Indian J. Pure Appl. Math., **47**(2): 357-388, June 2016

DOI: 10.1007/s13226-016-0192-5

$GI/M/1$ TYPE QUEUEING-INVENTORY SYSTEMS WITH POSTPONED WORK, RESERVATION, CANCELLATION AND COMMON LIFE TIME

A. Krishnamoorthy[1], Dhanya Shajin[1] and B. Lakshmy

Department of Mathematics, Cochin University of Science and Technology,

Cochin 682 022, *India*

e-mails: achyuthacusat@gmail.com; dhanya.shajin@gmail.com; lakshmykrishnaiyer@yahoo.com

(*Received* 22 *May* 2015; *accepted* 28 *September* 2015)

In this paper we analyze two single server queueing-inventory systems in which items in the inventory have a random common life time. On realization of common life time, all customers in the system are flushed out. Subsequently the inventory reaches its maximum level S through a (positive lead time) replenishment for the next cycle which follows an exponential distribution. Through cancellation of purchases, inventory gets added until their expiry time; where cancellation time follows exponential distribution. Customers arrive according to a Poisson process and service time is exponentially distributed. On arrival if a customer finds the server busy, then he joins a buffer of varying size. If there is no inventory, the arriving customer first try to queue up in a finite waiting room of capacity K. Finding that at full, he joins a pool of infinite capacity with probability γ $(0 < \gamma < 1)$; else it is lost to the system forever. We discuss two models based on 'transfer' of customers from the pool to the waiting room / buffer. In Model 1 when, at a service completion epoch the waiting room size drops to preassigned number $L - 1$ $(1 < L < K)$ or below, a customer is transferred from pool to waiting room with probability p $(0 < p < 1)$ and positioned as the last among the waiting customers. If at a departure epoch the waiting room turns out to be empty and there is at least one customer in the pool, then the one ahead of all waiting in the pool gets transferred to the waiting room with probability one. We introduce a totally different transfer mechanism in Model 2: when at a service completion epoch, the server turns idle with at least one item in the inventory, the pooled customer is immediately taken for service. At the time of a cancellation if the server is idle with none, one or more customers in the waiting room, then the head of the pooled customer go to the buffer directly for service. Also we assume that no customer joins the system when there is no item in the inventory. Several system performance measures are obtained. A cost function is discussed for each model and some numerical illustrations are presented. Finally a comparison of the two models are made.

Key words : Flush out; reservation; cancellation; common life time; queueing-inventory.

---

[1]Research supported by Kerala State Council for Science, Technology & Environment (No. 001/KESS/2013/CSTE).

1. Introduction

In this paper we consider a queueing-inventory problem with reservation of inventoried items, cancellation of the reservation and common life time (CLT) of items in the inventory. The common life time plays a crucial role unlike the perishable inventory problems investigated in literature. Perishability or decay of items need not affect simultaneously all stored and sold items in a cycle (for definition of cycle see the end of this section). However, this is not the case with the problem that we investigate in this paper. We consider the case of common life time for all items in the inventory along with those sold in the same cycle. In other words these perish simultaneously – they are no more usable. This is the case with air plane/train/bus tickets for journey by a specified flight/train/bus. Here the seats are considered as inventory. Whether they are sold or otherwise, once the flight/train/bus departs, the vacant seats and those which are sold but the passenger does not show up at the time of departure, all fall in the "no use category". Recently this type of problem is investigated in a very special case in Krishnamoorthy *et al*. [7]. Assuming all underlying distributions to be exponential, the authors analyze the problem as a quasi-birth-and-death (QBD) process where only the 'phases' and not 'levels' disappear from the system on realization of CLT. We elaborate on 'phases' and 'levels' a bit later. With the assumption of flush out of all customers from the system on realization of CLT, what we get is a $GI/M/1$ type Markov chain.

The flush out/reneging of all customers on realization of CLT could be seen in several day to day phenomena. For example, people waiting for tickets for travel by a specific train, for watching a specific show of a movie and so on, tend to leave the system when they do not get tickets for the intended purpose. Those who do not get tickets until realization of CLT leave the service area.

Also replenishment of items within a cycle (that is, starting from a fresh batch of inventory until realization of CLT) is meaningless since if we go for a replenishment when inventory goes to zero, the CLT of these fresh replenishments do not match with that of the earlier ones (of course, here we assumed CLT to be exponential and hence does not create serious problem). Further cancellation process may now lead to inventory level going above S. It may also be observed that in a bus/train/airplane the number of seats are fixed and so a replenishment of inventory in that context is meaningless.

For the model discussed in Krishnamoorthy *et al*. [7], the stability condition was to be investigated. It is obvious that the present system is stable because of the immediate departure of all waiting customers from the system on realization of CLT. A look at the infinitesimal generator of the process shows that the left most column (starting from the second row) has all elements positive. This is a special case of the $GI/M/1$ type situation. A general $GI/M/1$ type situation needs checking of stability. In our case the system is always stable since the CLT of inventory has a finite mean and the number of customers joining during this time is finite with probability 1. Further unsatisfied customers leave the system forever.

The main difference between the model(s) discussed in this paper and the one considered in Krishnamoorthy *et al.* [7], despite having several common features, are:

(i) In Krishnamoorthy *et al.* [7] orbital customers are not flushed out of the system on realization of CLT whereas customers in the pool are also flushed out in the model described in this paper. As a consequence, whereas the infinitesimal generator of the continuous time Markov chain (CTMC) in Krishnamoorthy *et al.* [7] has a quasi-Toeplitz structure with the repeating part starting in row 2, the Markov chain in the present case does not have this nice structure.

(ii) Model described by Krishnamoorthy *et al.* [7] could be stable only under prescribed conditions. In contrast the model described here is stable for any traffic intensity, however, large. The latter is the consequence of the flush out of customers from the pool as well, thereby the system become totally empty – that is to say the system is devoid of inventory and customers on realization of common life time. The new cycle then starts.

(iii) Whereas lead time for inventory replenishment in Krishnamoorthy *et al.* [7] is zero, in the present case it has exponential distribution. All these lead to a totally different structure for the function to be optimized in the case of the present model. The flush out of all customers from the system on realization of CLT is necessitated by the fact that they all want to have inventory in the present cycle.

Thus a much more complex system is studied in this paper. Going back to Krishnamoorthy *et al.* [7], we note that whereas the authors assume retrial of customers from an orbit when they are not able to get into the buffer or waiting room, we assume a more visible entity called pool for such customers in the present work. This has the advantage of customers in the pool knowing the status of the buffer as also waiting room as well as the server getting complete information on the pool. This aspect will enable the service system to design a transfer mechanism of customers from the pool.

Note that there were some errors in the expressions for E_{NC}, E_{NP}, P_{full} and P_{vacant} (see pages 8, 9 for definition) in the model described in Krishnamoorthy *et al*. [7]. The authors have subsequently corrected those.

Before proceeding further, we provide a brief survey on queueing-inventory models. Its origin dates back to 1992 with Sigman and Levi [17] introducing the M/G/1 queueing-inventory model with exponentially distributed lead time under light traffic. This was followed by contributions from Berman *et al*. [2], Berman and Kim [3], Berman and Sapna [4], Arivarignan *et al*. [1], Krishnamoorthy *et al*. [8] and by several other researchers. In Krishnamoorthy *et al*. [8] the authors provide a stochastic decomposition of the system under study; neverthless, it is not a big surprise since the inventory replenishment lead time is assumed to be zero though the N-policy is brought in. Thus Schwarz *et al*. [14] stand out as the first significant contribution providing stochastic decomposition of the system state of a queueing-inventory problem. Krishnamoorthy and Viswanath [9] brings in production of items for inventory thereby subsuming Schwarz *et al*. [14]. The latter is also subsumed by Saffari *et al*. [13] in that the lead time is arbitrarily distributed. Further contribution with stochastic decomposition results could be found in Schwarz *et al*. [16], Schwarz and Daduna [15], Krenzler and Daduna [6], Otten *et al*. [12].

Though our concern in this paper is not stochastic decomposition of system state, we wanted to bring to the notice of the readers some of the finest contributions in the queueing-inventory concerning stochastic decomposition.

This paper is organized as follows. The section to follow provides the model description. In fact two models are analyzed. Mathematical formulation of Model 1 is taken up in Section 2. That section also contains key performance characteristics of the system. This is followed by the Mathematical formulation of Model 2 in Section 3. Evaluation of its performance is also indicated in that section. Section 4 analyzes numerically an objective function – the objective being cost minimization / profit maximization.

Some notations, abbreviations and definitions used in the sequel:

- $N_1(t)$ = Number of customers with in the pool at time t.
- $N_2(t)$ = Number of customers in the waiting room at time t.
- $N_3(t)$ = Number of customers in the buffer (including in service) at time t.

- $I(t)$ = Number of items in the inventory at time t.
- $u(t) = \begin{cases} 0; & \text{if server is idle at time } t, \\ 1; & \text{pooled customer in service at time } t, \\ 2; & \text{customer in service not from the pool at time } t \end{cases}$.
- **e** = Column vector of $1'$s with appropriate order.
- $\mathbf{e}_i'$ = Row vector with 1 is in the i^{th} position and remaining elements are zero.
- $U_1 = (S+1)(S+2)/2 + K(S+1)$.
- $U_2 = K(S+1)$.
- $U_3 = (S+1)^2 + K(2S+1)$.
- $U_4 = S(S+1) + K(2S+1) + 1$.
- $CTMC$: continuous time Markov chain.
- QBD : quasi-birth and death process.
- CLT : common life time.
- **GI/M/1 type** queue: see Neuts [10], [11] for details.
- **Cycle**: The time duration from the epoch at which we start with maximum inventory level S at a replenishment epoch, to the moment when the common life time is realized.
- **Lead time**: On expiry of common life time, the inventory level reaches its maximum S through a replenishment for the next cycle. The time elapsed between realization of CLT of a batch to the epoch at which the replenishment takes place for the next cycle, is called lead time.

2. Mathematical Formulation: Model 1

We have a single commodity inventory system with S items at the beginning of a cycle. Customers arrive according to a Poisson process of rate λ demanding exactly one unit of item (extension to demand for more than one item by a customer is straight forward). To deliver the item to the customer in service, it requires an exponentially distributed time with parameter μ. The inventoried items have a common life time which means that they all perish together on realization of this time. Examples are indicated in the introduction (another example is drugs that are manufactured in a batch). We assume that this common life time is exponentially distributed with parameter α. On realization of common life time the process of ordering for inventory replenishment starts. The physical arrival of items takes an exponentially distributed amount of time having parameter η. The quantity of replenishment is S. A buffer of varying size, depending on the number of items in the inventory is available near the service counter. We call it varying size because at most as many customers as the number of items in the inventory are allowed to be in this buffer. In addition the possibility of cancellation of purchase (return of the item with a penalty) is introduced here. Inter cancellation time follows exponential distribution with parameter $i\beta$, when there are i items in the purchased list in the current cycle (that is, there are $(S-i)$ items are in the inventory). Next in order is a finite waiting space of capacity K. When the buffer is full further arrivals wait in this room; as and when inventory level in the buffer goes above (due to cancellation), the head in the waiting room moves to the buffer and positions himself as the last there. When the waiting room is also full, further arrivals are directed to a pool (of customers) having infinite capacity. Whereas customers join with probability one in the buffer and waiting room whenever there is a vacancy, it is not the case with the pool. An arrival finding waiting room also full joins the pool with probability $\gamma(0<\gamma<1)$ or balks with complementary probability.

We introduce a transfer mechanism of customers from pool to waiting room as follows: when, at a departure epoch the number of customers in the waiting room drops to a preassigned number $L-1, (1<L<K)$ or below, a customer is transferred from the pool to the waiting room with probability $p\ (0<p<1)$ and positioned as last among the waiting customers. If at a service completion epoch the waiting room turns out to be empty and there is at least one customer in the pool, the one ahead of all waiting in the pool gets transferred (with probability one) to the waiting room. Transfer of customers from a pool is introduced and analyzed in Deepak *et al.* [5].

It is in the transfer mechanism that the two models discussed in this paper differ. This mechanism for Model 2 is discussed at the appropriate place in Section 3.

Further all customers are flushed out from the system (finite buffer, waiting room and pool) when the common life time is realized.

