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ON GROUND STATES FOR THE SCHRÖDINGER-POISSON SYSTEM WITH PERIODIC POTENTIALS [1]

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This paper is concerned with the following Schrödinger-Poisson system

$$\begin{cases} -\Delta u + V(x)u - K(x)\phi(x)u = q(x)|u|^{p-2}u, & \text{in } \mathbb{R}^3, \\ -\Delta\phi = K(x)u^2, & \text{in } \mathbb{R}^3, \end{cases}$$

where $p \in (2, 6)$, $V(x) \in C(\mathbb{R}^3, \mathbb{R})$ is a general periodic function, $K(x)$ and $q(x)$ are non-periodic functions. Under suitable assumptions, we prove the existence of ground state solutions via variational methods for strongly indefinite problems.

Key words : Schrödinger-Poisson system; ground state solutions; variational methods; strongly indefinite functionals.

1. INTRODUCTION AND MAIN RESULT

In this paper, we study the following nonlinear system

$$\begin{cases} -\Delta u + V(x)u - K(x)\phi(x)u = q(x)|u|^{p-2}u, & \text{in } \mathbb{R}^3, \\ -\Delta\phi = K(x)u^2, & \text{in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $p \in (2, 6)$, $V(x) \in C(\mathbb{R}^3, \mathbb{R})$ is a general periodic function, $K(x)$ and $q(x)$ are non-periodic functions. Such a system, also called Schrödinger-Poisson equations, arises in an interesting physical

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context. In fact, according to a classical model, the interaction of a charge particle with an electromagnetic field can be described by coupling the nonlinear Schrödinger and Maxwell equations. For more details on the physical background, we refer to [9, 10, 39] and the references therein. In particular, if we are looking for electrostatic-type solutions, we just have to solve (1.1).

In recent years, the following system

$$\begin{cases} -\Delta u + V(x)u + K(x)\phi(x)u = f(x,u), & \text{in } \mathbb{R}^3, \\ -\Delta\phi = K(x)u^2, & \text{in } \mathbb{R}^3, \end{cases} \tag{1.2}$$

has been extensively investigated in the literature based on variant assumptions on V, K and $f(x,u)$. See for example [3-7, 11, 14, 15, 22, 29, 31, 34, 41, 43, 46] and the references therein. Since problem (1.2) is set on whole space $\mathbb{R}^3$, it is well known that the main difficulty of this problem is the lack of compactness for Sobolev's embedding theorem, and then it is usually difficult to prove that a minimizing sequence or a (PS) sequence is strongly convergent if we seek solutions of (1.2) by variational methods. A usual way to overcome this difficulty is working on the radically symmetric function space which possesses compact embedding, see, for example [3-5, 14, 15, 29, 31]. When V is not a constant and not radially symmetric, Wang and Zhou [41] considered the asymptotically linear case. Based on the main ideas of del Pino and Felmer [13], they proved that the corresponding functional satisfies the (PS) condition and obtained the existence of positive solutions. In [6], Azzollni and Pomponio proved the existence of a ground state solution by using concentration compactness argument for problem (1.2) with $f(x,u) = |u|^{p-1}u$ and $3 < p < 5$, in [46] for $2 < p \leq 3$.

Recently, Cerami and Vaira [11] studied system (1.2) with $f(x,u) = a(x)|u|^{p-1}u$ $(3 < p < 5)$, $V = 1$ and $K \in L^2(\mathbb{R}^3)$ satisfying $K(x) \to 0$ as $|x| \to \infty$. They proved the existence of positive ground state by minimization on Nehari manifold and concentration compactness method. Similar method was also used in Vaira [39] for system (1.1) and (1.2). Later, Sun *et al*. [34] generalized the results of [11] to the asymptotically linear case. For $V > 0$ is periodic or asymptotically periodic, Alves *et al*. [7] established the existence of positive ground state solutions by using the mountain pass theorem. In addition, when $V > 0$ and $f(x,u)$ are 1-periodic in x, Zhao [46] obtained the existence of infinitely many geometrically distinct solutions. Very recently, Zhao *et al*. [43] considered the following system with sign-changing potential

$$\begin{cases} -\Delta u + \lambda V(x)u + K(x)\phi(x)u = |u|^{p-2}u, & \text{in } \mathbb{R}^3, \\ -\Delta\phi = K(x)u^2, & \text{in } \mathbb{R}^3, \end{cases} \tag{1.3}$$

where $\lambda > 0$ is a parameter. With the aid of parameter λ ($\lambda > 0$ large enough), they proved that the variational functional satisfies (PS) condition and obtained the existence of solutions for the case $p \in (4,6)$.

