) i.e. almost surely, $0 \leq t < \eta$, $Y_t^1 = Y_t^2$. Letting $\lambda \uparrow \infty$ the pathwise uniqueness follows. □

Now we show that (3.2) is the SPDE associated with the SDE (2.1). Also note that the SPDE (3.2) is with coefficients given by the restrictions to $\mathcal{S}_{-\alpha+1}(\mathbb{R}^d)$ of $\hat{b}, \hat{\sigma}$ given by b, σ of (A1).

***Theorem* 3.2** — *Assume (A1). Let $\{X(t)|t \geq 0\}$ be a (strong) solution to the SDE (2.1) on the Wiener space $(\Omega, \mathcal{F}, P, W(\cdot))$ with $X(0) = 0\ a.s.$ Then the $\mathcal{S}_{-\alpha+1}(\mathbb{R}^d)$ valued process $\{Y(t)|t \geq 0\}$ given by*

$$Y_t\ :=\ \tau_{X_t} y, t \geq 0,$$

is a strong solution to the SPDE (3.2).

PROOF : The proof follows by an application of Itô's formula in [17].

Using the formula in Theorem 2.3 of [17], we get

$$\tau_{X_t} y \ = \ \tau_{X_0} y - \sum_{i=1}^{d} \int_0^t \partial_i(\tau_{X_s} y)\, dX_s^i + \frac{1}{2} \sum_{i,j=1}^{d} \int_0^t \partial_{ij}^2 (\tau_{X_s} y) \bar{a}_{ij}(X_s)\, ds,$$

where $X_t = (X_t^1, \cdots, X_t^d),\ t \geq 0$ and the equation holds in $\mathcal{S}_{-\alpha}$. i.e.,

$$\begin{aligned} Y_t \ &= \ y - \sum_{i=1}^{d} \int_0^t \bar{b}_i(X_s) \partial_i(Y_s)\, ds + \frac{1}{2} \sum_{i,j=1}^{d} \int_0^t \bar{a}_{ij}(X_s) \partial_{ij}^2 (Y_s)\, ds \\ &\quad - \sum_{i=1}^{d} \sum_{j=1}^{d} \int_0^t \bar{\sigma}_{ij}(X_s) \partial_i(Y_s)\, dW_j(s). \end{aligned}$$

Using (A1), we get

$$\begin{aligned} Y_t \ &= \ y - \sum_{i=1}^{d} \int_0^t \langle b_i, Y_s \rangle \partial_i(Y_s)\, ds + \frac{1}{2} \sum_{i,j=1}^{d} \sum_{k=1}^{d} \int_0^t \langle \sigma_{ik}, Y_s \rangle \langle \sigma_{kj}, Y_s \rangle \partial_{ij}^2 (Y_s)\, ds \\ &\quad - \sum_{i=1}^{d} \sum_{j=1}^{d} \int_0^t \langle \sigma_{ij}, Y_s \rangle \partial_i(Y_s)\, dW_j(s) \\ &= \ y + \int_0^t L(Y_s)\, ds + \int_0^t A(Y_s) \cdot dW_s \end{aligned}$$

where in the last equality we have used the fact $\hat{b}_i, \hat{\sigma}_{ij}$ are restrictions of b_i, σ_{ij} to $\mathcal{S}_{-\alpha+1}(\mathbb{R}^d)$, $i, j = 1, \cdots d$. Since $\tau_x : \mathcal{S}_{-\alpha+1}(\mathbb{R}^d) \to \mathcal{S}_{-\alpha+1}(\mathbb{R}^d)$ and the map $x \mapsto \tau_x y$ is continuous, it follows that $\{Y(t) | t \geq 0\}$ is a $\mathcal{S}_{-\alpha+1}(\mathbb{R}^d)$ valued strong solution to (3.2) on $[0, \infty)$. □

