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SOME GAUSSIAN BINOMIAL SUM FORMULÆ WITH APPLICATIONS

Emrah Kılıç* and Helmut Prodinger**

**TOBB University of Economics and Technology, Mathematics Department,*
06560 *Ankara, Turkey*

***Department of Mathematics, University of Stellenbosch*
7602 *Stellenbosch, South Africa*

e-mails: ekilic@etu.edu.tr; hproding@sun.ac.za

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We introduce and compute some Gaussian q-binomial sums formulæ. In order to prove these sums, our approach is to use q-analysis, in particular a formula of Rothe, and computer algebra. We present some applications of our results.

Key words : Gaussian binomial coefficients; Fibonacci numbers; q-analogues; sum formulæ; CAS.

1. Introduction

Let $\{U_n\}$ and $\{V_n\}$ be generalized Fibonacci and Lucas sequences, respectively, whose the Binet forms are

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \alpha^{n-1}\frac{1-q^n}{1-q} \quad \text{and} \quad V_n = \alpha^n + \beta^n = \alpha^n(1+q^n)$$

with $q = \beta/\alpha = -\alpha^{-2}$, so that $\alpha = \mathbf{i}/\sqrt{q}$.

When $\alpha = \frac{1+\sqrt{5}}{2}$ (or equivalently $q = (1-\sqrt{5})/(1+\sqrt{5})$), the sequence $\{U_n\}$ is reduced to the Fibonacci sequence $\{F_n\}$ and the sequence $\{V_n\}$ is reduced to the Lucas sequence $\{L_n\}$.

When $\alpha = 1+\sqrt{2}$ (or equivalently $q = (1-\sqrt{2})/(1+\sqrt{2})$), the sequence $\{U_n\}$ is reduced to the Pell sequence $\{P_n\}$ and the sequence $\{V_n\}$ is reduced to the Pell-Lucas sequence $\{Q_n\}$.

Throughout this paper we will use the following notations: the q-Pochhammer symbol $(x;q)_n = (1-x)(1-xq)\dots(1-xq^{n-1})$ and the Gaussian q-binomial coefficients

$$\begin{bmatrix} n \\ k \end{bmatrix}_z = \frac{(q^z;q^z)_n}{(q^z;q^z)_k(q^z;q^z)_{n-k}}.$$

The $z = 1$ case will be denoted by $\begin{bmatrix} n \\ k \end{bmatrix}$.

Furthermore, we will use *generalized Fibonomial coefficients*

$$\begin{Bmatrix} n \\ k \end{Bmatrix}_{U,t} = \frac{U_{nt}U_{(n-1)t}\dots U_{(n-k+1)t}}{U_tU_{2t}\dots U_{kt}}$$

with $\left\{\begin{smallmatrix} n \\ 0 \end{smallmatrix}\right\}_{U,t} = 1$ where U_n is the nth generalized Fibonacci number.

In the special case $t = 1$, the generalized Fibonomial coefficients are denoted by $\left\{\begin{smallmatrix} n \\ k \end{smallmatrix}\right\}_U$. When $U_n = F_n$, the generalized Fibonomial reduces to the Fibonomial coefficients denoted by $\left\{\begin{smallmatrix} n \\ k \end{smallmatrix}\right\}_F$:

$$\begin{Bmatrix} n \\ k \end{Bmatrix}_{F} = \frac{F_nF_{n-1}\dots F_{n-k+1}}{F_1F_2\dots F_k}.$$

Similarly, when $U_n = P_n$, the generalized Fibonomial reduces to the Pellnomial coefficients denoted by $\left\{\begin{smallmatrix} n \\ k \end{smallmatrix}\right\}_P$:

$$\begin{Bmatrix} n \\ k \end{Bmatrix}_{P} = \frac{P_nP_{n-1}\dots P_{n-k+1}}{P_1P_2\dots P_k}.$$

The link between the generalized Fibonomial and Gaussian q-binomial coefficients is

$$\begin{Bmatrix} n \\ k \end{Bmatrix}_{U,t} = \alpha^{tk(n-k)}\begin{bmatrix} n \\ k \end{bmatrix}_t \quad \text{with} \quad q = -\alpha^{-2}.$$