Moreover, the semiclassical limit of the system (1.2) was also discussed recently. More precisely, replacing $-\Delta$ by $-\varepsilon^2\Delta$, namely

$$\begin{cases} -\varepsilon^2\Delta u + V(x)u + K(x)\phi(x)u = |u|^{p-2}u, & \text{in } \mathbb{R}^3, \\ -\Delta\phi = K(x)u^2, & \text{in } \mathbb{R}^3. \end{cases} \tag{1.4}$$

In [3], Ambrosetti proved that the existence of spike-like solutions via perturbation methods. Ruiz [32] and D'Aprile and Wei [16] showed that system (1.4) with $V \equiv K \equiv 1$ possesses a family of solutions concentrating around a sphere when $\varepsilon \to 0$ for $p \in (2, 18/7)$. Their results were generalized in [26, 27] for the radial V and K. In [33], Ruiz and Vaira proved the existence of multi-bump solutions whose bumps concentrated around local minimums of the potential V. The proofs explored in [26, 27, 33] are based on a singular perturbation, essentially a Lyapunov-Schmitt reduction method. In [25], assume that V has a local minimum or maximum point x_0, Ianni and Vaira proved that (1.4) possesses a nontrivial solution u_ε for $\varepsilon > 0$ small. For other result of singularly perturbation problem and concentration phenomena of semiclassical states, we refer the readers to [20, 21, 23, 24] and the references therein.

Inspired by papers [11] and [39], we will consider system (1.1) with more general periodic potential V and the range of p and prove the existence of ground state solutions. More precisely, we make the following assumptions:

(V) $V \in C(\mathbb{R}^3, \mathbb{R})$ is 1-periodic in x_j for $j = 1, 2, 3$, and 0 lies in a gap of the spectrum of $-\Delta + V$;

(K) $K \in L^2(\mathbb{R}^3)$, $\lim_{|x|\to\infty} K(x) = 0$, $K(x) \geq 0$ for all $x \in \mathbb{R}^3$ and $K \not\equiv 0$;

(F) $q \in C(\mathbb{R}^3, \mathbb{R})$, there exists a constant $a > 0$ such that $q(x) > a$ for all $x \in \mathbb{R}^3$ and $\lim_{|x|\to\infty} q(x) = a$.

The main result of this paper is the following theorem.

***Theorem* 1.1** — *Suppose that (V), (K), (F) are satisfied. Then problem (1.1) has at least one ground state solutions.*

Remark 1.2 : Compared with the case $V = 1$ in [39], our assumption (V) is more general. To the best of our knowledge, there is no work focused on this case. Therefore, our result is new, and extend the corresponding one in [39].

Our argument is variational, which can be outlined as follows. The solutions of (1.1) are obtained as critical points of the energy functional Φ. Φ possesses the linking structure, however it does not satisfy the Palais-Smale condition in general. Thus we consider certain auxiliary problem related to the "limit equation" of (1.1) which is periodic problem and whose least energy solutions with least energy $\tilde{C}$ are known (see [30, 35, 36, 38, 42]). Based on the result of "limit equation", we establish the concentration compactness lemma and prove that Φ satisfies the Cerami condition $(C)_c$ at all levels $c < \tilde{C}$. Furthermore, by using a recent critical point theorem in [8] and [19], we obtain the existence of ground state solutions.

On the other hand, to obtain our results, we have to overcome several difficulties in using variational method. First, there is a lack of the compactness of the Sobolev embedding since the domain is the whole $\mathbb{R}^3$. Second, the energy functional Φ is strongly indefinite and it has more complex geometry structure than functionals which have mountain pass structure. Third, the appearance of a non-local term in our problem also brings us some difficulties under the strongly indefiniteness.

As a motivation we recall that there are a large number of literatures devoted to the study of the existence of ground state solutions. Ding and Wei [18] treated the nonperiodic Dirac equation with super-quadratic subcritical nonlinearities, Ding and Lee [17] also studied the nonperiodic superquadratic first-order Hamiltonian. By using the variational methods for strongly indefinite problems developed recently by Bartsch and Ding [8], they proved the existence of least energy solution, respectively. Very recently, based on the main ideas of [18], Chen and Zheng [12] considered the Maxwell-Dirac systems. Additionally, some authors have studied several different problems by different methods. Among these problems are the periodic Schrödinger equation in [30, 35-38, 42], and the Hamiltonian system in [28, 44, 45].