By the above assumptions $\Omega = \{(N_1(t), N_2(t), I(t), N_3(t))\,, t\geq 0\}$ is a $CTMC$. Its state space is given by

$$\{\Delta\}\bigcup\{(0,0,i,n_3)\,; 0\leq i\leq S, 0\leq n_3\leq i\}$$

$$\bigcup \{(n_1, n_2, i, n_3)\,; n_1 \geq 0, 1 \leq n_2 \leq K, 0 \leq i \leq S, n_3 = i\}$$

where $\{\Delta\}$ denotes the temporary absorbing state. The so called 'temporary absorbing' state $\{\Delta\}$ indicates that there is no inventory in the system consequent to realization of CLT and so no customer in the system due to flush out / reneging. The system stays there for an exponentially distributed amount of time (namely the lead time). It may happen that within a cycle (before realization of CLT) there is no inventory in the system as also no customer. However, this state of the system is different form $\{\Delta\}$. Thus we need to distinguish between the two, for which purpose the symbol $\{\Delta\}$ was brought in. The infinitesimal generator Q is of the form

$$Q = \begin{bmatrix} A_{\Delta} & A_{\Delta 0} & & & & \\ A_{0\Delta} & A_{00} & A_{01} & & & \\ A_2' & A_{10} & A_1 & A_0 & & \\ A_2' & & A_2 & A_1 & A_0 & \\ A_2' & & & A_2 & A_1 & A_0 \\ \vdots & & & & \ddots & \ddots & \ddots \end{bmatrix}.$$

The matrices A_0, A_1, A_2 are square matrices of the same order U_2 with A_0 representing transition from level n_1 to $n_1 + 1, n_1 \geq 1$, A_1 represents the transitions within the level $n_1, n_1 \geq 1$ and A_2 contains transition rates from level n_1 to $n_1 - 1, n_1 \geq 2$. Dimension of matrices A_{00}, A_{01}, A_{10} are $U_1 \times U_1, U_1 \times U_2, U_2 \times U_1$ respectively. Matrices $A_{0\Delta}, A_2'$ are column vectors of order U_1, U_2 respectively. $A_{\Delta 0}$ is a row vector of order U_1.

$$A_{\Delta} = -\eta, A_{\Delta 0} = \eta \mathbf{e}'_{S(S+1)/2+1}, A_{0\Delta} = \alpha \mathbf{e}, A_2' = \alpha \mathbf{e}.$$

Define $A_k^{(n_2,i,n_3:m_2,j,m_3)}$, $k = 00, 01, 10, 0, 1, 2$ as the transition rates from $(n_2, i, n_3) \rightarrow (m_2, j, m_3)$ where n_2, m_2 represent the number of customers in the waiting room, i, j represent the number of items in the inventory and n_3, m_3 represent the number of customers in the buffer. These transition rates are

$$A_{01}^{(n_2,i,n_3:m_2,j,m_3)} = \begin{cases} \gamma\lambda, & \text{for} \quad n_2 = K, 0 \leq i \leq S, n_3 = i; \\ & \qquad m_2 = n_2, j = i, m_3 = n_3, \\ 0, & \qquad \text{otherwise.} \end{cases}$$

$$A_{10}^{(n_2,i,n_3:m_2,j,m_3)} = \begin{cases} (S-i)\beta, & \text{for} \quad n_2 = 1, 0 \leq i \leq S-1, n_3 = i; \\ & \qquad m_2 = n_2, j = i+1, m_3 = n_3 + 1, \\ p(S-i)\beta, & \text{for} \quad 2 \leq n_2 \leq L, 0 \leq i \leq S-1, n_3 = i; \\ & \qquad m_2 = n_2, j = i+1, m_3 = n_3 + 1, \\ 0, & \qquad \text{otherwise.} \end{cases}$$

$$A_0^{(n_2,i,n_3:m_2,j,m_3)} = \begin{cases} \gamma\lambda, & \text{for} \quad n_2 = K, 0 \leq i \leq S, n_3 = i; \\ & \quad\quad m_2 = n_2, j = i, m_3 = n_3, \\ 0, & \quad\quad \text{otherwise}. \end{cases}$$

$$A_2^{(n_2,i,n_3:m_2,j,m_3)} = \begin{cases} (S-i)\beta, & \text{for} \quad n_2 = 1, 0 \leq i \leq S-1, n_3 = i; \\ & \quad\quad m_2 = n_2, j = i+1, m_3 = n_3+1, \\ p(S-i)\beta, & \text{for} \quad 2 \leq n_2 \leq L, 0 \leq i \leq S-1, n_3 = i; \\ & \quad\quad m_2 = n_2, j = i+1, m_3 = n_3+1, \\ 0, & \quad\quad \text{otherwise}. \end{cases}$$

$$A_1^{(n_2,i,n_3:m_2,j,m_3)} = \begin{cases} \lambda, & \text{for} \quad 1 \leq n_2 \leq K-1, 0 \leq i \leq S, n_3 = i; \\ & \quad\quad m_2 = n_2+1, j = i, m_3 = n_3, \\ \mu, & \text{for} \quad 1 \leq n_2 \leq K, 1 \leq i \leq S, n_3 = i; \\ & \quad\quad m_2 = n_2, j = i-1, m_3 = n_3-1, \\ (1-p)(S-i)\beta, & \text{for} \quad 2 \leq n_2 \leq L, 0 \leq i \leq S-1, n_3 = i; \\ & \quad\quad m_2 = n_2-1, j = i+1, m_3 = n_3+1, \\ (S-i)\beta, & \text{for} \quad L+1 \leq n_2 \leq K, 0 \leq i \leq S-1, n_3 = i; \\ & \quad\quad m_2 = n_2-1, j = i+1, m_3 = n_3+1, \\ -(\lambda + S\beta + \alpha), & \text{for} \quad 1 \leq n_2 \leq K-1, i = 0, n_3 = 0; \\ & \quad\quad m_2 = n_2, j = i, m_3 = n_3, \\ -(\lambda + \mu + (S-i)\beta + \alpha), & \text{for} \quad 1 \leq n_2 \leq K-1, 1 \leq i \leq S, n_3 = i; \\ & \quad\quad m_2 = n_2, j = i, m_3 = n_3, \\ -(\gamma\lambda + S\beta + \alpha), & \text{for} \quad n_2 = K, i = 0, n_3 = 0; \\ & \quad\quad m_2 = n_2, j = i, m_3 = n_3, \\ -(\gamma\lambda + \mu + (S-i)\beta + \alpha), & \text{for} \quad n_2 = K, 1 \leq i \leq S, n_3 = i; \\ & \quad\quad m_2 = n_2, j = i, m_3 = n_3, \\ 0, & \quad\quad \text{otherwise}. \end{cases}$$

$$A_{00}^{(n_2,i,n_3:m_2,j,m_3)} = \begin{cases} \lambda, & \text{for} \quad n_2 = 0, 0 \leq i \leq S, n_3 = i; \\ & \quad\quad m_2 = n_2+1, j = i, m_3 = n_3, \\ \lambda, & \text{for} \quad n_2 = 0, 1 \leq i \leq S, 0 \leq n_3 \leq i-1; \\ & \quad\quad m_2 = n_2, j = i, m_3 = n_3+1, \\ \lambda, & \text{for} \quad 1 \leq n_2 \leq K-1, 0 \leq i \leq S, n_3 = i; \\ & \quad\quad m_2 = n_2+1, j = i, m_3 = n_3, \\ \mu, & \text{for} \quad n_2 = 0, 1 \leq i \leq S, 1 \leq n_3 \leq i; \\ & \quad\quad m_2 = n_2, j = i-1, m_3 = n_3-1, \\ \mu, & \text{for} \quad 1 \leq n_2 \leq K, 1 \leq i \leq S, n_3 = i; \\ & \quad\quad m_2 = n_2, j = i-1, m_3 = n_3-1, \\ (S-i)\beta, & \text{for} \quad n_2 = 0, 0 \leq i \leq S-1, 0 \leq n_3 \leq i; \\ & \quad\quad m_2 = n_2, j = i+1, m_3 = n_3, \\ (S-i)\beta, & \text{for} \quad 1 \leq n_2 \leq K, 0 \leq i \leq S-1, n_3 = i; \\ & \quad\quad m_2 = n_2-1, j = i+1, m_3 = n_3+1, \\ -(\lambda + (S-i)\beta + \alpha), & \text{for} \quad n_2 = 0, 0 \leq i \leq S, n_3 = 0; \\ & \quad\quad m_2 = 0, j = i, m_3 = n_3, \\ -(\lambda + \mu + (S-i)\beta + \alpha), & \text{for} \quad n_2 = 0, 1 \leq i \leq S, 1 \leq n_3 \leq i; \\ & \quad\quad m_2 = 0, j = i, m_3 = n_3, \\ -(\lambda + S\beta + \alpha), & \text{for} \quad 1 \leq n_2 \leq K-1, i = 0, n_3 = 0; \\ & \quad\quad m_2 = n_2, j = i, m_3 = n_3, \\ -(\lambda + \mu + (S-i)\beta + \alpha), & \text{for} \quad 1 \leq n_2 \leq K-1, 1 \leq i \leq S, n_3 = i; \\ & \quad\quad m_2 = n_2, j = i, m_3 = n_3, \\ -(\gamma\lambda + S\beta + \alpha), & \text{for} \quad n_2 = K, i = 0, n_3 = 0; \\ & \quad\quad m_2 = n_2, j = i, m_3 = n_3, \\ -(\gamma\lambda + \mu + (S-i)\beta + \alpha), & \text{for} \quad n_2 = K, 1 \leq i \leq S, n_3 = i; \\ & \quad\quad m_2 = n_2, j = i, m_3 = n_3, \\ 0, & \quad\quad \text{otherwise}. \end{cases}$$

2.1 *Steady-state analysis*

In this section, we perform the steady-state analysis of the queueing-inventory model described above.

Let **x** be the steady-state probability vector of generator Q. Then we have

$$\mathbf{x}Q = 0, \quad \mathbf{x}\mathbf{e} = 1. \tag{1}$$

Partitioning **x** as $\mathbf{x} = (x_\Delta, \mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, ...)$ and then each of the sub-vectors as

$$\mathbf{x}_0 = (x_0(0, i, n_3), x_0(n_2, i, i); 0 \le i \le S, 0 \le n_3 \le i, 1 \le n_2 \le K),$$

$$\mathbf{x}_{n_1} = (x_{n_1}(n_2, i, i); 0 \le i \le S, 1 \le n_2 \le K), \text{ for } n_1 \ge 1,$$

we see that **x** is obtained as (see Neuts [10])

$$\mathbf{x}_{n_1} = \mathbf{x}_1 R^{n_1 - 1}, \quad n_1 \ge 2$$

where R is the minimal nonnegative solution to the matrix equation:

$$\sum_{k=0}^{2} R^k A_k = 0$$

and the boundary equations are given by

$$x_\Delta A_{\Delta 0} + \mathbf{x}_0 A_{00} + \mathbf{x}_1 A_{10} = 0,$$

$$\mathbf{x}_0 A_{01} + \mathbf{x}_1 [A_1 + R A_2] = 0,$$

$$x_\Delta = \frac{\alpha}{\eta} \sum_{n_1=0}^{\infty} \mathbf{x}_{n_1} \mathbf{e}.$$

The normalizing condition (1) gives

$$x_\Delta + \mathbf{x}_0 \mathbf{e} + \mathbf{x}_1 [I - R]^{-1} \mathbf{e} = 1.$$

The system state probabilities computed above provide the following useful information about the system.

