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CHARACTERIZATIONS OF SYMMETRIZED POLYDISC

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Let Γ_n, $n \geq 2$, denote the symmetrized polydisc in $\mathbb{C}^n$, and Γ_1 be the closed unit disc in $\mathbb{C}$. We provide some characterizations of elements in Γ_n. In particular, an element $(s_1, \ldots, s_{n-1}, p) \in \mathbb{C}^n$ is in Γ_n if and only if $s_j = \beta_j + \overline{\beta_{n-j}} p$, $j = 1, \ldots, n-1$, for some $(\beta_1, \ldots, \beta_{n-1}) \in \Gamma_{n-1}$, and $|p| \leq 1$.

Key words : Symmetrized polydisc; Schur theorem; positive definite matrix.

1. Introduction

In this paper we study and characterize the symmetrized polydisc in the n-complex plane $\mathbb{C}^n$, $n \geq 2$. We denote by $\Gamma_n := \overline{\mathbb{G}_n}$ the symmetrized polydisc in $\mathbb{C}^n$, where

$$\mathbb{G}_n = \{\pi_n(\mathbf{z}) : \mathbf{z} \in \mathbb{D}^n\},$$

$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, and $\pi_n : \mathbb{C}^n \to \mathbb{C}^n$ is the symmetrization map

$$\pi_n(\mathbf{z}) = (\sum_{1 \leq i \leq n} z_i, \sum_{1 \leq i_1 < i_2 \leq n} z_{i_1} z_{i_2}, \ldots, \prod_{i=1}^{n} z_i),$$

for all $\mathbf{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n$. It follows easily from the fundamental theorem of algebra that π_n is an onto map, that is, $\pi_n(\mathbb{C}^n) = \mathbb{C}^n$.

We first turn to the case of symmetrized bidisc Γ_2. The early study of symmetrized bidisc was motivated to a large extent by the theory of spectral Nevanlinna-Pick problem, μ-synthesis problem in control engineering and complex geometry of domains in $\mathbb{C}^n$ (cf. [4, 3, 2]). One of the most important tools in the investigation of function theory and operator theory on Γ_2 is the following classification result due to Agler and Young: Let $(s,p) \in \mathbb{C}^2$ and $|p| \leq 1$. Then

$$(s,p) \in \Gamma_2 \Leftrightarrow \exists \beta \in \overline{\mathbb{D}} \text{ such that } s = \beta + p\overline{\beta}.$$

Moreover, for a pair of commuting operators (S,P) acting on some separable Hilbert space $\mathcal{H}$, Γ_2 is a spectral set of (S,P) (see [5]) if and only if $S = X + PX^*$ for some operator X with some natural contractivity property in an appropriate sense (see [10]).

In this paper we establish some characterization results concerning the elements of Γ_n, $n \geq 2$. One of our main results is the following theorem.

***Theorem* 1.1** — *Let $(s_1, \ldots, s_{n-1}, p) \in \mathbb{C}^n$, $n \geq 2$, and $\Gamma_1 = \overline{\mathbb{D}}$. Then the following are equivalent:*

(i) $(s_1, \ldots, s_{n-1}, p) \in \Gamma_n$.

(ii) $|p| \leq 1$ *and* $s_j = \beta_j + \overline{\beta_{n-j}}p$, $j = 1, \ldots, n-1$, *for some* $(\beta_1, \ldots, \beta_{n-1}) \in \Gamma_{n-1}$.

In particular, for the case of Γ_2 we recover the characterization result of Agler and Young.

For the rest of the paper, we let $n \geq 2$ and denote the unit circle $\{z \in \mathbb{C} : |z| = 1\}$ by $\mathbb{T}$. Given a holomorphic function f on a domain $\Omega \subseteq \mathbb{C}^m$, we denote by Z_f the set of all zeroes of f in Ω.

The reminder of this paper is organized as follows. Section 2 below collects some general results and facts concerning zero sets of polynomials. The proofs of the main theorems are given in Section 3.