For the reader's convenience and later use, we recall Rothe's formula [1, 10.2.2(c)]:

$$\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}(-1)^k q^{\binom{k}{2}}x^k = (x;q)_n\,.$$

Recently, the authors of [2, 3, 5] computed certain Fibonomial sums with generalized Fibonacci and Lucas numbers as coefficients. For example, if n and m are both nonnegative integers, then

$$\sum_{k=0}^{2n}\begin{Bmatrix} 2n \\ k \end{Bmatrix}U_{(2m-1)k} = P_{n,m}\sum_{k=1}^{m}\begin{Bmatrix} 2m-1 \\ 2k-1 \end{Bmatrix}U_{(4k-2)n},$$

$$\sum_{k=0}^{2n+1}\begin{Bmatrix} 2n+1 \\ k \end{Bmatrix}U_{2mk} = P_{n,m}\sum_{k=0}^{m}\begin{Bmatrix} 2m \\ 2k \end{Bmatrix}U_{(2n+1)2k},$$

$$\sum_{k=0}^{2n}\begin{Bmatrix} 2n \\ k \end{Bmatrix}V_{(2m-1)k} = P_{n,m}\sum_{k=1}^{m}\begin{Bmatrix} 2m-1 \\ 2k-1 \end{Bmatrix}V_{(4k-2)n},$$

$$\sum_{k=0}^{2n+1}\begin{Bmatrix} 2n+1 \\ k \end{Bmatrix}V_{2mk} = P_{n,m}\sum_{k=0}^{m}\begin{Bmatrix} 2m \\ 2k \end{Bmatrix}V_{(2n+1)2k},$$

where

$$P_{n,m} = \begin{cases} \prod_{k=0}^{n-m} V_{2k} & \text{if } n \geq m, \\ \prod_{k=1}^{m-n-1} V_{2k}^{-1} & \text{if } n < m; \end{cases}$$

alternating analogues of these sums were also evaluated.

Recently Kılıç and Prodinger computed the following Gaussian q-binomial sums with a parametric rational weight function: For any positive integer w, any nonzero real number a, nonnegative integer n, integers t and r such that $t + n \geq 0$ and $r \geq -1$,

$$\begin{aligned}
&\sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q \frac{(-1)^j q^{\binom{j+1}{2}+jt}}{(aq^j; q^w)_{r+1}} \\
&= a^{-t}(q;q)_n \Bigg(\sum_{j=0}^{r} \frac{(-1)^j}{(q^w;q^w)_j (q^w;q^w)_{r-j}} \frac{q^{w\binom{j+1}{2}-twj}}{(aq^{wj};q)_{n+1}} \\
&\quad + (-1)^{r+1} \sum_{j=0}^{t-r-1} \begin{bmatrix} n+j \\ n \end{bmatrix}_q \begin{bmatrix} t-1-j \\ r \end{bmatrix}_{q^w} q^{w\binom{r+1}{2}+(j-t)rw} a^j \Bigg).
\end{aligned}$$

In this paper we derive some Gaussian q-binomial sums. Then we present some applications of our results.

2. The Main Results

We start with our first result:

***Theorem* 1** — *For any* $n \geq 1$,

$$\sum_{k=1}^{n} \begin{bmatrix} 2n \\ n+k \end{bmatrix} q^{\frac{1}{2}k(k-1)} \left(1 - q^k\right) = (1 - q^n) \begin{bmatrix} 2n-1 \\ n \end{bmatrix}$$

and its Fibonomial corollary:

$$\sum_{k=1}^{n} \begin{Bmatrix} 2n \\ n+k \end{Bmatrix}_{U,t} (-1)^{\binom{k}{2}} U_{tk} = U_{tn} \begin{Bmatrix} 2n-1 \\ n \end{Bmatrix}_{U,t}.$$

PROOF : Let

$$S = \sum_{k=-n}^{n} \begin{bmatrix} 2n \\ n+k \end{bmatrix} q^{\frac{1}{2}k(k-1)} \left(1 - q^k\right).$$