This paper is organized as follows. In Section 2, we formulate the variational setting. In Section 3, we introduce the least energy solutions of the associated limit problem, and recall some critical point theorems required. In Section 4, we will use the linking and concentration compactness arguments to prove our main theorems.

2. Variational Setting

Hereafter we use the following notation:

- $D^{1,2}(\mathbb{R}^3)$ is the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm

$$\|u\|^2_{D^{1,2}} = \int_{\mathbb{R}^3} |\nabla u|^2 dx.$$

- $|\cdot|_s$ denotes the usual L^s- norm, $1 \leq s \leq \infty$.
- $(\cdot,\cdot)_2$ denotes the usual L^2 inner product.

- C, C_i, c_i are different positive constants.
- E^* denotes the dual space of E.
- S is the best Sobolev constant for the embedding of $D^{1,2}(\mathbb{R}^3)$ in $L^6(\mathbb{R}^3)$, that is

$$S = \inf_{u \in D^{1,2} \setminus \{0\}} \frac{\|u\|_{D^{1,2}}}{|u|_6}.$$

In what follows, we give the variational setting for problem (1.1). Let $A := -\Delta + V$, then A is formally self-adjoint operator acting on $L^2 := L^2(\mathbb{R}^3, \mathbb{R})$ with domain $D(A) = H^2(\mathbb{R}^3, \mathbb{R})$. In virtue of assumption (V), we have the orthogonal decomposition

$$L^2 = L^- \oplus L^+, \ \ u = u^- + u^+,$$

such that A is negative definite on L^- and positive definite on L^+. Let $|A|$ denote the absolute of A and $|A|^{\frac{1}{2}}$ be the square root of $|A|$. Let $E := \mathcal{D}(|A|^{\frac{1}{2}})$ be the domain of the self-adjoint operator $|A|^{\frac{1}{2}}$ which is a Hilbert space equipped with the inner product

$$(u, v) = (|A|^{\frac{1}{2}} u, |A|^{\frac{1}{2}} v)_2$$

and norm $\|u\| = (u, u)^{\frac{1}{2}}$. By (V), $E = H^1 := H^1(\mathbb{R}^3, \mathbb{R})$ with equivalent norm (see [19]). Therefore E embeds continuously into L^s for all $s \in [2, 6]$, and compactly into L^s_{loc} for all $s \in [2, 6)$. Thus for all $s \in [2, 6]$, there exists $c_s > 0$ such that

$$|u|_s \leq c_s \|u\|, \ \ \text{for } u \in E.$$

In addition, we have the following decomposition

$$E = E^- \oplus E^+, \ \ \text{where} \ \ E^{\pm} = E \cap L^{\pm},$$

orthogonal with respect to both $(\cdot, \cdot)_2$ and $(\cdot, \cdot)$. This decomposition also induces a natural decomposition of L^s, $s \in (2, 6)$, hence there exists β_s such that

$$\beta_s |u^+|_s^s \leq |u|_s^s \ \ \text{for all } u \in E. \tag{2.1}$$

It is well known that problem (1.1) can be reduced to a single equation with nonlocal term. Actually, for each $u \in E$, the linear functional T_u in $D^{1,2}(\mathbb{R}^3)$ defined by

$$T_u(v) = \int_{\mathbb{R}^3} K(x) u^2 v dx, \ \ v \in D^{1,2}(\mathbb{R}^3)$$

is continuous. In fact, Hölder inequality and Sobolev inequality imply that

$$\begin{aligned} |T_u(v)| = \left|\int_{\mathbb{R}^3} K(x)u^2 v dx\right| &\leq |K|_2|u^2|_3|v|_6 \\ &\leq S^{-1}|K|_2|u|_6^2\|v\|_{D^{1,2}}. \end{aligned} \tag{2.2}$$