***Theorem* 3.3** — *Assume (A1). The SDE (2.1) has at most one pathwise unique solution.*

PROOF : Let $X_i(t, x_0), i = 1, 2$ be two solutions of (2.1) on the Wiener space $(\Omega, \mathcal{F}, P, W(\cdot))$ with same initial condition say x_0 and coefficients $\bar{b}_i, \bar{\sigma}_{ij}$ satisfying (A1) with 'factors' $(b_i, y), (\sigma_{ij}, y)$ respectively. Then we can write $X_i(t, x_0) = x_0 + X_i(t, 0)$ where $X_i(t, 0)$ is a strong solution of (2.1) with $\bar{b}_i, \bar{\sigma}_{ij}$ replaced by $\bar{b}_i(x_0 + \cdot), \bar{\sigma}_{ij}(x_0 + \cdot)$. Note that these latter coefficients satisfy (A1) with corresponding factors b_i, σ_{ij} and $\tau_{x_0} y$. Set

$$Y_i(t) \ := \tau_{X_i(t,0)} \tau_{x_0} y,\ t \geq 0,\ \ i = 1, 2.$$

Hence using Theorem 3.2, it follows that $Y_i(\cdot), i = 1, 2$ are strong solutions to (3.2) with same initial condition $\tau_{x_0} y \in \mathcal{S}_{-\alpha+1}(\mathbb{R}^d)$. Then using Theorem 3.1, it follows that

$$P(Y_1(t) = Y_2(t), \forall\, t \geq 0) \ = \ 1.$$

Hence

$$P(X_1(t,0) = X_2(t,0), \forall\, t \geq 0) \;=\; 1.$$

This completes the proof. □

Now we prove our general theorem on existence of unique pathwise solution.

***Theorem* 3.4** — *Assume (A1) and (A2). Then the SDE (2.1) has a pathwise unique strong solution.*

PROOF : First we prove the existence of a weak solution. The proof closely mimics the arguments in [12, p.166-169]. Let $\mathcal{D}(\mathbb{R}^d;\mathbb{R}^d)$ denotes the class of $\mathbb{R}^d$-valued infinitely differentiable functions with compact support and the definition of $\mathcal{D}(\mathbb{R}^d;\mathbb{R}^{d\times d})$ is similar. Let $b_n \in \mathcal{D}(\mathbb{R}^d;\mathbb{R}^d)$ and $\sigma_n \in \mathcal{D}(\mathbb{R}^d;\mathbb{R}^{d\times d}), n \geq 1$ be such that $b_n \to \bar{b},\ \sigma_n \to \bar{\sigma}$ uniformly on compact sets of $\mathbb{R}^d$ and $\sup_n\{\|b_n\|_\infty, \|\sigma_n\|_\infty\} < \infty$. Consider the SDE

$$dX_n(t) \;=\; b_n(X(t))dt + \sigma_n(X(t))dW(t),\ X_n(0) = x. \tag{3.7}$$

The SDE (3.7) has a unique strong solution for each $n \geq 1$. Consider

$$\begin{aligned}
E|X_n(t) - X_n(s)|^4 \;&\leq\; 64\Big[E\Big|\int_s^t b_n(X_n(u))du\Big|^4 + E\Big|\int_s^t \sigma_n(X_n(u))dW(s)\Big|^4\Big]\\
&\leq\; 64|t-s|^3 E\int_s^t |b_n(X_n(u))|^4 du + 64E\Big|\int_s^t \sigma_n(X_n(u))dW(u)\Big|^4\Big]\\
&\leq\; 64|t-s|^3 E\int_s^t |b_n(X_n(u))|^4 du + 64K_1 E\Big|\int_s^t \|\sigma_n(X_n(u))\|^2 du\Big|^2\Big]\\
&\leq\; K_2[|t-s|^2 + |t-s|^4]\ \leq K|t-s|^2,\ 0 \leq s \leq t \leq T. \qquad (3.8)
\end{aligned}$$

where $K_2 > 0$ depends on K_1 and the uniform bounds of b_n, σ_n and K depends on K_2 and T. Here we use Jensen's inequality for the second inequality and the Burkholder-Davis-Gundy inequality for the third inequality.