ADDED IN PROOF : After completing our work, we became aware of the work of Constantin Costara (see [8] and [7]) which overlaps considerably with ours. In particular, one of our main results, Theorem 1.1, have already appeared in Costara's work (see [7] and Theorem 3.7 in [8]). However, our proof of Theorem 1.1 is simple and different from that of Constantin Costara. Moreover, Theorem 3.1 and the equivalence of (ii) and (iii) of Theorem 3.2 are new. We are indebted to Nicholas Young and Constantin Costara for pointing out these references to us.

2. PREPARATORY RESULTS

In this section we will prove a couple of auxiliary results that will be used in the proofs of the theorems. We will also list some results from the literature about the location of zeros of polynomials.

Lemma 2.1 — Let $(s_1, \dots, s_{n-1}, p) \in \mathbb{C}^n$ with $|p| < 1$. Then there is an element $(\beta_1, \dots, \beta_{n-1}) \in \mathbb{C}^{n-1}$ such that $s_j = \beta_j + \overline{\beta_{n-j}}p$, $j = 1, \dots, n-1$. Moreover, $(\beta_1, \dots, \beta_{n-1})$ is given by the following identity

$$\beta_j = \frac{s_{n-j} - \overline{s_j}p}{1 - |p|^2}, \qquad j = 1, \dots, n-1.$$

PROOF : The conclusion amounts to solve the following set of equations:

$$s_j = \beta_j + \overline{\beta_{n-j}}p, \qquad j = 1, \dots, n-1,$$

for $(\beta_1, \dots, \beta_{n-1}) \in \mathbb{C}^{n-1}$. We now consider the above $n-1$ equations in pairs, that is:

$$s_k = \beta_k + \overline{\beta_{n-k}}p \qquad \text{and} \qquad s_{n-k} = \beta_{n-k} + \overline{\beta_k}p, \tag{2.1}$$

where $k = 1, \dots, [n/2]$, if n is odd, and $k = 1, \dots, n/2$, if n is even. Since $|p| < 1$, by solving each pair of equations in (2.1), we get that

$$\beta_j = \frac{s_{n-j} - \overline{s_j}p}{1 - |p|^2}, \qquad j = 1, \dots, n-1.$$

This completes the proof. ■

The following lemma is an application of the Rouche's theorem. This will play an important role in our approach to the symmetrized polydisc.

Lemma 2.2 — Let $n \geq 2$, $(a_1, \dots, a_n) \in \mathbb{C}^n$, $|a_n| < 1$, and $(b_1, \dots, b_{n-1}) \in \mathbb{C}^{n-1}$. Furthermore, let $a_j = b_j + \overline{b_{n-j}}a_n$, $j = 1, \dots, n-1$, and $f(z) = z^n - a_1 z^{n-1} + \cdots + (-1)^{n-1}a_{n-1}z + (-1)^n a_n$ and $g(z) = z^{n-1} - b_1 z^{n-2} + \cdots + (-1)^{n-2}b_{n-2}z + (-1)^{n-1}b_{n-1}$. Then

(i) The number of zeros of f in $\mathbb{D}$ is the same as the number of zeros of zg in $\mathbb{D}$.

(ii) $Z_f \cap \mathbb{T} = Z_g \cap \mathbb{T}$.

PROOF : For $a_j = b_j + \overline{b_{n-j}}a_n$, $j = 1, \dots, n-1$, as above we have

$$\begin{aligned} f(z) &= z^n - a_1 z^{n-1} + \cdots + (-1)^{n-1}a_{n-1}z + (-1)^n a_n \\ &= z^n - (b_1 + \overline{b_{n-1}}a_n)z^{n-1} + \cdots + (-1)^{n-1}(b_{n-1} + \overline{b_1}a_n)z + (-1)^n a_n. \end{aligned}$$

It follows that

$$|f(z) - zg(z)| = |a_n||{b_{n-1}}^{n-1}\bar{z}^{n-1} - \cdots + (-1)^{n-2}b_1\bar{z} + (-1)^{n-1}|. \tag{2.2}$$