Thus

$$\begin{aligned}
S &= \sum_{k=-n}^{n} \begin{bmatrix} 2n \\ n+k \end{bmatrix} q^{\frac{1}{2}k(k+1)} \left(1 - q^{-k}\right) \\
&= \sum_{k=-n}^{n} \begin{bmatrix} 2n \\ n+k \end{bmatrix} q^{\frac{1}{2}k(k-1)} \left(q^k - 1\right) = -S,
\end{aligned}$$

so $S=0$. Let

$$F\left(n,m\right)=\sum_{k=-n}^{m}\begin{bmatrix}2n\\n+k\end{bmatrix}q^{\frac{1}{2}k(k-1)}\left(1-q^{k}\right).$$

We need $-F\left(n,0\right)$ to evaluate our sum. Define

$$G\left(n,m\right):=-(1-q^{n})\begin{bmatrix}2n-1\\n+m\end{bmatrix}q^{m(m+1)/2}.$$

Then we have

$$G\left(n,m\right)=F(n,m),$$

which follows from

$$G(n,m)-G(n,m-1)=\begin{bmatrix}2n\\n+m\end{bmatrix}q^{\frac{1}{2}m(m-1)}\left(1-q^{n}\right).$$

Therefore our answer is

$$-F(n,0)=-G(n,0)=(1-q^{n})\begin{bmatrix}2n-1\\n\end{bmatrix},$$

as claimed.

The Fibonacci corollary follows by first replacing q by q^{t} and then translating. $\square$

For example, when $t=1$ and $\alpha=1+\sqrt{2}$ (or equivalently $q=\frac{1-\sqrt{2}}{1+\sqrt{2}}$), we have the following Pellnomial-Pell sum identity:

$$\sum_{k=1}^{n}\begin{Bmatrix}2n\\n+k\end{Bmatrix}_{P}(-1)^{\binom{k}{2}}P_{k}=P_{n}\begin{Bmatrix}2n-1\\n\end{Bmatrix}_{P}.$$

When $t=3$ and $\alpha=\frac{1+\sqrt{5}}{2}$ (or equivalently $q=\frac{1-\sqrt{5}}{1+\sqrt{5}}$), then we have the following Fibonomial-Fibonacci sum identity:

$$\sum_{k=1}^{n}\begin{Bmatrix}2n\\n+k\end{Bmatrix}_{F,3}(-1)^{\binom{k}{2}}F_{3k}=F_{3n}\begin{Bmatrix}2n-1\\n\end{Bmatrix}_{F,3}.$$

Our second result is:

Theorem 2 — *For all n such that $2n-1\geq r$ we have*

$$\sum_{k=1}^{n}\begin{bmatrix}2n\\n+k\end{bmatrix}(-1)^{k}q^{\frac{1}{2}\left(k^{2}-k(2r+1)\right)}\left(1+q^{k}\right)^{2r+1}=-2^{2r}\begin{bmatrix}2n\\n\end{bmatrix},$$

and its generalized Fibonomial-Lucas corollary:

$$\sum_{k=1}^{n}\begin{Bmatrix}2n\\n+k\end{Bmatrix}_{U,t}(-1)^{\frac{k(k+(-1)^{r})}{2}}V_{kt}^{2r+1}=-4^{r}\begin{Bmatrix}2n\\n\end{Bmatrix}_{U,t}.$$

PROOF : Define

$$S := \sum_{k=1}^{n} \begin{bmatrix} 2n \\ n+k \end{bmatrix} (-1)^k \, q^{\frac{1}{2}k(k-(2r+1))} \left(1+q^k\right)^{2r+1}.$$

Then we write

$$2S = \sum_{k\neq 0} \begin{bmatrix} 2n \\ n+k \end{bmatrix} (-1)^k \, q^{\frac{1}{2}k(k-(2r+1))} \left(1+q^k\right)^{2r+1}$$

and so

$$2S + 2^{2r+1} \begin{bmatrix} 2n \\ n \end{bmatrix} = \sum_{k=-n}^{n} \begin{bmatrix} 2n \\ n+k \end{bmatrix} (-1)^k \, q^{\frac{1}{2}k(k-(2r+1))} \left(1+q^k\right)^{2r+1}.$$