- Expected number of customers in the pool before realization of common life time

$$E_P(N) = \sum_{n_1=1}^{\infty} n_1 \mathbf{x}_{n_1} \mathbf{e}.$$

- Expected number of customers in the waiting room before realization of common life time

$$E_W(N) = \sum_{n_1=0}^{\infty} \sum_{n_2=1}^{K} \sum_{i=0}^{S} n_2 x_{n_1}(n_2, i, i).$$

- Expected number of customers in the buffer before realization of common life time

$$E_B(N) = \sum_{n_1=0}^{\infty} \sum_{n_2=1}^{K} \sum_{i=1}^{S} i x_{n_1}(n_2, i, i) + \sum_{i=1}^{S} \sum_{n_3=1}^{i} n_3 x_0(0, i, n_3).$$

- Expected number of items in the inventory before realization of common life time

$$E_I(N) = \sum_{n_1=0}^{\infty} \sum_{n_2=1}^{K} \sum_{i=1}^{S} i x_{n_1}(n_2, i, i) + \sum_{i=1}^{S} \sum_{n_3=0}^{i} i x_0(0, i, n_3).$$

- Expected number of items in the inventory immediately on realization of common life time

$$E_I'(N) = \sum_{n_1=0}^{\infty} \sum_{n_2=1}^{K} \sum_{i=1}^{S} i \frac{\alpha}{\alpha + \lambda + \mu + (S-i)\beta} x_{n_1}(n_2, i, i) +$$

$$\sum_{i=1}^{S} i \frac{\alpha}{\alpha + \lambda + (S-i)\beta} x_0(0, i, 0) + \sum_{i=1}^{S} \sum_{n_3=1}^{i} i \frac{\alpha}{\alpha + \lambda + \mu + (S-i)\beta} x_0(0, i, n_3).$$

- Rate of addition to the pool is

$$\gamma\lambda \sum_{n_1=0}^{\infty} \sum_{i=0}^{S} x_{n_1}(K, i, i).$$

- The probability that a customer enters service immediately on arrival

$$\sum_{i=1}^{S} x_0(0, i, 0).$$

- The rate at which pooled customers are transferred to the waiting room

$$E_{PW}(R) = \sum_{n_1=1}^{\infty} \sum_{i=0}^{S-1} (S-i)\beta \left[\sum_{n_2=2}^{L} p x_{n_1}(n_2, i, i) + x_{n_1}(1, i, i) \right].$$

- The rate at which customers abandon the system on arrival

$$E_{WL}(R) = (1-\gamma)\lambda \sum_{n_1=0}^{\infty} \sum_{i=0}^{S} x_{n_1}(K, i, i).$$

- Expected cancellation rate

$$E_C(R) = \sum_{n_1=0}^{\infty} \sum_{n_2=1}^{K} \sum_{i=0}^{S-1} (S-i)\beta x_{n_1}(n_2, i, i) + \sum_{i=0}^{S-1} \sum_{n_3=0}^{i} (S-i)\beta x_0(0, i, n_3).$$

- Expected inventory depletion rate

$$E_P(R) = \mu \left\{ \sum_{n_1=0}^{\infty} \sum_{n_2=1}^{K} \sum_{i=1}^{S} x_{n_1}(n_2, i, i) + \sum_{i=1}^{S} \sum_{n_3=1}^{i} x_0(0, i, n_3) \right\}.$$

- Expected number of cancellations in a cycle

$$E_{NC} = \frac{1}{\alpha} \left\{ \sum_{n_1=0}^{\infty} \sum_{n_2=1}^{K} \sum_{i=0}^{S-1} (S-i)\beta x_{n_1}(n_2, i, i) + \sum_{i=0}^{S-1} \sum_{n_3=0}^{i} (S-i)\beta x_0(0, i, n_3) \right\}.$$

- Expected number of purchases in a cycle

$$E_{NP} = \frac{1}{\alpha} \mu \left\{ \sum_{n_1=0}^{\infty} \sum_{n_2=1}^{K} \sum_{i=1}^{S} x_{n_1}(n_2, i, i) + \sum_{i=1}^{S} \sum_{n_3=1}^{i} x_0(0, i, n_3) \right\}.$$

- Expected number of transfers from the pool to the waiting room

$$E_{PW}(N) = \frac{1}{\alpha} \sum_{n_1=1}^{\infty} \sum_{i=0}^{S-1} (S-i)\beta \left[\sum_{n_2=2}^{L} p x_{n_1}(n_2, i, i) + x_{n_1}(1, i, i) \right].$$

- The probability that the system has S items in the inventory at the time of realization of common life time

$$P_{vacant} = \sum_{n_3=1}^{S} x_0(0, S, n_3) + \sum_{n_1=0}^{\infty} \sum_{n_2=1}^{K} x_{n_1}(n_2, S, S) + x_0(0, S, 0).$$

  This is equal to the probability, for example,, that a bus with S seats depart without any passenger on board.

- The probability that the system is left with no item in the inventory at the time of realization of common life time

$$P_{full} = x_0(0, 0, 0) + \sum_{n_1=0}^{\infty} \sum_{n_2=1}^{K} x_{n_1}(n_2, 0, 0).$$

  This is equivalent to the probability that bus referred to in the previous item leaves with full capacity.

3. Mathematical Formulation: Model 2

In this section we describe an inventory problem in which further restrictions are imposed on model 1 described in Section 2. The following are the additional restrictions:

(a) No customer joins the system when inventory level is zero.

(b) When a cancellation of purchased inventory occurs with none, one or more customers waiting in the waiting room and server idle due to no item in the inventory the head of the pool is transferred to the buffer for immediate service.

(c) If at a service completion epoch, there is no customer in the buffer as well as the waiting room, then again the head of the pool is transferred to the buffer for immediate service.

(d) No transfer takes place from the pool/waiting room to the buffer when no item in the inventory.

All other assumptions in model 1 hold for the present model. Assumptions (b), (c) are to reduce the waiting time of customers in the pool to same extent. Thus we get a $CTMC$ $\{(N_1(t), N_2(t), I(t), N_3(t), u(t)), t \geq 0\}$ with state space

$$\Omega' = \{\Delta\} \bigcup \{(0,0,i,0,0)\,; 0 \leq i \leq S\} \bigcup \{(n_1, n_2, 0, 0, 0)\,; n_1 \geq 0, 1 \leq n_2 \leq K\}$$
$$\bigcup \{(n_1, 0, i, n_3, k)\,; n_1 \geq 0, 1 \leq i \leq S, 1 \leq n_3 \leq i, k = 1, 2\}$$
$$\bigcup \{(n_1, n_2, i, i, k)\,; n_1 \geq 0, 1 \leq n_2 \leq K, 1 \leq i \leq S, k = 1, 2\}$$

where $\{\Delta\}$ denotes the temporary absorbing state. In this model $u(t)$ is brought in to identify whether the current service, if any, is of a pooled customer. This is introduced to explicitly compute certain system performance index that would help to control the number of customers in the pool. Thus the infinitesimal generator Q' is of form

$$Q' = \begin{bmatrix} B_\Delta & B_{\Delta 0} & & & & & \\ B_{0\Delta} & B_{00} & B_{01} & & & & \\ B'_2 & B_{10} & B_1 & B_0 & & & \\ B'_2 & & B_2 & B_1 & B_0 & & \\ B'_2 & & & B_2 & B_1 & B_0 & \\ \vdots & & & & \ddots & \ddots & \ddots \end{bmatrix}.$$

The matrices B_0, B_1, B_2 are square matrices of the same order U_4 with B_0 representing transition from level n_1 to $n_1 + 1, n_1 \geq 1$, B_1 represents the transitions within the level $n_1, n_1 \geq 1$ and B_2 contains transition rates from level n_1 to $n_1 - 1, n_1 \geq 2$. Dimension of matrices B_{00}, B_{01}, B_{10} are $U_3 \times U_3, U_3 \times U_4, U_4 \times U_3$ respectively. Matrices $B_{0\Delta}, B'_2$ are column vectors of order U_3, U_4 respectively. $B_{\Delta 0}$ is a row vector of order U_3.

$$B_\Delta = -\eta, B_{\Delta 0} = \eta \mathbf{e}'_{S^2+1}, B_{0\Delta} = \alpha \mathbf{e}, B'_2 = \alpha \mathbf{e}.$$

Define $B_l^{(n_2,i,n_3,k_1:m_2,j,m_3,k_2)}$, $l = 00, 01, 10, 0, 1, 2$ as the transition rates from $(n_2, i, n_3, k_1) \to (m_2, j, m_3, k_2)$ where n_2, m_2 represent the number of customers in the waiting room, i, j represent the number of items in the inventory, n_3, m_3 represent the number of customers in the buffer and k_1, k_2 represent the status of server. These transition rates are

$$B_{01}^{(n_2,i,n_3,k_1:m_2,j,m_3,k_2)} = \begin{cases} \gamma\lambda, & \text{for} \quad n_2 = K, 1 \leq i \leq S, n_3 = i, k_1 = 1, 2; \\ & \qquad m_2 = n_2, j = i, m_3 = n_3, k_2 = k_1, \\ 0, & \qquad \text{otherwise.} \end{cases}$$

$$B_{10}^{(n_2,i,n_3,k_1:m_2,j,m_3,k_2)} = \begin{cases} S\beta, & \text{for} \quad 0 \leq n_2 \leq K, i = 0, n_3 = i, k_1 = 0; \\ & \qquad m_2 = n_2, j = i + 1, m_3 = n_3 + 1, k_2 = 1, \\ \mu, & \text{for} \quad n_2 = 0, 2 \leq i \leq S, n_3 = 1, k_1 = 1, 2; \\ & \qquad m_2 = n_2, j = i - 1, m_3 = n_3, k_2 = 1, \\ 0, & \qquad \text{otherwise.} \end{cases}$$

$$B_{0}^{(n_2,i,n_3,k_1:m_2,j,m_3,k_2)} = \begin{cases} \gamma\lambda, & \text{for} \quad n_2 = K, 1 \leq i \leq S, n_3 = i, k_1 = 1, 2; \\ & \qquad m_2 = n_2, j = i, m_3 = n_3, k_2 = k_1, \\ 0, & \qquad \text{otherwise.} \end{cases}$$

$$B_{00}^{(n_2,i,n_3,k_1:m_2,j,m_3,k_2)} = \begin{cases} \lambda, & \text{for} \quad n_2 = 0, 1 \leq i \leq S, n_3 = i, k_1 = 1, 2; \\ & \qquad m_2 = n_2 + 1, j = i, m_3 = n_3, k_2 = k_1, \\ \lambda, & \text{for} \quad n_2 = 0, 2 \leq i \leq S, 1 \leq n_3 \leq i - 1, k_1 = 1, 2; \\ & \qquad m_2 = n_2, j = i, m_3 = n_3 + 1, k_2 = k_1, \\ \lambda, & \text{for} \quad n_2 = 0, 1 \leq i \leq S, n_3 = 0, k_1 = 0; \\ & \qquad m_2 = n_2, j = i, m_3 = n_3 + 1, k_2 = 2, \\ \lambda, & \text{for} \quad 1 \leq n_2 \leq K - 1, 1 \leq i \leq S, n_3 = i, k_1 = 1, 2; \\ & \qquad m_2 = n_2 + 1, j = i, m_3 = n_3, k_2 = k_1, \\ \mu, & \text{for} \quad n_2 = 0, 1 \leq i \leq S, n_3 = 1, k_1 = 1, 2; \\ & \qquad m_2 = n_2, j = i - 1, m_3 = n_3 - 1, k_2 = 0, \\ \mu, & \text{for} \quad n_2 = 0, 2 \leq i \leq S, 2 \leq n_3 \leq i, k_1 = 1, 2; \\ & \qquad m_2 = n_2, j = i - 1, m_3 = n_3 - 1, k_2 = 2, \\ \mu, & \text{for} \quad 1 \leq n_2 \leq K, i = 1, n_3 = i, k_1 = 1, 2; \\ & \qquad m_2 = n_2, j = i - 1, m_3 = n_3 - 1, k_2 = 0, \\ \mu, & \text{for} \quad 1 \leq n_2 \leq K, 2 \leq i \leq S, n_3 = i, k_1 = 1, 2; \\ & \qquad m_2 = n_2, j = i - 1, m_3 = n_3 - 1, k_2 = 2, \\ (S-i)\beta, & \text{for} \quad n_2 = 0, 0 \leq i \leq S - 1, n_3 = 0, k_1 = 0; \\ & \qquad m_2 = n_2, j = i + 1, m_3 = n_3, k_2 = 0, \\ (S-i)\beta, & \text{for} \quad n_2 = 0, 1 \leq i \leq S - 1, 1 \leq n_3 \leq i, k_1 = 1, 2; \\ & \qquad m_2 = n_2, j = i + 1, m_3 = n_3, k_2 = k_1, \\ S\beta, & \text{for} \quad 1 \leq n_2 \leq K, i = 0, n_3 = i, k_1 = 0; \\ & \qquad m_2 = n_2 - 1, j = i + 1, m_3 = n_3 + 1, k_2 = 2, \\ (S-i)\beta, & \text{for} \quad 1 \leq n_2 \leq K, 1 \leq i \leq S - 1, n_3 = i, k_1 = 1, 2; \\ & \qquad m_2 = n_2 - 1, j = i + 1, m_3 = n_3 + 1, k_2 = k_1, \\ -(S\beta + \alpha), & \text{for} \quad 0 \leq n_2 \leq K, i = 0, n_3 = i, k_1 = 0; \\ & \qquad m_2 = n_2, j = i, m_3 = n_3, k_2 = k_1, \\ -(\lambda + (S-i)\beta + \alpha), & \text{for} \quad n_2 = 0, 1 \leq i \leq S, n_3 = 0, k_1 = 0; \\ & \qquad m_2 = 0, j = i, m_3 = n_3, k - 2 = k_1, \\ -(\lambda + \mu + (S-i)\beta + \alpha), & \text{for} \quad n_2 = 0, 1 \leq i \leq S, 1 \leq n_3 \leq i, k_= 1, 2; \\ & \qquad m_2 = 0, j = i, m_3 = n_3, k_2 = k_1, \\ -(\lambda + \mu + (S-i)\beta + \alpha), & \text{for} \quad 1 \leq n_2 \leq K - 1, 1 \leq i \leq S, n_3 = i, k_1 = 1, 2; \\ & \qquad m_2 = n_2, j = i, m_3 = n_3, k_2 = k_1, \\ -(\gamma\lambda + \mu + (S-i)\beta + \alpha), & \text{for} \quad n_2 = K, 1 \leq i \leq S, n_3 = i, k_1 = 1, 2; \\ & \qquad m_2 = n_2, j = i, m_3 = n_3, k_2 = k_1, \\ 0, & \qquad \text{otherwise.} \end{cases}$$