It follows from the Lax-Milgram theorem that there exists a unique $\phi_u \in D^{1,2}(\mathbb{R}^3)$ such that

$$\int_{\mathbb{R}^3} \nabla\phi_u \nabla v dx = \int_{\mathbb{R}^3} K(x)u^2 v dx \ \ \forall v \in D^{1,2}(\mathbb{R}^3), \tag{2.3}$$

that is ϕ_u is a weak solution of $-\Delta\phi = K(x)u^2$, and ϕ_u can be represented by

$$\phi_u(x) = \int_{\mathbb{R}^3} \frac{K(y)u^2(y)}{|x-y|} dy.$$

By (2.2) and (2.3), it is easy to see that

$$\begin{aligned} \|\phi_u\|_{D^{1,2}}^2 = \int_{\mathbb{R}^3} K(x)\phi_u u^2 dx &\leq |K|_2|u^2|_3|\phi_u|_6 \\ &\leq S^{-1}c_6^2|K|_2\|u\|^2\|\phi_u\|_{D^{1,2}}. \end{aligned} \tag{2.4}$$

It can be proved that $(u,\phi) \in E \times D^{1,2}(\mathbb{R}^3)$ is a solution of (1.1) if and only if $u \in E$ is a critical point of the functional $\Phi : E \to \mathbb{R}$ defined by

$$\Phi(u) = \frac{1}{2}\int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2) dx - \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_u u^2 dx - \frac{1}{p}\int_{\mathbb{R}^3} q(x)|u|^p dx, \tag{2.5}$$

and $\phi = \phi_u$. In virtue of (2.4), we know that Φ is well defined. Furthermore, our hypotheses imply that $\Phi \in C^1(E,\mathbb{R})$ (see [40]).

From the decomposition of E, then (2.5) is equivalent to the following functional

$$\Phi(u) = \frac{1}{2}(\|u^+\|^2 - \|u^-\|^2) - \Gamma(u) - \Psi(u), \ \ u \in E, \tag{2.6}$$

where

$$\Gamma(u) = \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_u u^2 dx \ \text{ and } \ \Psi(u) = \frac{1}{p}\int_{\mathbb{R}^3} q(x)|u|^p dx.$$

Thus assumption (V) implies that (2.6) is strongly indefinite functional, such type of functionals have appeared extensively in the study of differential equations via critical point theory, see for example [17-19, 30, 35, 40, 42, 44, 45] and the references therein. Moreover, it is not difficult to compute that, for all $u, \varphi, \eta \in E$,

$$\Gamma'(u)\varphi = \int_{\mathbb{R}^3} K(x)\phi_u u\varphi dx,$$

$$\Gamma''(u)[\varphi,\eta] = \int_{\mathbb{R}^3} K(x)\phi_u\varphi\eta dx + 2\int_{\mathbb{R}^3} K(x)(\int_{\mathbb{R}^3} \frac{K(y)}{|x-y|}u(y)\eta(y)dy)u\varphi dx,$$

$$\Psi'(u)\varphi = \int_{\mathbb{R}^3} q(x)u^{p-2}u\varphi dx,$$

and

$$\Psi''(u)[\varphi,\eta] = (p-1)\int_{\mathbb{R}^3} q(x)u^{p-2}\varphi\eta dx.$$

Hence, Φ is C^2 in E.

3. Some Preliminaries and Limit Problem

For the convenience of discussion, we define the operator $L : E \to D^{1,2}(\mathbb{R}^3)$ as

$$L[u] = \phi_u.$$

In the following, we give some properties about the functional L.

Lemma 3.1 — (1) L is continuous and $\phi_u > 0$ if $u \neq 0$;

(2) L maps bounded sets into bounded sets;

(3) Let $K \in L^2(\mathbb{R}^3)$. If $u_n \rightharpoonup u$ in E, then up to a subsequence,

$$\phi_{u_n} \to \phi_u \text{ in } D^{1,2}(\mathbb{R}^3).$$

Proof : The proofs were given in [11] and [43], here we omit the details. □

Lemma 3.2 — Let $K \in L^2(\mathbb{R}^3)$. If $u_n \rightharpoonup u$ in E, up to a subsequence, then as $n \to \infty$,

$$\int_{\mathbb{R}^3} K(x)\phi_{u_n}u_n^2 dx = \int_{\mathbb{R}^3} K(x)\phi_u u^2 dx + o(1), \tag{3.1}$$

$$\int_{\mathbb{R}^3} K(x)\phi_{u_n}u_n\varphi dx = \int_{\mathbb{R}^3} K(x)\phi_u u\varphi dx + o(1), \forall\varphi \in E. \tag{3.2}$$