If we restrict the above equation on $\mathbb{T}$, we get

$$|f(z) - zg(z)| = |a_n||z^{n-1} - b_1 z^{n-2} + \cdots + (-1)^{n-2}b_{n-2}z + (-1)^{n-1}b_{n-1}|.$$

By virtue of $|a_n| < 1$ this yields

$$|(f - zg)(w)| = |a_n||g(w)| = |a_n||(zg)(w)| < |(zg)(w)|,$$

for all $|w| = 1$. Then Rouche's theorem shows that f and zg have the same number of zeroes inside $\mathbb{D}$. This completes the proof of (i)

We now turn to (ii). Let $f(\lambda) = 0$ for some $\lambda \in \mathbb{T}$. Hence, by (2.2)

$$|g(\lambda)| = |a_n||g(\lambda)|.$$

If $g(\lambda) \neq 0$ then we have $|g(\lambda)| < |g(\lambda)|$, which is a contradiction. This implies that $g(\lambda) = 0$.

Conversely, suppose $g(\lambda) = 0$ for some $\lambda \in \mathbb{T}$. Using (2.2) again it follows that

$$|f(\lambda)| = 0.$$

This completes the proof of (ii). ■

Now, as an easy consequence we obtain:

Proposition 2.3 — Let $n \geq 2$, $(a_1, \dots, a_n) \in \mathbb{C}^n$, $|a_n| < 1$, and $(b_1, \dots, b_{n-1}) \in \mathbb{C}^{n-1}$. Furthermore, let $a_j = b_j + \overline{b_{n-j}} a_n$, $j = 1, \dots, n-1$, and $f(z) = z^n - a_1 z^{n-1} + \cdots + (-1)^{n-1} a_{n-1} z + (-1)^n a_n$ and $g(z) = z^{n-1} - b_1 z^{n-2} + \cdots + (-1)^{n-2} b_{n-2} z + (-1)^{n-1} b_{n-1}$. Then all the zeros of f lies in $\overline{\mathbb{D}}$ if and only if all the zeros of g lies in $\overline{\mathbb{D}}$.

PROOF : The proof follows directly from Lemma 2.2. ■

Among the applications to domains in $\mathbb{C}^n$ in which positive definite forms played an important role, the following one, which will be useful in the sequel, stand out as particularly impressive [11] (see also [9]):

***Theorem* 2.4** (*Schur*) — *Given a polynomial* $p(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n$, $a_0 \neq 0$, *the zero set* Z_p *is a subset of* $\mathbb{D}$ *if and only if the Hermitian form*

$$H(x) = \sum_{j=1}^{n} |\overline{a_0} x_j + \overline{a_1} x_{j+1} + \cdots + \overline{a_{n-j}} x_n|^2 - \sum_{j=1}^{n} |a_n x_j + a_{n-1} x_{j+1} + \cdots + a_{n-j} x_n|^2$$

is positive definite. ■

3. MAIN RESULTS

With this background in place, we now state and prove the main results of this paper. Some of our results are new even in the case of Γ_2.

We first note that a necessary condition for $\mathbf{w} \in \mathbb{C}^n$ to be in $\mathbb{G}_n$ is that $|w_j| < \binom{n}{j}$, $j = 1, \ldots, n$. Let us assume $\mathbf{w} = \pi_n(\mathbf{z})$ for some $\mathbf{z} \in \mathbb{C}^n$. It follows from the definition of $\mathbb{G}_n$ that: *if* $\pi(\mathbf{z}) \in \mathbb{G}_n$, *for* $\mathbf{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n$, *then* $\pi(z_1, \ldots, z_j) \in \mathbb{G}_j$, *for any* $1 < j < n$. Hence, it is also necessary that, for each $1 < k < n$ and $1 \leq m_1 < \cdots < m_k \leq n$

$$(w^{(k)}_{m_1}, \ldots, w^{(k)}_{m_k}) = \pi_k(z_{m_1}, \ldots, z_{m_k}), \tag{3.1}$$

we have $|w^{(k)}_{m_l}| < \binom{k}{l}$ for all $1 \leq l \leq k$.