$$B_1^{(n_2,i,n_3,k_1:m_2,j,m_3,k_2)} = \begin{cases} \lambda, & \text{for} \quad n_2 = 0, 1 \le i \le S, n_3 = i, k_1 = 1,2; \\ & \qquad m_2 = n_2 + 1, j = i, m_3 = n_3, k_2 = k_1, \\ \lambda, & \text{for} \quad n_2 = 0, 2 \le i \le S, 1 \le n_3 \le i-1, k_1 = 1,2; \\ & \qquad m_2 = n_2, j = i, m_3 = n_3 + 1, k_2 = k_1, \\ \lambda, & \text{for} \quad 1 \le n_2 \le K-1, 1 \le i \le S, n_3 = i, k_1 = 1,2; \\ & \qquad m_2 = n_2 + 1, j = i, m_3 = n_3, k_2 = k_1, \\ \mu, & \text{for} \quad n_2 = 0, i = 1, n_3 = 1, k_1 = 1,2; \\ & \qquad m_2 = n_2, j = i-1, m_3 = n_3 - 1, k_2 = 0, \\ \mu, & \text{for} \quad n_2 = 0, 2 \le i \le S, 2 \le n_3 \le i, k_1 = 1,2; \\ & \qquad m_2 = n_2, j = i-1, m_3 = n_3 - 1, k_2 = 2, \\ \mu, & \text{for} \quad 1 \le n_2 \le K, i = 1, n_3 = i, k_1 = 1,2; \\ & \qquad m_2 = n_2, j = i-1, m_3 = n_3 - 1, k_2 = 0, \\ \mu, & \text{for} \quad 1 \le n_2 \le K, 2 \le i \le S, n_3 = i, k_1 = 1,2; \\ & \qquad m_2 = n_2, j = i-1, m_3 = n_3 - 1, k_2 = 2, \\ (S-i)\beta, & \text{for} \quad n_2 = 0, 1 \le i \le S-1, 1 \le n_3 \le i, k_1 = 1,2; \\ & \qquad m_2 = n_2, j = i+1, m_3 = n_3, k_2 = k_1, \\ (S-i)\beta, & \text{for} \quad 1 \le n_2 \le K, 1 \le i \le S-1, n_3 = i, k_1 = 1,2; \\ & \qquad m_2 = n_2 - 1, j = i+1, m_3 = n_3 + 1, k_2 = k_1, \\ -(S\beta + \alpha), & \text{for} \quad 0 \le n_2 \le K, i = 0, n_3 = i, k_1 = 0; \\ & \qquad m_2 = n_2, j = i, m_3 = n_3, k_2 = k_1, \\ -(\lambda + \mu + (S-i)\beta + \alpha), & \text{for} \quad n_2 = 0, 1 \le i \le S, 1 \le n_3 \le i, k_= 1,2; \\ & \qquad m_2 = 0, j = i, m_3 = n_3, k_2 = k_1, \\ -(\lambda + \mu + (S-i)\beta + \alpha), & \text{for} \quad 1 \le n_2 \le K-1, 1 \le i \le S, n_3 = i, k_1 = 1,2; \\ & \qquad m_2 = n_2, j = i, m_3 = n_3, k_2 = k_1, \\ -(\gamma\lambda + \mu + (S-i)\beta + \alpha), & \text{for} \quad n_2 = K, 1 \le i \le S, n_3 = i, k_1 = 1,2; \\ & \qquad m_2 = n_2, j = i, m_3 = n_3, k_2 = k_1, \\ 0, & \qquad \text{otherwise.} \end{cases}$$

$$B_2^{(n_2,i,n_3,k_1:m_2,j,m_3,k_2)} = \begin{cases} S\beta, & \text{for} \quad 0 \le n_2 \le K, i = 0, n_3 = i, k_1 = 0; \\ & \qquad m_2 = n_2, j = i+1, m_3 = n_3 + 1, k_2 = 1, \\ \mu, & \text{for} \quad n_2 = 0, 2 \le i \le S, n_3 = 1, k_1 = 1,2; \\ & \qquad m_2 = n_2, j = i-1, m_3 = n_3, k_2 = 1, \\ 0, & \qquad \text{otherwise.} \end{cases}$$

3.1 *Steady-state analysis*

Note that the system described is always stable since realization of common life time results in all customers in the system being flushed out. In this section, we perform the steady-state analysis of the queueing-inventory model.

Let **y** be the steady-state probability vector of generator Q'. Then we have

$$\mathbf{y}Q' = 0, \quad \mathbf{ye} = 1. \tag{2}$$

Partitioning **y** as $\mathbf{y} = (y_\Delta, \mathbf{y}_0, \mathbf{y}_1, \mathbf{y}_2, ...)$ and then each of the sub-vectors as

$$\mathbf{y}_0 = ((y_0(0,i,0,0), 0 \le i \le S); (y_0(0,i,n_3,k), 1 \le i \le S, 1 \le n_3 \le i, k = 1,2);$$

$$(y_0(n_2,0,0,0), 1 \le n_2 \le K); (y_0(n_2,i,i,k), 1 \le n_2 \le K, 1 \le i \le S, k = 1,2))$$

$$\mathbf{y}_{n_1} = ((y_{n_1}(n_2,i,i,k), 1 \le i \le S, 1 \le n_2 \le K, k = 1,2);$$

$$(y_{n_1}(0,i,n_3,k), 1 \le i \le S, 1 \le n_3 \le i, k = 1,2); (y_{n_1}(n_2,0,0,0), 1 \le n_2 \le K) \ \text{for} \ \ n_1 \ge 1,$$

we see that **y** is obtained as (see Neuts [10])

$$\mathbf{y}_{n_1} = \mathbf{y}_1 R^{n_1-1}, \quad n_1 \geq 2$$

where R is the minimal nonnegative solution to the matrix equation:

$$\sum_{k=0}^{2} R^k B_k = 0$$

and the boundary equations are given by

$$y_{\Delta} B_{\Delta 0} + \mathbf{y}_0 B_{00} + \mathbf{y}_1 B_{10} = 0,$$

$$\mathbf{y}_0 B_{01} + \mathbf{y}_1 [B_1 + R B_2] = 0,$$

$$y_{\Delta} = \frac{\alpha}{\eta} \sum_{n_1=0}^{\infty} \mathbf{y}_{n_1} \mathbf{e}.$$

The normalizing condition (2) gives

$$y_{\Delta} + \mathbf{y}_0 \mathbf{e} + \mathbf{y}_1 [I - R]^{-1} \mathbf{e} = 1.$$

3.2 *A random walk*

We consider the model with negligible service time; reservation, cancellation and realization of common life time on the set $\{0, 1, 2, ..., S\}$, the set of possible states of the inventory level process. No customer joins when the inventory level is zero and so there will be none in the waiting room and pool. The arrival process, cancellation and CLT are as described in Section 2. Let $I(t)$ be the inventory level at time t. Then $\{I(t), t \geq 0\}$ is a Markov chain on state space $\{0, 1, 2, ..., S\} \bigcup \left\{\tilde{\Delta}\right\}$ where $\left\{\tilde{\Delta}\right\}$ is an absorbing state which denotes the realization of common life time. Thus the infinitesimal generator is

$$\tilde{W} = \begin{bmatrix} \tilde{T} & \tilde{T}^0 \\ \mathbf{0} & 0 \end{bmatrix}$$

where

$$\tilde{T} = \begin{array}{c} \\ 0 \\ 1 \\ \vdots \\ S-2 \\ S-1 \\ S \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & \dots & S-2 & S-1 & S \end{array} \\ \begin{pmatrix} h_S & S\beta & & & & \\ \lambda & h_{S-1} & (S-1)\beta & & & \\ & \ddots & \ddots & \ddots & & \\ & & \lambda & h_2 & 2\beta & \\ & & & \lambda & h_1 & \beta \\ & & & & \lambda & h_0 \end{pmatrix} \end{array}, \ \tilde{T}^0 = \alpha \mathbf{e}$$

with $h_i = -(\lambda + i\beta + \alpha),\ 0 \le i \le (S-1)$ and $h_S = -(S\beta + \alpha)$. The expected time $E_{\mathbf{T}}$ until absorption follows a Phase type distribution with representation $(\xi, \tilde{T})$ where $\xi = (0, ..., 0, 1)$ is the initial probability vector of order $(S+1)$. Hence $E_{\mathbf{T}} = -\xi\tilde{T}^{-1}\mathbf{e}$.

3.3 *Expected number of pooled customers getting service in a cycle*

In order to compute the number of pooled customers getting service in a cycle, we consider the case of a finite pool. For numerical procedure the truncation level P_L (size of the pool) is taken such that the probability of the number of customers in the pool going above the truncation size is of the order less than ϵ (here ϵ is taken as 10^{-6}). Consider the Markov chain $\{(N(t), N_1'(t), N_2(t), I(t), N_3(t), u(t)),$ $t \ge 0\}$ where $N(t)$ = number of pooled customers who received service upto time in the present cycle and $N_1'(t)$ = number of customers in the finite pool at time t. Its state space is

$$\{\Delta'\} \bigcup \{(n, n_1, 0, i, n_3, k)\,; n \ge 0, 0 \le n_1 \le \boldsymbol{P}_L, 1 \le i \le S, 1 \le n_3 \le i, k = 1, 2\} \bigcup$$
$$\{(n, 0, 0, i, 0, 0)\,; n \ge 0, 0 \le i \le S\} \bigcup \{(n, n_1, n_2, 0, 0, 0)\,; n \ge 0, 0 \le n_1 \le \boldsymbol{P}_L, 1 \le n_2 \le K\}$$
$$\bigcup \{(n, n_1, n_2, i, i, k)\,; n \ge 0, 0 \le n_1 \le \boldsymbol{P}_L, 1 \le n_2 \le K, 1 \le i \le S, k = 1, 2\}$$

where $\{\Delta'\}$ is an absorbing state which denotes the realization of common life time. The infinitesimal generator of the Markov chain is

$$\mathcal{N}_{P_{\mathcal{L}}} = \begin{bmatrix} 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots \\ \mathcal{H} & \mathcal{H}_1 & \mathcal{H}_0 & & & \\ \mathcal{H} & & \mathcal{H}_1 & \mathcal{H}_0 & & \\ \mathcal{H} & & & \mathcal{H}_1 & \mathcal{H}_0 & \\ \vdots & & & & \ddots & \ddots \end{bmatrix}$$

where $\mathcal{H}_0, \mathcal{H}_1$ are square matrices of order $U_3 + \boldsymbol{P}_L U_4$ with $\mathcal{H} = \alpha\mathbf{e}$.