And if $\varphi_n \rightharpoonup \varphi$ in E, the

$$\int_{\mathbb{R}^3} K(x)\phi_{u_n}u_n\varphi_n dx = \int_{\mathbb{R}^3} K(x)\phi_u u\varphi dx + o(1). \tag{3.3}$$

Proof : First, we prove the conclusion (3.1). By Sobolev embedding, $\phi_{u_n} \rightharpoonup \phi_u$ in $D^{1,2}(\mathbb{R}^3)$ implies that

$$\phi_{u_n} \rightharpoonup \phi_u \text{ in } L^6(\mathbb{R}^3),$$

and then

$$\int_{\mathbb{R}^3} K(x)u^2(\phi_{u_n} - \phi_u)dx \to 0, \tag{3.4}$$

since $K(x)u^2 \in L^{\frac{6}{5}}(\mathbb{R}^3)$ by Hölder inequality and assumption (K). Moreover, from $u_n \rightharpoonup u$ in E, we can assume that, up to a subsequence,

$$u_n \rightharpoonup u \text{ in } L^s(\mathbb{R}^3), \;\; u_n \to u \text{ in } L^s_{loc}(\mathbb{R}^3), \text{ for } 2 \le s < 6.$$

Thus, by Hölder inequality, we have

$$\begin{aligned}\int_{\mathbb{R}^3} (K(x)\phi_{u_n} u_n^2 - K(x)\phi_{u_n} u^2)dx &= \int_{\mathbb{R}^3} K(x)\phi_{u_n}(u_n^2 - u^2)dx \\ &\le |\phi_{u_n}|_6 |u_n + u|_6 \left(\int_{\mathbb{R}^3} |K(x)(u_n - u)|^{\frac{3}{2}} dx\right)^{\frac{2}{3}} \\ &\le C_1 \left(\int_{\mathbb{R}^3} |K(x)(u_n - u)|^{\frac{3}{2}} dx\right)^{\frac{2}{3}} \to 0 \end{aligned} \tag{3.5}$$

as $n \to \infty$, since the sequence $v_n := (u_n - u)^{\frac{3}{2}} \rightharpoonup 0$ in $L^4(\mathbb{R}^3)$ and $K(x)^{\frac{3}{2}} \in L^{\frac{4}{3}}(\mathbb{R}^3)$. Thus, by (3.4) and (3.5), we have

$$\begin{aligned}&\int_{\mathbb{R}^3} (K(x)\phi_{u_n} u_n^2 - K(x)\phi_u u^2)dx \\ =\;& \int_{\mathbb{R}^3} K(x)u^2(\phi_{u_n} - \phi_u)dx + \int_{\mathbb{R}^3} K(x)\phi_{u_n}(u_n^2 - u^2)dx \to 0\end{aligned}$$

as $n \to \infty$.

Next, we prove the conclusion (3.2). It suffices to show that

$$\int_{\mathbb{R}^3} (K(x)\phi_{u_n} u_n \varphi - K(x)\phi_u u\varphi)dx \to 0$$

uniformly for any $\varphi \in E$ with $\|\varphi\| \le 1$ as $n \to \infty$. In fact, similar to (3.5), we have

$$\begin{aligned}\int_{\mathbb{R}^3} K(x)\phi_{u_n}(u_n - u)\varphi dx &\le c_6 \|\varphi\| |\phi_{u_n}|_6 \left(\int_{\mathbb{R}^3} |K(x)(u_n - u)|^{\frac{3}{2}} dx\right)^{\frac{2}{3}} \\ &\le C_2 \|\varphi\| \left(\int_{\mathbb{R}^3} |K(x)(u_n - u)|^{\frac{3}{2}} dx\right)^{\frac{2}{3}} \to 0\end{aligned} \tag{3.6}$$

as $n \to \infty$. On the other hand, by Lemma 3.1 and Hölder inequality, it is easy to check that

$$\int_{\mathbb{R}^3} K(x)(\phi_{u_n} - \phi_u)u\varphi dx \to 0, \tag{3.7}$$

as $n \to \infty$. Thus, by (3.6) and (3.7), we have

$$\begin{aligned} & \int_{\mathbb{R}^3} (K(x)\phi_{u_n} u_n \varphi - K(x)\phi_u u \varphi) dx \\ = & \int_{\mathbb{R}^3} K(x)(\phi_{u_n} - \phi_u) u \varphi dx + \int_{\mathbb{R}^3} K(x)\phi_{u_n}(u_n - u)\varphi dx \to 0 \end{aligned}$$

as $n \to \infty$.