We are now ready for the first characterization result.

***Theorem* 3.1** — *Let* $\mathbf{z}, \mathbf{w} \in \mathbb{C}^n$, $\pi_n(\mathbf{z}) = \mathbf{w}$ *and* $|w_j| < \binom{n}{j}$, $j = 1, \ldots, n$. *Also assume that* $|w^{(2)}_{m_1}| < 2$ *and* $|w^{(2)}_{m_2}| < 1$ *for all* $1 \leq m_1 < m_2 \leq n$, *where* $w^{(2)}_{m_1}, w^{(2)}_{m_2}$ *are as in the notation of (3.1). Then* $\mathbf{w} \in \mathbb{G}_n$ *if and only if*

$$\prod_{j=1}^{n} (1 - |z_j|^2) > 0.$$

PROOF : Let $\mathbf{z}, \mathbf{w} \in \mathbb{C}^n$, $\pi_n(\mathbf{z}) = \mathbf{w}$ and $|w_j| < \binom{n}{j}$, $j = 1, \ldots, n$.

If $\Pi^n_{j=1}(1 - |z_j|^2) > 0$, then $|z_j| < 1$ for some $1 \leq j \leq n$. If not, then $|\Pi^n_{j=1} z_j| > 1$, which contradicts the fact that $|w_n| < \binom{n}{n} = 1$. We claim that $z_i \in \mathbb{D}$ for all $i = 1, \ldots, n$. If not, then $|z_l| > 1$ for some l, $1 \leq l \leq n$. From this and the fact that $\Pi^n_{j=1}(1 - |z_j|^2) > 0$, it readily follows that $|z_m| > 1$ for some $m = 1, \ldots, n$ and $l \neq m$. We may assume without loss of generality that $l = 1$ and $m = 2$. That is,

$$|z_1| > 1 \quad \text{and} \quad |z_2| > 1. \tag{3.2}$$

We now consider $(w^{(2)}_1, w^{(2)}_2) = \pi(z_1, z_2)$. By equation (3.2) we get that $|w^{(2)}_2| > 1$, which contradicts the assumption that $|w^{(2)}_2| < 1$.

The necessary part follows immediately from the definition of $\mathbb{G}_n$. ■

The following result (see [1] and [5]) provides a useful way to characterize the elements of $\mathbb{G}_2$: Let $(s, p) \in \mathbb{C}^2$. Then $(s, p) \in \mathbb{G}_2$ if and only if

$$\begin{bmatrix} 1 - |p|^2 & -\bar{s} + s\bar{p} \\ -s + \bar{s}p & 1 - |p|^2 \end{bmatrix} > 0,$$

if and only if

$$s = \beta + p\bar{\beta},$$

for some $\beta \in \mathbb{G}_1 := \mathbb{D}$.

We generalize the above fact in the following sense:

***Theorem* 3.2** — *Let $(s_1, s_2, \dots, s_{n-1}, p) \in \mathbb{C}^n$ with $|p| < 1$. Then the following are equivalent.*

(i) $(s_1, s_2, \dots, s_{n-1}, p) \in \mathbb{G}_n$.

ii) There exists $(\beta_1, \dots, \beta_{n-1}) \in \mathbb{G}_{n-1}$ such that $s_j = \beta_j + \overline{\beta_{n-j}}p$, $j = 1, \dots, n-1$.