Consider

$$\begin{aligned} &\sum_{k=-n}^{n} \begin{bmatrix} 2n \\ n+k \end{bmatrix} (-1)^k \, q^{\frac{1}{2}k(k-(2r+1))} z^k \\ &= \sum_{k=0}^{2n} \begin{bmatrix} 2n \\ k \end{bmatrix} (-1)^{k-n} \, q^{\frac{1}{2}(k-n)(k-n-(2r+1))} z^{k-n} \\ &= (-1)^n z^{-n} q^{\frac{n^2+n(2r+1)}{2}} \sum_{k=0}^{2n} \begin{bmatrix} 2n \\ k \end{bmatrix} (-1)^k \, q^{\binom{k}{2}} (zq^{-n-r})^k \\ &= (-1)^n z^{-n} q^{\binom{n+1}{2}+nr} (zq^{-n-r}; q)_{2n}, \end{aligned}$$

according to formula 10.2.2(c) (Rothe's formula) in [1]. In order to obtain our claimed sum S, we use this formula for $z = 1, q, q^2, \ldots, q^{2r+1}$. Hence they are all 0 provided that $r \leq 2n-1$. Therefore

$$\sum_{k=1}^{n} \begin{bmatrix} 2n \\ n+k \end{bmatrix} (-1)^k \, q^{\frac{1}{2}k(k-(2r+1))} \left(1+q^k\right)^{2r+1} = -2^{2r} \begin{bmatrix} 2n \\ n \end{bmatrix},$$

as claimed. □

We can now replace q by q^t to obtain some Fibonomial type corollaries.

As an example, when $t = 3$, $r = 2$ and $\alpha = \frac{1+\sqrt{5}}{2}$ (or equivalently $q = \frac{1-\sqrt{5}}{1+\sqrt{5}}$), then we have the following Fibonomial-Lucas sum identity:

$$\sum_{k=1}^{n} \begin{Bmatrix} 2n \\ n+k \end{Bmatrix}_{F,3} (-1)^{\binom{k+1}{2}} L_{3t}^5 = -16 \begin{Bmatrix} 2n \\ n \end{Bmatrix}_{F,3}.$$

Our third result is a list of formulæ that can be obtained automatically by using the q-Zeilberger algorithm, in particular the version that was developed at the Risc center in Linz [9].

***Theorem* 3** — *For* $n \geq 1$

$$\sum_{k=0}^{n} \begin{bmatrix} 2n \\ n+k \end{bmatrix} q^{\frac{1}{2}k(k-(2b+1))} \left(1 - q^{(2b+1)k}\right) = \frac{X_b}{q^{\binom{b+1}{2}} \prod_{j=1}^{b}(1+q^{n-j})} (1-q^n) \begin{bmatrix} 2n-1 \\ n \end{bmatrix},$$

and the polynomials X_b *are getting more and more involved.*

We give a list of the first few:

$$\begin{aligned}
X_0 &= 1,\\
X_1 &= 2 + q + q^n + 2q^{n+1},\\
X_2 &= 2 + 2q + q^3 + 2q^n + q^{2n} + 3q^{n+1} + 3q^{n+2} + 2q^{n+3} + 2q^{2n+2} + 2q^{2n+3},\\
X_3 &= 2 + 2q + 2q^3 + q^6\\
&\quad + 2q^n + 2q^{2n} + q^{3n} + 4q^{1+n} + 4q^{2+n} + 5q^{3+n} + 3q^{4+n} + q^{5+n} + 2q^{6+n}\\
&\quad + q^{1+2n} + 3q^{2+2n} + 5q^{3+2n} + 4q^{4+2n} + 4q^{5+2n} + 2q^{6+2n}\\
&\quad + 2q^{3+3n} + 2q^{5+3n} + 2q^{6+3n},\\
X_4 &= 2 + 2q + 2q^3 + 2q^6 + q^{10}\\
&\quad + 2q^n + 2q^{2n} + 2q^{3n} + q^{4n} + 4q^{1+n} + 4q^{2+n} + 6q^{3+n} + 6q^{4+n} + 4q^{5+n}\\
&\qquad + 3q^{6+n} + 3q^{7+n} + q^{8+n} + q^{9+n} + 2q^{10+n}\\
&\quad + 2q^{1+2n} + 4q^{2+2n} + 7q^{3+2n} + 7q^{4+2n} + 10q^{5+2n} + 7q^{6+2n} + 7q^{7+2n}\\
&\qquad + 4q^{8+2n} + 2q^{9+2n} + 2q^{10+2n}\\
&\quad + q^{1+3n} + q^{2+3n} + 3q^{3+3n} + 3q^{4+3n} + 4q^{5+3n} + 6q^{6+3n} + 6q^{7+3n}\\
&\qquad + 4q^{8+3n} + 4q^{9+3n} + 2q^{10+3n}\\
&\quad + 2q^{4+4n} + 2q^{7+4n} + 2q^{9+4n} + 2q^{10+4n}.
\end{aligned}$$