The entries in $\mathcal{H}_0$ and $\mathcal{H}_1$ are as under:

$$\mathcal{H}_1 = \begin{bmatrix} B_{00}' & B_{01} & & & & \\ B_{10}' & B_1' & B_0 & & & \\ & B_2' & B_1' & B_0 & & \\ & & \ddots & \ddots & \ddots & \\ & & & B_2' & B_1' & B_0 \\ & & & & B_2' & B_1^0 \end{bmatrix}, \mathcal{H}_0 = \begin{bmatrix} M_1' & & & & \\ M_2' & M_1 & & & \\ & M_2 & M_1 & & \\ & & \ddots & \ddots & \\ & & & M_2 & M_1 \end{bmatrix}$$

where

$$B_{00}^{\prime(n_2,i,n_3,k_1:m_2,j,m_3,k_2)} = \begin{cases} \lambda, & \text{for} \quad n_2=0, 1\le i\le S, n_3=i, k_1=1,2; \\ & \qquad m_2=n_2+1, j=i, m_3=n_3, k_2=k_1, \\ \lambda, & \text{for} \quad n_2=0, 2\le i\le S, 1\le n_3\le i-1, k_1=1,2; \\ & \qquad m_2=n_2, j=i, m_3=n_3+1, k_2=k_1, \\ \lambda, & \text{for} \quad n_2=0, 1\le i\le S, n_3=0, k_1=0; \\ & \qquad m_2=n_2, j=i, m_3=n_3+1, k_2=2, \\ \lambda, & \text{for} \quad 1\le n_2\le K-1, 1\le i\le S, n_3=i, k_1=1,2; \\ & \qquad m_2=n_2+1, j=i, m_3=n_3, k_2=k_1, \\ \mu, & \text{for} \quad n_2=0, 1\le i\le S, n_3=1, k_1=2; \\ & \qquad m_2=n_2, j=i-1, m_3=n_3-1, k_2=0, \\ \mu, & \text{for} \quad n_2=0, 2\le i\le S, 2\le n_3\le i, k_1=2; \\ & \qquad m_2=n_2, j=i-1, m_3=n_3-1, k_2=2, \\ \mu, & \text{for} \quad 1\le n_2\le K, i=1, n_3=i, k_1=2; \\ & \qquad m_2=n_2, j=i-1, m_3=n_3-1, k_2=0, \\ \mu, & \text{for} \quad 1\le n_2\le K, 2\le i\le S, n_3=i, k_1=2; \\ & \qquad m_2=n_2, j=i-1, m_3=n_3-1, k_2=2, \\ (S-i)\beta, & \text{for} \quad n_2=0, 0\le i\le S-1, n_3=0, k_1=0; \\ & \qquad m_2=n_2, j=i+1, m_3=n_3, k_2=0, \\ (S-i)\beta, & \text{for} \quad n_2=0, 1\le i\le S-1, 1\le n_3\le i, k_1=1,2; \\ & \qquad m_2=n_2, j=i+1, m_3=n_3, k_2=k_1, \\ S\beta, & \text{for} \quad 1\le n_2\le K, i=0, n_3=i, k_1=0; \\ & \qquad m_2=n_2-1, j=i+1, m_3=n_3+1, k_2=2, \\ (S-i)\beta, & \text{for} \quad 1\le n_2\le K, 1\le i\le S-1, n_3=i, k_1=1,2; \\ & \qquad m_2=n_2-1, j=i+1, m_3=n_3+1, k_2=k_1, \\ -(S\beta+\alpha), & \text{for} \quad 0\le n_2\le K, i=0, n_3=i, k_1=0; \\ & \qquad m_2=n_2, j=i, m_3=n_3, k_2=k_1, \\ -(\lambda+(S-i)\beta+\alpha), & \text{for} \quad n_2=0, 1\le i\le S, n_3=0, k_1=0; \\ & \qquad m_2=0, j=i, m_3=n_3, k-2=k_1, \\ -(\lambda+\mu+(S-i)\beta+\alpha), & \text{for} \quad n_2=0, 1\le i\le S, 1\le n_3\le i, k_=1,2; \\ & \qquad m_2=0, j=i, m_3=n_3, k_2=k_1, \\ -(\lambda+\mu+(S-i)\beta+\alpha), & \text{for} \quad 1\le n_2\le K-1, 1\le i\le S, n_3=i, k_1=1,2; \\ & \qquad m_2=n_2, j=i, m_3=n_3, k_2=k_1, \\ -(\gamma\lambda+\mu+(S-i)\beta+\alpha), & \text{for} \quad n_2=K, 1\le i\le S, n_3=i, k_1=1,2; \\ & \qquad m_2=n_2, j=i, m_3=n_3, k_2=k_1, \\ 0, & \qquad \text{otherwise}. \end{cases}$$

$$B_{1}^{\prime(n_2,i,n_3,k_1:m_2,j,m_3,k_2)} = \begin{cases} \lambda, & \text{for} \quad n_2=0, 1\le i\le S, n_3=i, k_1=1,2; \\ & \qquad m_2=n_2+1, j=i, m_3=n_3, k_2=k_1, \\ \lambda, & \text{for} \quad n_2=0, 2\le i\le S, 1\le n_3\le i-1, k_1=1,2; \\ & \qquad m_2=n_2, j=i, m_3=n_3+1, k_2=k_1, \\ \lambda, & \text{for} \quad 1\le n_2\le K-1, 1\le i\le S, n_3=i, k_1=1,2; \\ & \qquad m_2=n_2+1, j=i, m_3=n_3, k_2=k_1, \\ \mu, & \text{for} \quad n_2=0, i=1, n_3=1, k_1=2; \\ & \qquad m_2=n_2, j=i-1, m_3=n_3-1, k_2=0, \\ \mu, & \text{for} \quad n_2=0, 2\le i\le S, 2\le n_3\le i, k_1=2; \\ & \qquad m_2=n_2, j=i-1, m_3=n_3-1, k_2=2, \\ \mu, & \text{for} \quad 1\le n_2\le K, i=1, n_3=i, k_1=2; \\ & \qquad m_2=n_2, j=i-1, m_3=n_3-1, k_2=0, \\ \mu, & \text{for} \quad 1\le n_2\le K, 2\le i\le S, n_3=i, k_1=2; \\ & \qquad m_2=n_2, j=i-1, m_3=n_3-1, k_2=2, \\ (S-i)\beta, & \text{for} \quad n_2=0, 1\le i\le S-1, 1\le n_3\le i, k_1=1,2; \\ & \qquad m_2=n_2, j=i+1, m_3=n_3, k_2=k_1, \\ (S-i)\beta, & \text{for} \quad 1\le n_2\le K, 1\le i\le S-1, n_3=i, k_1=1,2; \\ & \qquad m_2=n_2-1, j=i+1, m_3=n_3+1, k_2=k_1, \\ -(S\beta+\alpha), & \text{for} \quad 0\le n_2\le K, i=0, n_3=i, k_1=0; \\ & \qquad m_2=n_2, j=i, m_3=n_3, k_2=k_1, \\ -(\lambda+\mu+(S-i)\beta+\alpha), & \text{for} \quad n_2=0, 1\le i\le S, 1\le n_3\le i, k_=1,2; \\ & \qquad m_2=0, j=i, m_3=n_3, k_2=k_1, \\ -(\lambda+\mu+(S-i)\beta+\alpha), & \text{for} \quad 1\le n_2\le K-1, 1\le i\le S, n_3=i, k_1=1,2; \\ & \qquad m_2=n_2, j=i, m_3=n_3, k_2=k_1, \\ -(\gamma\lambda+\mu+(S-i)\beta+\alpha), & \text{for} \quad n_2=K, 1\le i\le S, n_3=i, k_1=1,2; \\ & \qquad m_2=n_2, j=i, m_3=n_3, k_2=k_1, \\ 0, & \qquad \text{otherwise}. \end{cases}$$

$$B_{10}^{\prime(n_2,i,n_3,k_1:m_2,j,m_3,k_2)} = \begin{cases} S\beta, & \text{for} \quad 0 \le n_2 \le K, i=0, n_3=i, k_1=0; \\ & \qquad m_2=n_2, j=i+1, m_3=n_3+1, k_2=1, \\ \mu, & \text{for} \quad n_2=0, 2 \le i \le S, n_3=1, k_1=2; \\ & \qquad m_2=n_2, j=i-1, m_3=n_3, k_2=1, \\ 0, & \qquad \text{otherwise.} \end{cases}$$

$$B_{1}^{0(n_2,i,n_3,k_1:m_2,j,m_3,k_2)} = \begin{cases} \lambda, & \text{for} \quad n_2=0, 1 \le i \le S, n_3=i, k_1=1,2; \\ & \qquad m_2=n_2+1, j=i, m_3=n_3, k_2=k_1, \\ \lambda, & \text{for} \quad n_2=0, 2 \le i \le S, 1 \le n_3 \le i-1, k_1=1,2; \\ & \qquad m_2=n_2, j=i, m_3=n_3+1, k_2=k_1, \\ \lambda, & \text{for} \quad 1 \le n_2 \le K-1, 1 \le i \le S, n_3=i, k_1=1,2; \\ & \qquad m_2=n_2+1, j=i, m_3=n_3, k_2=k_1, \\ \mu, & \text{for} \quad n_2=0, i=1, n_3=1, k_1=2; \\ & \qquad m_2=n_2, j=i-1, m_3=n_3-1, k_2=0, \\ \mu, & \text{for} \quad n_2=0, 2 \le i \le S, 2 \le n_3 \le i, k_1=2; \\ & \qquad m_2=n_2, j=i-1, m_3=n_3-1, k_2=2, \\ \mu, & \text{for} \quad 1 \le n_2 \le K, i=1, n_3=i, k_1=2; \\ & \qquad m_2=n_2, j=i-1, m_3=n_3-1, k_2=0, \\ \mu, & \text{for} \quad 1 \le n_2 \le K, 2 \le i \le S, n_3=i, k_1=2; \\ & \qquad m_2=n_2, j=i-1, m_3=n_3-1, k_2=2, \\ (S-i)\beta, & \text{for} \quad n_2=0, 1 \le i \le S-1, 1 \le n_3 \le i, k_1=1,2; \\ & \qquad m_2=n_2, j=i+1, m_3=n_3, k_2=k_1, \\ (S-i)\beta, & \text{for} \quad 1 \le n_2 \le K, 1 \le i \le S-1, n_3=i, k_1=1,2; \\ & \qquad m_2=n_2-1, j=i+1, m_3=n_3+1, k_2=k_1, \\ -(S\beta+\alpha), & \text{for} \quad 0 \le n_2 \le K, i=0, n_3=i, k_1=0; \\ & \qquad m_2=n_2, j=i, m_3=n_3, k_2=k_1, \\ -(\lambda+\mu+(S-i)\beta+\alpha), & \text{for} \quad n_2=0, 1 \le i \le S, 1 \le n_3 \le i, k_{=}1,2; \\ & \qquad m_2=0, j=i, m_3=n_3, k_2=k_1, \\ -(\lambda+\mu+(S-i)\beta+\alpha), & \text{for} \quad 1 \le n_2 \le K-1, 1 \le i \le S, n_3=i, k_1=1,2; \\ & \qquad m_2=n_2, j=i, m_3=n_3, k_2=k_1, \\ -(\mu+(S-i)\beta+\alpha), & \text{for} \quad n_2=K, 1 \le i \le S, n_3=i, k_1=1,2; \\ & \qquad m_2=n_2, j=i, m_3=n_3, k_2=k_1, \\ 0, & \qquad \text{otherwise.} \end{cases}$$

$$B_{2}^{\prime(n_2,i,n_3,k_1:m_2,j,m_3,k_2)} = \begin{cases} S\beta, & \text{for} \quad 0 \le n_2 \le K, i=0, n_3=i, k_1=0; \\ & \qquad m_2=n_2, j=i+1, m_3=n_3+1, k_2=1, \\ \mu, & \text{for} \quad n_2=0, 2 \le i \le S, n_3=1, k_1=2; \\ & \qquad m_2=n_2, j=i-1, m_3=n_3, k_2=1, \\ 0, & \qquad \text{otherwise.} \end{cases}$$