The inequality (3.8) implies the tightness of the sequence of probability laws $\{P_n | n \geq 1\}$, $(P_n := \text{Law of } X_n(\cdot))$ in the space of probability measures on $C[0,T]$ which we denote by $\mathcal{P}(C[0,\ T];\mathbb{R}^d)$. Let μ be a limit point in $\mathcal{P}(C[0,\ T];\mathbb{R}^d)$ of the sequence $\{P_n | n \geq 1\}$. Now using the Skorohod representation theorem, there exists processes $\tilde{X}_n(\cdot), n \geq 1$ and $X(\cdot)$ on a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ with laws P_n and μ respectively such that $\tilde{X}_n(\cdot) \to X(\cdot)$ a.s. in $C([0,\ T];\mathbb{R}^d)$. Now for each $f \in C_b^2(\mathbb{R}^d),\ g \in C_b^2(\mathbb{R}^d \times \cdots \times \mathbb{R}^d), 0 \leq t_1 < t_2 < \cdots < t_m \leq s \leq t$, we have

$$E^{\tilde{P}}\Big[\Big(f(\tilde{X}_n(t)) - f(\tilde{X}_n(s) - \int_s^t L_n f(\tilde{X}_n(u))du\Big) g(\tilde{X}_n(t_1), \cdots, \tilde{X}_n(t_m))\Big] = 0, \tag{3.9}$$

where

$$L_n f = \langle b_n(x), \nabla f \rangle + \frac{1}{2} trace(\sigma_n \sigma_n^t \nabla^2 f).$$

Now by letting $n \to \infty$ in (3.9) we see that $X(\cdot)$ is a solution to the Martingale problem corresponding to (2.1) with initial distribution δ_x. This proves the existence of a weak solution to (2.1).

Now from the Yamada-Watanabe theorem (see Corollary 3.23 of [14], p. 310-311), which says that weak existence together with pathwise uniqueness implies the existence of a pathwise unique strong solution, we conclude that (2.1) has a pathwise unique strong solution. □

Remark 3.1 : We note that the theorem implies in particular the strong Markov property for the solutions $(X(t))$ of equation (2.1).

4. A Class of Coefficients Satisfying (A1)

In this section, we explore the condition (A1) and give classes of SDEs which come under the framework of Theorem 3.4. Since we are interested in irregular coefficients we will restrict the range of α to $0 \le \alpha \le 1$ (see remarks after statement of (A1) in Sect. 2). First we give a sufficient condition in terms of the Fourier transform of the coefficients. We use Hermite-Sobolev spaces over the complex field $\mathcal{S}_\alpha(\mathbb{R}^d; \mathbb{C})$ (see the remarks following the definition of Hermite-Sobolev spaces in Sect. 2 for a definition). We use the notation $\hat{\phi}$ of the Fourier transform of a distribution ϕ. Recall that for $\phi \in \mathcal{S}, \hat{\phi}(\xi) := (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-i\xi \cdot x} \phi(x)\, dx$.

***Theorem* 4.1** — *For $0 \le \alpha \le 1$, the following is a sufficient condition for (A1). There exists $y \in \mathcal{S}_{-\alpha+1}(\mathbb{R}^d)$ such that for $1 \le i, j \le d$, $\bar{f} := \bar{b}_i, \bar{\sigma}_{ij}$, satisfies*

$$\bar{f} \in \mathcal{S}_\alpha(\mathbb{R}^d),\ \frac{\hat{\bar{f}}}{\hat{y^-}} \in \mathcal{S}_\alpha(\mathbb{R}^d; \mathbb{C}),$$

where $y^-(u) = y(-u), u \in \mathbb{R}^d$.

Proof : Define f as the inverse Fourier transform of $\frac{\hat{\bar{f}}}{\hat{y^-}}$. Hence from the hypothesis, it follows that $f \in \mathcal{S}_\alpha(\mathbb{R}^d)$; here we use the fact that the Fourier transform $\phi \to \hat{\phi} : \mathcal{S}_\alpha(\mathbb{R}^d; \mathbb{C}) \to \mathcal{S}_\alpha(\mathbb{R}^d; \mathbb{C})$ is an onto isometry, see [26], Lemma (1.1.3) for example. Hence

$$\hat{f}\hat{y^-} = \hat{\bar{f}}.$$

Taking inverse Fourier transform we get

$$\bar{f}(x) \ = \ y^- * f(x), x \in \mathbb{R}^d.$$

Since $\tau_x y \in \mathcal{S}_{-\alpha+1}(\mathbb{R}^d) \subseteq L^2(\mathbb{R}^d)$ and $f \in \mathcal{S}_\alpha(\mathbb{R}^d) \subseteq L^2(\mathbb{R}^d)$, it follows that

$$\bar{f}(x) \ = \ \langle f, \tau_x y \rangle, \ x \in \mathbb{R}^d,$$

where the dual bracket is given by the duality pair $(\mathcal{S}_\alpha(\mathbb{R}^d), \ \mathcal{S}_{-\alpha}(\mathbb{R}^d))$. □

We illustrate the conditions in Theorem 4.1 with the following example.