As an example, we state the general Fibonomial-Lucas-Fibonacci instance for $b = 1$:

$$\sum_{k=0}^{n} \begin{Bmatrix} 2n \\ n+k \end{Bmatrix}_{U,t} (-1)^{\frac{1}{2}tk(k-3)} U_{3kt} = \frac{\left(2V_{t(n+1)} + (-1)^t V_{t(n-1)}\right) U_{nt}}{(-1)^t V_{(n-1)t}} \begin{Bmatrix} 2n-1 \\ n \end{Bmatrix}_{U,t}.$$

For example, when $\alpha = \left(1+\sqrt{5}\right)/2$ (or equivalently $q = \frac{1-\sqrt{5}}{1+\sqrt{5}}$) and $t = 1$, then we have the following Fibonomial-Lucas-Fibonacci sum identity:

$$\sum_{k=0}^{n} \begin{Bmatrix} 2n \\ n+k \end{Bmatrix}_{F} (-1)^{\frac{1}{2}k(k-3)} F_{3k} = -\frac{L_{n+2}F_n}{L_{n-1}} \begin{Bmatrix} 2n-1 \\ n \end{Bmatrix}_{F}.$$

We give another Fibonomial-Lucas-Fibonacci corollary (the instance $b = 2$); more complicated ones can be obtained by replacing q by q^t and taking larger b's.

$$\sum_{k=0}^{n} \begin{Bmatrix} 2n \\ n+k \end{Bmatrix}_U (-1)^{\binom{k}{2}} U_{5k}$$
$$= (2V_{2n+1} + V_{2n-3} - 2V_{2n+3} + 3(-1)^n V_1 - 2(-1)^n V_3) \times \frac{U_n}{V_{n-1}V_{n-2}} \begin{Bmatrix} 2n-1 \\ n \end{Bmatrix}_U .$$

Note that $2V_{2n+1}+V_{2n-3}-2V_{2n+3}$ could still simplified a bit using the recursion, but the recursion depends on α.

For example, when $\alpha = \left(1+\sqrt{5}\right)/2$

$$\sum_{k=0}^{n} \begin{Bmatrix} 2n \\ n+k \end{Bmatrix}_F (-1)^{\binom{k}{2}} F_{5k} = \frac{F_n \left(L_{2n+1} - 4L_{2n} - 5(-1)^n\right)}{L_{n-1}L_{n-2}} \begin{Bmatrix} 2n-1 \\ n \end{Bmatrix}_F .$$

Now we state our next result:

***Theorem* 4** — *For* $n \geq 1$

$$\sum_{k=0}^{n} \begin{bmatrix} 2n \\ n+k \end{bmatrix} q^{\frac{1}{2}k(k-3)} \left(1-q^k\right)^3 = 2 \begin{bmatrix} 2n-3 \\ n-1 \end{bmatrix} \frac{(1-q)}{q} (1-q^n)\left(1-q^{2n-1}\right),$$

and its Fibonomial-Fibonacci corollary

$$\sum_{k=0}^{n} \begin{Bmatrix} 2n \\ n+k \end{Bmatrix}_{U,t} (-1)^{\frac{1}{2}tk(k-3)} U_{tk}^3 = (-1)^t \, 2U_t U_{tn} U_{t(2n-1)} \begin{Bmatrix} 2n-3 \\ n-1 \end{Bmatrix}_{U,t} .$$

PROOF : One can produce a proof similar to our first theorem, but we gain no insight from it; and a computer can prove it without any effort. □