$$M_{2}^{\prime(n_2,i,n_3,k_1:m_2,j,m_3,k_2)} = \begin{cases} \mu, & \text{for} \quad n_2=0, 2 \le i \le S, n_3=1, k_1=1; \\ & \qquad m_2=n_2, j=i-1, m_3=n_3, k_2=1, \\ 0, & \qquad \text{otherwise.} \end{cases}$$

$$M_1'^{(n_2,i,n_3,k_1:m_2,j,m_3,k_2)} = \begin{cases} \mu, & \text{for} \quad n_2 = 0, 1 \le i \le S, n_3 = 1, k_1 = 1; \\ & \qquad m_2 = n_2, j = i-1, m_3 = n_3 - 1, k_2 = 0, \\ \mu, & \text{for} \quad n_2 = 0, 2 \le i \le S, 2 \le n_3 \le i, k_1 = 1; \\ & \qquad m_2 = n_2, j = i-1, m_3 = n_3 - 1, k_2 = 2, \\ \mu, & \text{for} \quad 1 \le n_2 \le K, i = 1, n_3 = i, k_1 = 1; \\ & \qquad m_2 = n_2, j = i-1, m_3 = n_3 - 1, k_2 = 0, \\ \mu, & \text{for} \quad 1 \le n_2 \le K, 2 \le i \le S, n_3 = i, k_1 = 1; \\ & \qquad m_2 = n_2, j = i-1, m_3 = n_3 - 1, k_2 = 2, \\ 0, & \qquad \text{otherwise.} \end{cases}$$

$$M_1^{(n_2,i,n_3,k_1:m_2,j,m_3,k_2)} = \begin{cases} \mu, & \text{for} \quad n_2 = 0, i = 1, n_3 = 1, k_1 = 1; \\ & \qquad m_2 = n_2, j = i-1, m_3 = n_3 - 1, k_2 = 0, \\ \mu, & \text{for} \quad n_2 = 0, 2 \le i \le S, 2 \le n_3 \le i, k_1 = 1; \\ & \qquad m_2 = n_2, j = i-1, m_3 = n_3 - 1, k_2 = 2, \\ \mu, & \text{for} \quad 1 \le n_2 \le K, i = 1, n_3 = i, k_1 = 1; \\ & \qquad m_2 = n_2, j = i-1, m_3 = n_3 - 1, k_2 = 0, \\ \mu, & \text{for} \quad 1 \le n_2 \le K, 2 \le i \le S, n_3 = i, k_1 = 1; \\ & \qquad m_2 = n_2, j = i-1, m_3 = n_3 - 1, k_2 = 2, \\ 0, & \qquad \text{otherwise.} \end{cases}$$

$$M_2^{(n_2,i,n_3,k_1:m_2,j,m_3,k_2)} = \begin{cases} \mu, & \text{for} \quad n_2 = 0, 2 \le i \le S, n_3 = 1, k_1 = 1; \\ & \qquad m_2 = n_2, j = i-1, m_3 = n_3, k_2 = 1, \\ 0, & \qquad \text{otherwise.} \end{cases}$$

If p_k is the probability that absorption occurs with exactly k pooled customers getting service, then

$$p_k = \delta_{P_L}(-\mathcal{H}^{-1}\mathcal{H}_0)^k(-\mathcal{H}_1^{-1}\mathcal{H}), \quad k \ge 0$$

with $\delta_{P_L} = (\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_{P_L})$ is a row vector of order $U_3 + P_L U_4$. Therefore the expected number of pooled customers getting service before realization of common life time is

$$E_{P_L}(N) = \sum_{k=0}^{\infty} k p_k$$

(see Krishnamoorthy *et al*. [7]).

3.4 *Additional performance measures*

1. Expected number of customers in the pool before realization of common life time

$$E_P(N) = \sum_{n_1=1}^{\infty} n_1 \mathbf{y}_{n_1} \mathbf{e}.$$

2. Expected number of customers in the waiting room before realization of common life time

$$E_W(N) = \sum_{n_1=0}^{\infty} \sum_{n_2=1}^{K} n_2 \left\{ y_{n_1}(n_2, 0, 0, 0) + \sum_{i=1}^{S} [y_{n_1}(n_2, i, i, 1) + y_{n_1}(n_2, i, i, 2)] \right\}.$$

3. Expected number of customers in the buffer before realization of common life time

$$E_B(N) = \sum_{n_1=0}^{\infty} \left\{ \sum_{n_2=1}^{K} \sum_{i=1}^{S} i[y_{n_1}(n_2, i, i, 1) + y_{n_1}(n_2, i, i, 2)] \right.$$

$$\left. + \sum_{i=1}^{S} \sum_{n_3=1}^{i} n_3 [y_{n_1}(0, i, n_3, 1) + y_{n_1}(0, i, n_3, 2)] \right\}.$$

4. Expected number of items in the inventory before realization of common life time

$$E_I(N) = \sum_{n_1=0}^{\infty} \sum_{i=1}^{S} i \left\{ \sum_{n_2=1}^{K} [y_{n_1}(n_2, i, i, 1) + y_{n_1}(n_2, i, i, 2)] \right.$$

$$\left. + \sum_{n_3=0}^{i} [y_{n_1}(0, i, n_3, 1) + y_{n_1}(0, i, n_3, 2)] \right\} + \sum_{i=1}^{S} i y_0(0, i, 0, 0).$$

5. Expected number of items in the inventory immediately on realization of common life time

$$E_I'(N) = \sum_{n_1=0}^{\infty} \sum_{i=1}^{S} i \frac{\alpha}{\alpha + \lambda + \mu + (S-i)\beta} \left\{ \sum_{n_2=1}^{K} [y_{n_1}(n_2, i, i, 1) + y_{n_1}(n_2, i, i, 2)] \right.$$

$$\left. + \sum_{n_3=0}^{i} [y_{n_1}(0, i, n_3, 1) + y_{n_1}(0, i, n_3, 2)] \right\} + \sum_{i=1}^{S} i \frac{\alpha}{\alpha + \lambda + (S-i)\beta} y_0(0, i, 0, 0).$$

6. Rate of addition to the pool is

$$\gamma\lambda \sum_{n_1=0}^{\infty} \sum_{i=1}^{S} [y_{n_1}(K, i, i, 1) + y_{n_1}(K, i, i, 2)].$$

7. The probability that a customer enters service immediately on arrival

$$\sum_{i=1}^{S} y_0(0,i,0,0).$$

8. The rate at which pooled customers are transferred to the buffer

$$E_{PB}(R) = \sum_{n_1=1}^{\infty}\left[\sum_{i=2}^{S}\mu[y_{n_1}(0,i,1,1)+y_{n_1}(0,i,1,2)] + \sum_{n_2=0}^{K} S\beta y_{n_1}(n_2,0,0,0)\right].$$

9. The rate at which customers abandon the system on arrival

$$E_{WL}(R) = (1-\gamma)\lambda\sum_{n_1=0}^{\infty}\sum_{i=1}^{S}[y_{n_1}(K,i,i,1)+y_{n_1}(K,i,i,2)].$$

10. Expected cancellation rate

$$E_C(R) = \sum_{n_1=0}^{\infty}\sum_{i=1}^{S-1}(S-i)\beta\left[\sum_{n_2=1}^{K}[y_{n_1}(n_2,i,i,1)+y_{n_1}(n_2,i,i,2)]\right.$$
$$\left.+\sum_{n_3=1}^{i}[y_{n_1}(0,i,n_3,1)+y_{n_1}(0,i,n_3,2)]\right] + \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{K} S\beta y_{n_1}(n_2,0,0,0)$$
$$+\sum_{i=1}^{S-1}(S-i)\beta y_0(0,i,0,0).$$

11. Expected inventory depletion rate

$$E_P(R) = \mu\sum_{n_1=0}^{\infty}\sum_{i=1}^{S}\left\{\sum_{n_2=1}^{K}[y_{n_1}(n_2,i,i,1)+y_{n_1}(n_2,i,i,2)]\right.$$
$$\left.+\sum_{n_3=1}^{i}[y_{n_1}(0,i,n_3,1)+y_{n_1}(0,i,n_3,2)]\right\}.$$

12. Expected number of cancellations in a cycle

$$E_{NC} = \frac{1}{\alpha}\left\{\sum_{n_1=0}^{\infty}\sum_{i=1}^{S-1}(S-i)\beta\left[\sum_{n_2=1}^{K}[y_{n_1}(n_2,i,i,1)+y_{n_1}(n_2,i,i,2)]\right.\right.$$
$$\left.+\sum_{n_3=1}^{i}[y_{n_1}(0,i,n_3,1)+y_{n_1}(0,i,n_3,2)]\right] + \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{K} S\beta y_{n_1}(n_2,0,0,0)$$
$$\left.+\sum_{i=1}^{S-1}(S-i)\beta y_0(0,i,0,0)\right\}.$$

13. Expected number of purchases in a cycle

$$E_{NP} = \frac{1}{\alpha}\mu \sum_{n_1=0}^{\infty} \sum_{i=1}^{S} \left\{ \sum_{n_2=1}^{K} [y_{n_1}(n_2, i, i, 1) + y_{n_1}(n_2, i, i, 2)] + \sum_{n_3=1}^{i} [y_{n_1}(0, i, n_3, 1) + y_{n_1}(0, i, n_3, 2)] \right\}.$$

14. Expected number of transfers from the pool to the buffer

$$E_{PB}(N) = \frac{1}{\alpha} \sum_{n_1=1}^{\infty} \left[\sum_{i=2}^{S} \mu[y_{n_1}(0, i, 1, 1) + y_{n_1}(0, i, 1, 2)] + \sum_{n_2=0}^{K} S\beta y_{n_1}(n_2, 0, 0, 0) \right].$$

15. The probability that the system has S items in the inventory at the time of realization of common life time

$$P_{vacant} = \sum_{n_1=0}^{\infty} \left[\sum_{n_3=1}^{S} [y_{n_1}(0, S, n_3, 1) + y_{n_1}(0, S, n_3, 2)] + \sum_{n_2=1}^{K} [y_{n_1}(n_2, S, S, 1) + y_{n_1}(n_2, S, S, 2)] \right] + y_0(0, S, 0, 0).$$

16. The probability that the system is left with no item in the inventory at the time of realization of common life time

$$P_{full} = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{K} y_{n_1}(n_2, 0, 0, 0).$$

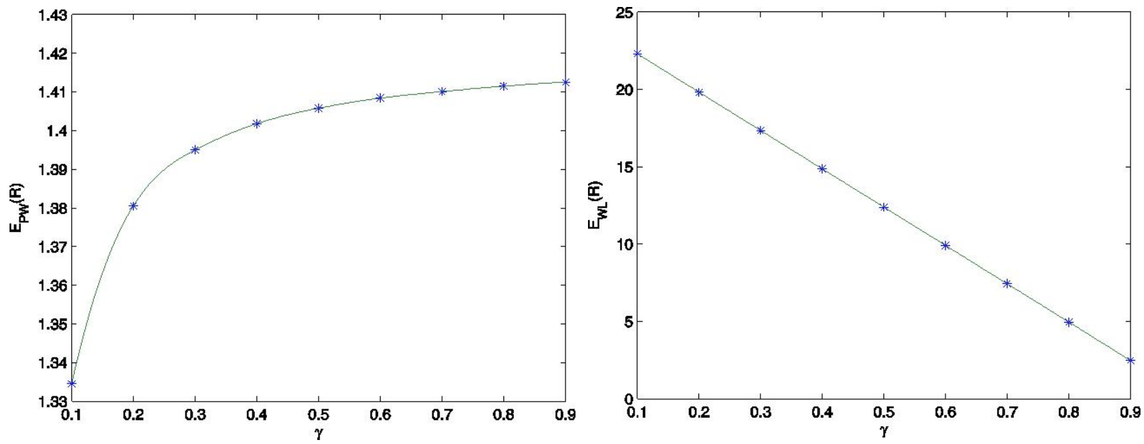
4. Numerical Illustrations

In this section we provide numerical illustration of the system performance with variation in values of underlying parameters.