(iii) The following matrix is positive definite:

$$\begin{bmatrix} 1-|p|^2 & -s_1+\overline{s}_{n-1}p & \dots & (-1)^{n-1}s_{n-1}+(-1)^n\overline{s}_1p \\ -\overline{s}_1+s_{n-1}\overline{p} & 1+|s_1|^2-|s_{n-1}|^2-|p|^2 & \dots & (-1)^{n-2}s_{n-2}+(-1)^{n-1}\overline{s}_2p \\ \vdots & \vdots & \dots & \vdots \\ (-1)^{n-1}\overline{s}_{n-1}+(-1)^n s_1\overline{p} & (-1)^{n-2}\overline{s}_{n-2}+(-1)^{n-1}s_2\overline{p} & \dots & 1-|p|^2 \end{bmatrix}.$$

PROOF : We first prove $(i) \iff (ii)$. Since $(s_1, \dots, s_{n-1}, p) \in \mathbb{C}^n$ with $|p| < 1$, by Lemma 2.1 there exists $(\beta_1, \dots, \beta_{n-1}) \in \mathbb{C}^{n-1}$ such that

$$s_j = \beta_j + \overline{\beta_{n-j}}p, \qquad j = 1, \dots, n-1.$$

By Proposition 2.3, $(s_1, \dots, s_{n-1}, p) \in \mathbb{G}_n$ if and only if $(\beta_1, \dots, \beta_{n-1}) \in \mathbb{G}_{n-1}$.

$(i) \iff (iii)$ follows from Theorem 2.4, by unfolding the positive definiteness of the Hermitian form in term of the Hermitian matrix that it corresponds to. ■

To proceed further, we recall the following results on the distinguished boundary of Γ_n (see Theorem 2.4, [6]). Recall also that the distinguished boundary of Γ_n is given by $\{\pi_n(\mathbf{z}) : \mathbf{z} \in \mathbb{T}^n\}$.

***Theorem* 3.3** — *Let $(s_1, \dots, s_{n-1}, p) \in \mathbb{C}^n$. Then the following statements are equivalent:*

(i) $(s_1, \dots, s_{n-1}, p)$ lies in the distinguished boundary of Γ_n.

(ii) $|p| = 1$ and $s_j = \beta_j + \overline{\beta_{n-j}}p$, $j = 1, \dots, n-1$ for some $(\beta_1, \dots, \beta_{n-1})$ in the distinguished boundary of Γ_{n-1}.

We now proceed to prove the main theorem.

PROOF OF THEOREM 1.1 : Note that by virtue of Theorem 3.3 stated above, we only have to consider the case $|p| < 1$.

Let $(s_1, \dots, s_{n-1}, p) \in \Gamma_n$. First assume that $|p| < 1$. Invoke Lemma 2.1 to conclude that

$$s_j = \beta_j + \overline{\beta_{n-j}}p, \quad j = 1, \dots, n-1, \tag{3.3}$$

for some $(\beta_1, \dots, \beta_{n-1}) \in \mathbb{C}^{n-1}$. Consider $f, g \in \mathbb{C}[z]$ as $\beta_1, \dots, \beta_{n-1}$:

$$f(z) := z^n - s_1 z^{n-1} + \dots + (-1)^{n-1}s_{n-1}z + (-1)^n p$$
$$g(z) := z^{n-1} - \beta_1 z^{n-2} + \dots + (-1)^{n-1}\beta_{n-1}.$$

Since $(s_1, \ldots, s_{n-1}, p) \in \Gamma_n$, $Z_f \subset \overline{\mathbb{D}}$. Then by Proposition 2.3 it follows that all the zeros of g lies in $\overline{\mathbb{D}}$. Hence, $(\beta_1, \ldots, \beta_{n-1}) \in \Gamma_{n-1}$.

Conversely, suppose (ii) holds. Let $(s_1, \ldots, s_{n-1}, p) \in \mathbb{C}^n$ and there exists $(\beta_1, \ldots, \beta_{n-1}) \in \Gamma_{n-1}$ such that the equations in (3.3) hold. As before, we only need to treat the case $|p| < 1$. We will again apply Proposition 2.3 to on the polynomials:

$$f(z) := z^n - s_1 z^{n-1} + \cdots + (-1)^{n-1} s_{n-1} z + (-1)^n p$$
$$g(z) := z^{n-1} - \beta_1 z^{n-2} + \cdots + (-1)^{n-1} \beta_{n-1}.$$

We conclude from here that $(s_1, \ldots, s_{n-1}, p) \in \Gamma_n$. ■

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