For example, if we take $t = 5$ and $\alpha = \frac{1+\sqrt{5}}{2}$ (or equivalently $q = \frac{1-\sqrt{5}}{1+\sqrt{5}}$), then we have the following Fibonomial-Fibonacci sum identity :

$$\sum_{k=0}^{n} \begin{Bmatrix} 2n \\ n+k \end{Bmatrix}_{F,5} (-1)^{\frac{1}{2}k(k-3)} F_{5k}^3 = -2 \begin{Bmatrix} 2n-3 \\ n-1 \end{Bmatrix}_{F,5} F_5 F_{5n} F_{5(2n-1)} .$$

Now we state our next results including the 5th and 7th powers of $\left(1-q^k\right)$:

***Theorem* 5** — *For* $n \geq 1$

$$\sum_{k=0}^{n} \begin{bmatrix} 2n \\ n+k \end{bmatrix} q^{\frac{1}{2}k(k-5)} \left(1-q^k\right)^5 = \frac{2(1-q)^2(1-q^n)^2(1+3q-3q^n-q^{n+1})}{q^3(1+q^{n-1})(1+q^{n-2})} \begin{bmatrix} 2n-1 \\ n \end{bmatrix},$$

and its Fibonomial-Fibonacci corollary

$$\sum_{k=0}^{n}\left\{\begin{matrix}2n\\n+k\end{matrix}\right\}_{U,t}(-1)^{t\binom{k}{2}}U_{tk}^{5}=\frac{(-1)^{t}\,2U_t^2U_{tn}^2\left(U_{t(n+1)}+3\,(-1)^{t}\,U_{t(n-1)}\right)}{V_{t(n-1)}V_{t(n-2)}}\left\{\begin{matrix}2n-1\\n\end{matrix}\right\}_{U,t}.$$

PROOF : Again, this is best done by a computer. □

For example, when $t=1$ and $\alpha=\left(1+\sqrt{5}\right)/2$, we get the following Fibonomial-Fibonacci corollary:

$$\sum_{k=0}^{n}\left\{\begin{matrix}2n\\n+k\end{matrix}\right\}_{F}(-1)^{\binom{k}{2}}F_{k}^{5}=\frac{2F_n^2F_{n-3}}{L_{n-1}L_{n-2}}\left\{\begin{matrix}2n-1\\n\end{matrix}\right\}_{F}.$$

We also give the next instance; after that, the terms get too involved:

***Theorem* 6** — *For* $n\geq 1$

$$\begin{aligned}\sum_{k=0}^{n}\begin{bmatrix}2n\\n+k\end{bmatrix}q^{\frac{1}{2}k(k-7)}\left(1-q^{k}\right)^{7}&=\frac{2(1-q)^{3}\,(1-q^{n})^{2}}{q^{6}(1+q^{n-1})(1+q^{n-2})(1+q^{n-3})}\begin{bmatrix}2n-1\\n\end{bmatrix}\\&\times(1+4q+9q^{2}+10q^{3}+10q^{2n}+9q^{2n+1}+4q^{2n+2}\\&+q^{2n+3}-5q^{n}-19q^{n+1}-19q^{n+2}-5q^{n+3}),\end{aligned}$$

and its Fibonomial-Fibonacci-Lucas corollary

$$\begin{aligned}&\sum_{k=0}^{n}\left\{\begin{matrix}2n\\n+k\end{matrix}\right\}_{U}(-1)^{\frac{1}{2}k(k-7)}U_{k}^{7}\\&\quad=\left(V_{2n+3}-4V_{2n+1}+9V_{2n-1}-10V_{2n-3}-5\,(-1)^{n}\,V_{3}+19\,(-1)^{n}\,V_{1}\right)\\&\quad\times\frac{2U_1^3U_n^2}{5V_{n-1}V_{n-2}V_{n-3}}\left\{\begin{matrix}2n-1\\n\end{matrix}\right\}_{U}.\end{aligned}$$

For example, when $\alpha=\left(1+\sqrt{5}\right)/2$, we get

$$\sum_{k=0}^{n}\left\{\begin{matrix}2n\\n+k\end{matrix}\right\}_{F}(-1)^{\frac{1}{2}k(k-7)}F_{k}^{7}=\frac{2F_n^2(L_{2n-2}+4L_{2n-4}-(-1)^{n})}{5L_{n-1}L_{n-2}L_{n-3}}\left\{\begin{matrix}2n-1\\n\end{matrix}\right\}_{F}.$$

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