Model 1

Effect of γ *on* $E_{PW}(R)$ *and* $E_{WL}(R)$

We consider the following values for the parameters $S = 12, K = 10, L = 6, \lambda = 20, \mu = 25, p = 0.75, \alpha = 0.25, \beta = 5, \eta = 5$. For this set of parameter values, Figure 1 shows that the impact of the probability γ on measures $E_{PW}(R)$ and $E_{WL}(R)$. From the Figure 1, it is clear that $E_{PW}(R)$ is increasing and the loss rate $E_{WL}(R)$ is monotonically decreasing in γ. This is due to the fact that as γ increases inflow rate to the pool increases, thus the loss rate decreases. Also, as γ increases transfer rate from pool to waiting room increases. However, this increase is marginal.



(a) Effect of γ on $E_{PW}(R)$ (b) Effect of γ on $E_{WL}(R)$

Figure 1: Effect of γ on $E_{PW}(R)$ and $E_{WL}(R)$

Effect of the arrival rate λ

From Table 1, we observe that an increase in the arrival rate makes a decrease in measures like expected number of items in the inventory before realization of common life time and expected number of items in the inventory immediately on realization of common life time. However, the expected number of customers in the pool, waiting room and buffer, expected number of cancellations, expected number of purchases and rate of transfer from the pool to waiting room increase. These are on expected lines.

λ	$E_P(N)$	$E_W(N)$	$E_B(N)$	$E_I(N)$	$E'_I(N)$	$E_{PW}(R)$	E_{NP}	E_{NC}
15	60.1708	5.0662	0.6412	0.9744	0.0069	1.3788	7.3557	2.6579
20	74.3319	5.1698	0.6746	0.9534	0.0059	1.4108	7.5985	2.6662
25	88.5280	5.2447	0.7000	0.9397	0.0052	1.4321	7.8095	2.6717
30	102.7461	5.3015	0.7198	0.9301	0.0046	1.4470	7.9834	2.6756
35	116.9786	5.3460	0.7357	0.9230	0.0042	1.4578	8.1304	2.6784
40	131.2199	5.3819	0.7488	0.9176	0.0038	1.4660	8.2517	2.6806

Table 1: Effect of the arrival rate: $S = 8, K = 6, L = 4, \mu = 10, \eta = 5, \alpha = 0.25, \beta = 0.1, p = 0.75, \gamma = 0.75$

Effect of the service time parameter μ

Table 2 indicates that increase in μ makes expected number of customers in the pool, waiting room and buffer, expected number of items in the inventory before realization of common life time and expected number of items immediately on realization of common life time, all decrease. However, as μ increases, rate of transfer from pool to waiting room, expected number of purchases and expected number of cancellations increase: higher realization time more the number of customers served out.

μ	$E_P(N)$	$E_W(N)$	$E_B(N)$	$E_I(N)$	$E'_I(N)$	$E_{PW}(N)$	E_{NP}	E_{NC}
15	74.2543	5.1679	0.4101	0.6893	0.0038	1.4461	7.4800	2.7719
20	74.2166	5.1666	0.2804	0.5599	0.0028	1.4576	7.5623	2.8237
25	74.1952	5.1658	0.2061	0.4858	0.0022	1.4619	7.8061	2.8533
30	74.1818	5.1651	0.1595	0.4398	0.0018	1.4636	8.1040	2.8719
35	74.1729	5.1647	0.1281	0.4081	0.0016	1.4644	8.3909	2.8844
40	74.1667	5.1644	0.1061	0.3862	0.0014	1.4649	8.6412	2.8931

Table 2: Effect of the service time parameter: $S = 8, K = 6, L = 4, \lambda = 30, \eta = 5, \alpha = 0.25, \beta = 0.1, p = 0.75, \gamma = 0.75$

Effect of the common life time parameter α

From Table 3, we observe that an increase in α results in a decrease in measures like expected number of customers in the pool and also in the waiting room, expected number of purchase, expected number of cancellations and rate of transfer from pool to waiting room. This is so since the mean value of common life time decreases with increase in value of α. However, the expected number of customers in the buffer, expected number of items in the inventory immediately on realization of common life time and expected number of items in the inventory before realization of common life time, all increase. These are also on expected lines.

α	$E_P(N)$	$E_W(N)$	$E_B(N)$	$E_I(N)$	$E'_I(N)$	$E_{PW}(N)$	E_{NP}	E_{NC}
0.1	204.8010	5.6348	0.3449	0.4618	0.0011	1.5978	12.8634	7.3807
0.2	95.9983	5.3198	0.5721	0.7985	0.0039	1.4703	8.5541	3.4469
0.3	59.9611	5.0246	0.7703	1.1000	0.0081	1.3541	6.9144	2.1491
0.4	42.1545	4.7481	0.9430	1.3697	0.0135	1.2481	5.9629	1.5094
0.5	31.6269	4.4891	1.0931	1.6112	0.0198	1.1513	5.3009	1.3231
0.6	24.7288	4.2463	1.2233	1.8273	0.0269	1.0629	4.7939	0.8859

Table 3: Effect of α: $S = 8, K = 6, L = 4, \lambda = 30, \mu = 10, \eta = 5, \beta = 0.1, p = 0.75, \gamma = 0.75$

Effect of the cancellation rate β

Table 4 shows that the expected number of customers in the pool, that in the waiting room and rate of transfer from pool to waiting room decrease with increase in β value. Here expected number customers in the buffer, expected number of purchase, expected number of cancellation, expected number of items in the inventory before realization of common life time and immediately on realization of common life time show a sharp upward trend. This is expected for higher cancellation rate.

β	$E_P(N)$	$E_W(N)$	$E_B(N)$	$E_I(N)$	$E'_I(N)$	$E_{PW}(R)$	E_{NP}	E_{NC}
0.15	73.3620	5.1572	0.7203	0.9996	0.0062	0.5527	8.8455	3.9717
0.20	72.4069	5.1446	0.7682	1.0479	0.0064	0.7368	10.0703	5.2596
0.25	71.4672	5.1320	0.8182	1.0984	0.0067	0.9208	11.2715	6.5207
0.30	70.5440	5.1195	0.8705	1.1510	0.0070	1.1048	12.4478	7.7616
0.35	69.6382	5.1070	0.9250	1.2060	0.0073	1.2888	13.5977	8.9783
0.40	68.7507	5.0946	0.9817	1.2632	0.0076	1.4727	14.7199	10.1694

Table 4: Effect of β: $S = 8, K = 6, L = 4, \lambda = 30, \eta = 5, \alpha = 0.25, \mu = 10, p = 0.75, \gamma = 0.75$

Effect of α, β on P_{full} and P_{vacant}

For $\beta = 0$, varying over α, we notice from Table 5 that, P_{full} decreases with increasing value of α – shorter the life time, lesser the chance for inventory being completely sold. Thus P_{vacant} increases with increasing value of α.

α	0.1	0.12	0.14	0.16	0.18	0.2	0.22	0.24	0.26
P_{full}	0.9398	0.9283	0.9169	0.9057	0.8947	0.8838	0.8730	0.8624	0.8520
P_{vacant}	0.0081	0.0097	0.0112	0.0128	0.0143	0.0158	0.0173	0.0188	0.0203

Table 5: Effect of α on P_{full}, P_{vacant}: ($\beta = 0, S = 7, K = 5, L = 3, \lambda = 30, \eta = 5, \mu = 20, p = 0.75, \gamma = 0.75$)

Table 6 shows the effect of β for fixed α value. It tells that higher cancellation rate results in reduction in probability of system being full (in the context of the bus / train / air plane leaving with all seats occupied). However, the extreme case of P_{vacant} does not increase with increase in value of β. Rather P_{vacant} stays constant. This could be attributed to high arrival rate $(\lambda = 30)$ and moderately high service rate $(\mu = 20)$; cancelled items are resold before common life time realization.

β	0.1	0.12	0.14	0.16	0.18	0.2	0.22	0.24	0.26
P_{full}	0.9066	0.9000	0.8935	0.8869	0.8804	0.8739	0.8674	0.8609	0.8545
P_{vacant}	0.0082	0.0082	0.0082	0.0082	0.0082	0.0082	0.0082	0.0082	0.0082

Table 6: Effect of β on P_{full}, P_{vacant}: ($\alpha = 0.1, S = 7, K = 5, L = 3, \lambda = 30, \eta = 5, \mu = 20, p = 0.75, \gamma = 0.75$)

Model 2

Effect of the arrival rate λ

Table 7 indicates that the increase in λ makes a decrease in measures like expected number of purchases, expected number of items in the inventory before realization of common life time and immediately on realization of common life time. As λ increases there is a moderate increase in the expected number of cancellations, expected number of customers in the pool and waiting room. The column on $E_{PB}(R)$ shows increase in value with λ increasing which could be attributed to increase in number of customers in the pool. There are some surprises in the column corresponding to E_{NP}. It shows an increasing trend with increase in value of λ upto a certain level and then it starts decreasing with further increase in value of λ. Still surprising is that the expected number of cancellations (E_{NC}) monotonically increase with λ. We do not have an explanation for these strange behaviour of E_{NP} and E_{NC}. However, in Model 1 this trend is not seen.

λ	$E_P(N)$	$E_W(N)$	$E_B(N)$	$E_I(N)$	$E'_I(N)$	$E_{PB}(R)$	E_{NP}	E_{NC}
5	0.0001	0.1987	0.3052	1.3901	0.0522	0.0001	7.3867	2.1106
10	0.0109	0.7014	0.3044	0.8075	0.0140	0.0045	8.1730	2.3437
15	0.1044	1.4440	0.2974	0.6139	0.0068	0.0333	8.2475	2.4211
20	0.4091	2.2433	0.2928	0.5250	0.0043	0.1002	7.8961	2.4567
25	1.0033	2.9272	0.2910	0.4772	0.0032	0.1908	7.2815	2.4758
30	1.8779	3.4396	0.2911	0.4486	0.0025	0.2819	6.5808	2.4872
35	2.9776	3.7991	0.2924	0.4301	0.0021	0.3597	5.9085	2.4946
40	4.2413	4.0449	0.2943	0.4174	0.0019	0.4205	5.3123	2.4997

Table 7: Effect of the arrival rate λ: $S = 7, K = 5, \mu = 20, \eta = 5, \alpha = 0.25, \beta = 0.1, \gamma = 0.75$

Effect of the service time parameter μ

From Table 8 we observe that as μ increases there is a moderate decrease in expected number of customers in the pool, waiting room and buffer, expected number of items in the inventory before realization of common life time and immediately on realization of common life time. But as μ increases there is a sharp increase in expected number of purchases and expected number of cancellations. $E_{PB}(R)$ decreases with increase in value of μ (see column $E_{PB}(R)$ of Table 8). The reason for this is the increase in probability of the server becoming idle with positive inventory in the system.

μ	$E_P(N)$	$E_W(N)$	$E_B(N)$	$E_I(N)$	$E'_I(N)$	$E_{PB}(R)$	E_{NP}	E_{NC}
22	1.3822	3.2058	0.2655	0.4219	0.0024	0.2367	7.0040	2.4979
24	1.0221	2.9720	0.2443	0.4002	0.0022	0.1962	7.3616	2.5066
26	0.7597	2.7450	0.2266	0.3823	0.0021	0.1610	7.6591	2.5137
28	0.5678	2.5292	0.2115	0.3674	0.0020	0.1312	7.9034	2.5197
30	0.4268	2.3277	0.1985	0.3549	0.0020	0.1065	8.1022	2.5247
32	0.3229	2.1417	0.1871	0.3443	0.0019	0.0862	8.2690	2.5290

Table 8: Effect of μ: $S = 7, K = 5, \lambda = 30, \eta = 5, \alpha = 0.25, \beta = 0.1, \gamma = 0.75$

Effect of common life time parameter α

From Table 9 we observe that as α increases there is high decrease in expected number of customers in the pool and that in the waiting room, rate of transfer from pool to buffer, expected number of cancellations and expected number of purchases. However, expected number of customers in the buffer, expected number of items in the inventory before realization of common life time and immediately on realization of common life time show a sharper upward trend. This is a consequence of higher rate of realization of CLT.