Proposition 4.1 — Assume (A2) and suppose the coefficients $\bar{b}_i, \bar{\sigma}_{ij} \in \mathcal{S}_1(\mathbb{R}^d), \ i,j = 1, \cdots, d$ and

$$\Pi_{k=1}^d \frac{u_k}{\sin u_k} \hat{\bar{b}}_i, \ \Pi_{k=1}^d \frac{u_k}{\sin u_k} \hat{\bar{\sigma}}_{ij} \in \mathcal{S}_1(\mathbb{R}^d; \mathbb{C}), \ i,j = 1, \cdots, d.$$

Then the SDE (2.1) has a pathwise unique strong solution.

PROOF : Take $\alpha = 1$ in Theorem 4.1. For $y \in L^2(\mathbb{R}^d)$, a sufficient condition for (A1) is

$$\bar{b}_i, \bar{\sigma}_{ij} \in \mathcal{S}_1(\mathbb{R}^d), \frac{\hat{\bar{b}}_i}{\hat{y^-}}, \frac{\hat{\bar{\sigma}}_{ij}}{\hat{y^-}} \in \mathcal{S}_1(\mathbb{R}^d; \mathbb{C}), \ i,j = 1, \cdots d.$$

In particular choose

$$y(x) \ = \ \Pi_{i=1}^d I_{[-1, \ 1]}(x_i), \ x = (x_1, \cdots, x_d) \in \mathbb{R}^d,$$

where $I_{[-1, \ 1]}$ denote the indicator function for the interval $[-1, \ 1]$. Then clearly $y \in L^2(\mathbb{R}^d), \ y^- = y$ and

$$\hat{y}(u) \ = \ 2^d \Pi_{i=1}^d \frac{\sin u_i}{u_i}, \ u = (u_1, \cdots, u_d) \in \mathbb{R}^d.$$

Hence the following becomes a sufficient condition for (A1).

The coefficients $\bar{b}_i, \bar{\sigma}_{ij} \in \mathcal{S}_1(\mathbb{R}^d), \ i,j = 1, \cdots, d$ and

$$\Pi_{k=1}^d \frac{u_k}{\sin u_k} \hat{\bar{b}}_i, \ \Pi_{k=1}^d \frac{u_k}{\sin u_k} \hat{\bar{\sigma}}_{ij} \in \mathcal{S}_1(\mathbb{R}^d; \mathbb{C}), \ i,j = 1, \cdots, d.$$

Now from Theorem 3.4, it follows that (2.1) has a pathwise unique strong solution. □

Remark 4.1 : The regularity of coefficients is often expressed in terms of the classical Sobolev spaces $W^{p,q}(\mathbb{R}^d)$. See for example the results in [15] and [16]. In these articles the existence of unique solutions in the weak sense are established but with coefficients whose regularity is expressed in terms of these spaces. In [15] it is assumed that $\bar{b} \in (W^{1,1}_{loc}(\mathbb{R}^d))^d, \ \nabla \cdot \bar{b} \in L^\infty(\mathbb{R}^d), \ \frac{\bar{b}}{1+|x|} \in L^1(\mathbb{R}^d) \bigcap L^\infty(\mathbb{R}^d)$ and $\bar{\sigma} \equiv I_d, \ I_d$ denotes the $d \times d$ identity matrix. In [16], the authors prove the existence of pathwise unique solutions when $d = 1, b = 0$ but $\bar{\sigma} \in W^{1,2}(\mathbb{R})$. In our case, it is easy to see that $\mathcal{S}_1(\mathbb{R}^d) \subseteq W^{2,2}(\mathbb{R}^d)$.

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