α	$E_P(N)$	$E_W(N)$	$E_B(N)$	$E_I(N)$	$E'_I(N)$	$E_{PB}(R)$	E_{NP}	E_{NC}
0.1	2.8942	3.9126	0.1443	0.2108	0.0005	0.3610	8.4422	6.6519
0.2	2.0721	3.5680	0.2441	0.3723	0.0017	0.3009	6.9775	3.1792
0.3	1.7348	3.3261	0.3361	0.5221	0.0035	0.2662	6.2731	2.0272
0.4	1.5289	3.1281	0.4208	0.6610	0.0060	0.2409	5.8045	1.4551
0.5	1.3794	2.9555	0.4989	0.7900	0.0090	0.2204	5.4446	1.1147
0.6	1.2607	2.8005	0.5707	0.9097	0.0124	0.2029	5.1467	0.8901

Table 9: Effect of α: $S = 7, K = 5, \lambda = 30, \mu = 20, \eta = 5, \beta = 0.1, \gamma = 0.75$

Effect of the cancellation rate β

Table 10 indicates that an increase in β makes expected number of customers in the pool, waiting room and buffer, expected number of items in the inventory before realization of common life time, rate of transfer from pool to buffer, expected number of cancellations and expected number of purchases, all increase. The high rate of arrival of customers results in the waiting room always occupied. Consequently pooled customers get very little absence to the buffer, as per the transfer policy.

β	$E_P(N)$	$E_W(N)$	$E_B(N)$	$E_I(N)$	$E'_I(N)$	$E_{PB}(R)$	E_{NP}	E_{NC}
0.05	1.6070	3.3412	0.2738	0.4302	0.0025	0.1326	6.0277	1.2473
0.06	1.6617	3.3636	0.2772	0.4339	0.0025	0.1616	6.1444	1.4959
0.07	1.7158	3.3845	0.2807	0.4375	0.0025	0.1909	6.2579	1.7442
0.08	1.7700	3.4040	0.2841	0.4412	0.0025	0.2208	6.3684	1.9921
0.09	1.8240	3.4223	0.2876	0.4449	0.0025	0.2511	6.4759	2.2398
0.10	1.8779	3.4396	0.2911	0.4486	0.0025	0.2819	6.5808	2.4872

Table 10: Effect of β: $S = 7, K = 5, \lambda = 30, \eta = 5, \alpha = 0.25, \mu = 20, \gamma = 0.75$

Effect of α, β on P_{full} and P_{vacant}

The interpretation of results in Tables 11 and 12 are on the same lines as in Model 1 (see Tables 5, 6).

α	0.1	0.12	0.14	0.16	0.18	0.2	0.22	0.24	0.26
P_{full}	0.9398	0.9283	0.9169	0.9057	0.8947	0.8838	0.8730	0.8624	0.8520
P_{vacant}	0.0114	0.0136	0.0158	0.0179	0.0201	0.0222	0.0243	0.0264	0.0284

Table 11: Effect of α on P_{full}, P_{vacant}: ($\beta = 0, S = 7, K = 5, \lambda = 30, \eta = 5, \mu = 20, \gamma = 0.75$)

β	0.1	0.12	0.14	0.16	0.18	0.2	0.22	0.24	0.26
P_{full}	0.9057	0.8990	0.8923	0.8857	0.8790	0.8724	0.8658	0.8592	0.8526
P_{vacant}	0.00114	0.0114	0.0114	0.0115	0.0115	0.0115	0.0115	0.0115	0.0116

Table 12: Effect of β on P_{full}, P_{vacant}: ($\alpha = 0.1, S = 7, K = 5, \lambda = 30, \eta = 5, \mu = 20, \gamma = 0.75$)

4.1 *Cost analysis*

Based on the above performance measures we construct a cost function for checking the optimality of the waiting room capacity K. It may be noted that we cannot arrive at an analytical form for the cost function since system state probabilities are not available in compact form.

We define a profit/revenue function as $\mathcal{F}(K,S)$ as

$$\mathcal{F}(K,S) = C_1E_C(R) + C_2E_P(R) - C_3E_B(N) - C_4E_W(N) - C_5E_P(N) - C_6E_I(N)$$

where

C_1 =Revenue to the system due to per unit cancellation of inventory purchased

C_2 =Revenue to the system due to per unit purchase of item in the inventory

C_3 =Holding cost of customer per unit per unit time in the buffer

C_4 =Holding cost of customer per unit per unit time in the waiting room

C_5 =Holding cost of customer per unit per unit time in the pool

C_6 =Holding cost per unit time per item in the inventory

In order to study the variation in different parameters on profit function we first fix the costs $C_1 = \$50, C_2 = \$200, C_3 = \$4, C_4 = \$7, C_5 = \$2, C_6 = \$10$.

4.1.1 *Effect of variation in S and K in Model 1*

We assign the following values to the parameters: $\lambda = 30, \mu = 20, \beta = 0.1, \eta = 5, \alpha = 0.25, p = 0.75, \gamma = 0.75, L = 3$. For different values of S and K, the expected profit is calculated and presented in Table 13. This table shows that the profit function decreases when K increases and increases for S.

K/S	6	7	8	9	10
5	121.2477	179.2218	237.3408	295.4081	353.2731
6	106.9024	163.3576	220.0913	276.9073	333.6499
7	94.3398	149.3635	204.7516	260.3190	315.9145
8	83.2438	136.9571	191.0771	245.4376	299.8984
9	73.3350	125.8753	178.8292	232.0525	285.4205
10	64.3745	115.8827	167.7849	219.9586	272.2969

Table 13: Effect of S and K on expected revenue

4.1.2 *Effect of variation in p, γ on expected revenue in Model 1*

We assign the following values to the parameters: $S = 8, K = 6, L = 3, \lambda = 30, \mu = 20$, $\beta = 0.1, \eta = 5, \alpha = 0.25, p = 0.75, \gamma = 0.75$. In Fig. 2, each curve is drawn keeping the choice for other parameters fixed; these graphs show that there is decreasing though marginal in revenue with increase in value of p. With γ increasing $\mathcal{F}(K,S)$ shows an increasing trend.

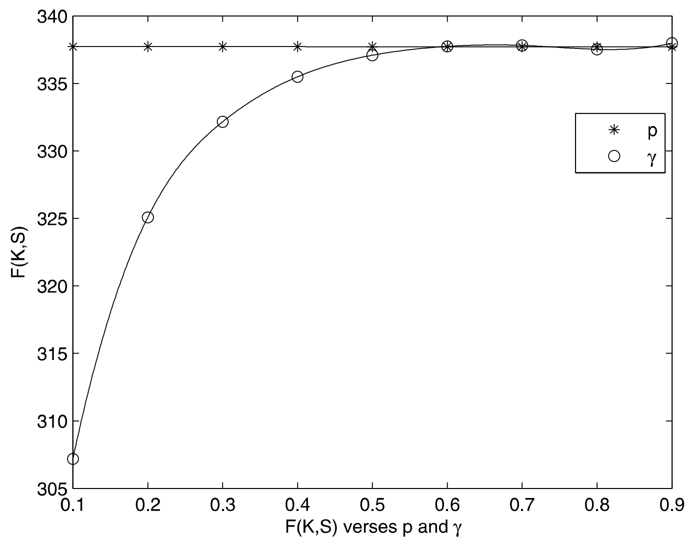


Figure 2: Effect of p and γ on expected revenue

4.1.3 *Effect of variation in S and K in Model 2*

We assign the following values to the parameters: $\lambda = 30, \mu = 20, \beta = 0.1, \eta = 5, \alpha = 0.25, \gamma = 0.75$. For different values of S and K, the expected revenue is calculated (see Table 14). This table shows that the profit function increases when S and K increases.

K/S	3	4	5	6	7
5	157.9565	204.3360	247.6024	288.2351	326.6477
6	163.5330	212.5247	258.3252	301.3083	341.8402
7	167.6997	218.9368	267.0368	312.2501	354.8707
8	170.7896	223.9125	274.0451	321.3158	365.9338
9	173.0665	227.7434	279.6345	328.7579	375.2396
10	174.7352	230.6729	284.0582	334.8184	383.0017

Table 14: Effect of S and K on expected revenue

A comparison between Model 1 and 2 is in order. A look at the values in Tables 13 and 14 indicate that the expected revenue is lower for Model 1. This is due to the transfer policy. In Model 1 the expected number of customers in the pool is relatively larger than that in Model 2. This results in higher holding cost of customers in the former and hence a reduced revenue from that results (see columns 2 and 3 of Table 13 and last two columns of Table 14). It is interesting to note that in Model 1 $\mathcal{F}(K, S)$ decreases with increase in value of K; however, this trend is reversed in Model 2. These are consequences of the transfer policies adopted: in Model 1 (based on number of customers in the waiting room) and Model 2, transfer from the pool only when server is idle with positive level of inventory on hand. With $\beta = 0$, P_{full} have the same values for different α values for both models; however, P_{vacant} do not show any similarity in behaviour.

CONCLUSIONS

In this paper we analyzed and compared two queueing-inventory models. These models defer only with respect to the transfer policy of customers from pool of postponed demands. Some unexpected results were seen (see Table 7). However, these surprises may have bearing on input values. Revenue wise Model 2 perform better. The problem discussed here finds application in advanced reservation system.

In a follow up paper we extend the present note to the case of Markovian arrival process and phase type service time with phase type CLT.

REFERENCES

1. G. Arivarignan, C. Elango and N. Arumugam, *A continuous review perishable inventory control system at service facilities*, In: Artalejo J. R., Krishnamoorthy A. (eds.) *Advances in Stochastic Modelling*, pp. 2940. Notable Publications, NJ, USA. (2002), 29-40.

2. O. Berman, E. H. Kaplan and D. G. Shimshak, Deterministic approximations for inventory management at service facilities, *IIE Trans.*, **25**(5) (1993), 98-104.

3. O. Berman and E. Kim, Stochastic models for inventory managements at service facilities, *Commun. Statist. Stochastic Models*, **15**(4) (1999), 695-718.

4. O. Berman and K. P. Sapna, Optimal service rates of a service facility with perishable inventory items, *Naval Research Logistic*, **49** (2002), 464-482.

5. T. G. Deepak, V. C. Joshua and A. Krishnamoorthy, Queues with postponed work, *TOP - Spanish Journal of Statistics and Operational Research*, **12** (2004), 375-398.

6. R. Krenzler and H. Daduna, Loss systems in a random environment steady-state analysis, *Queueing Syst*, (2014). DOI 10.1007/s11134-014-9426-6.

7. A. Krishnamoorthy, Dhanya Shajin and B. Lakshmy, On a queueing-inventory with reservation, cancellation, common life time and retrial, *Annals of Operation Research*, (2015). DOI 10.1007/s10479-015-1849-x.

8. A. Krishnamoorthy, V. C. Narayanan, T. G. Deepak and K. Vineetha, Effective utilization of server idle time in an (s, S) inventory with positive service time, *Journal of Applied Mathematics and Stochastic Analysis*, (2006). doi:10.1155/JAMSA/2006/69068.

9. A. Krishnamoorthy and N. C. Viswanath, Stochastic decomposition in production inventory with service time, *EJOR*, (2013). http://dx.doi.org/10.1016/j.ejor.2013.01.041

10. M. F. Neuts, *Matrix-geometric solutions in stochastic models: an algorithmic approach*, The Johns Hopkins University Press, Baltimore, (1981), [1994 version is Dover Edition].

11. M. F. Neuts, *Structured stochastic matrices of M/G/1 type and their applications*: Marcel Dekkar (1989).

12. S. Otten, R. Krenzler and H. Daduna, *Integrated models for production-inventory systems*, (2014), http://preprint.math.uni-hamburg.de/public/papers/prst/ prst2014-01.pdf.

13. M. Saffari, S. Asmussen and R. Haji, *The M/M/1 queue with inventory, lost sale and general lead times*, (2013), Queueing Systems, DOI:10.1007/s11134-012-9337-3.

14. M. Schwarz, C. Sauer, H. Daduna, R. Kulik and R. Szekli, M/M/1 queueing systems with inventory, *Queueing Syst*., **54** (2006), 55-78.

15. M. Schwarz and H. Daduna, Queueing systems with inventory management with random lead times and with backordering, *Mathematical Methods of Operations Research*, **64** (2006), 383-414.

16. M. Schwarz, C. Wichelhaus and H. Daduna, Product form models for queueing networks with an inventory, *Stochastic Models*, **23**(4) (2007), 627-663.

17. K. Sigman and D. Simchi-Levi, Light traffic heuristic for an M/G/1 queue with limited inventory, *Annals of Operation Research*, **40** (1992), 371-380.