Finally, we prove the conclusion (3.3). By the similar argument, we have

$$\int_{\mathbb{R}^3} K(x)\phi_{u_n} u_n \varphi_n dx \to \int_{\mathbb{R}^3} K(x)\phi_{u_n} u_n \varphi dx, \tag{3.8}$$

$$\int_{\mathbb{R}^3} K(x)\phi_{u_n} u_n \varphi dx \to \int_{\mathbb{R}^3} K(x)\phi_{u_n} u \varphi dx, \tag{3.9}$$

and

$$\int_{\mathbb{R}^3} K(x)\phi_{u_n} u \varphi dx \to \int_{\mathbb{R}^3} K(x)\phi_u u \varphi dx, \tag{3.10}$$

as $n \to \infty$. Thus, by (3.8)-(3.10), we have

$$\int_{\mathbb{R}^3} K(x)\phi_{u_n} u_n \varphi_n dx \to \int_{\mathbb{R}^3} K(x)\phi_u u \varphi dx$$

as $n \to \infty$. The proof is complete. □

In order to find critical points of Φ, we will use the following abstract theorem which is taken from [19] and [8].

Let E be a Banach space with direct sum decomposition $E = X \oplus Y$, $u = x + y$ and corresponding projections P_X, P_Y onto X, Y, respectively. For a functional $\Phi \in C^1(E, \mathbb{R})$ we write $\Phi_b = \{u \in E : \Phi(u) \geq b\}$. Recall that a sequence $\{u_n\} \subset E$ is said to be a $(C)_c$-sequence (respectively, $(PS)_c$-sequence) if $\Phi(u_n) \to c$ and $(1 + \|u_n\|)\Phi'(u_n) \rightharpoonup 0$ (respectively,$\Phi'(u_n \to 0)$. Φ is said to satisfy the $(C)_c$-condition (respectively, $(PS)_c$-condition) if any $(C)_c$-sequence (respectively, $(PS)_c$-sequence) has a convergent subsequence.

Now we assume that X is separable and reflexive, and we fix a countable dense subset $\mathcal{S} \subset \mathcal{X}^*$. For each $s \in \mathcal{S}$ there is a semi-norm on E defined by

$$p_s : E \to \mathbb{R},\ p_s(u) : |s(x)| + \|y\| \text{ for } u = x + y \in E.$$

We denote by $\mathcal{T}_S$ the topology induced by semi-norm family $\{p_s\}$, w^* denote the weak*-topology on E^*. Suppose

(Φ_0) for any $c \in \mathbb{R}$, superlevel Φ_c is $\mathcal{T}_S$-closed, and $\Phi' : (\Phi_c, \mathcal{T}_S) \to (E^*, w^*)$ is continuous;

(Φ_1) for any $c > 0$, there exists $\xi > 0$ such that $\|u\| < \xi \|P_Y u\|$ for all $u \in \Phi_c$;

(Φ_2) there exists $\rho > 0$ such that $\kappa := \inf \Phi(S_\rho \cap Y) > 0$, where $S_\rho := \{u \in E : \|u\| = \rho\}$;

The following theorem is a special case of Theorem 3.4 in [8], see also Theorem 4.3 in [19].

***Theorem* 3.3** — *Let* $(\Phi_0) - (\Phi_2)$ *be satisfied and suppose there are* $R > \rho > 0$ *and* $e \in Y$ *with* $\|e\| = 1$ *such that* $\sup \Phi(\partial Q) \leq \kappa$ *where* $Q := \{u = x + te : x \in X, t \geq 0, \|u\| < R\}$. *Then* Φ *has a* $(C)_c$*-sequence with* $\kappa \leq c \leq \sup \Phi(Q)$.

The following lemma is useful to verify (Φ_0) (see [19] or [8]).

Lemma 3.4 — Suppose $\Phi \in C^1(E, \mathbb{R})$ is of the form

$$\Phi(u) = \frac{1}{2}(\|y\|^2 - \|x\|^2) - \Psi(u) \ \text{ for } u = x + y \in E = X \oplus Y$$

such that

(i) $\Psi \in C^1(E, \mathbb{R})$ is bounded from blow;

(ii) $\Psi : (E, \mathcal{T}_w) \to \mathbb{R}$ is sequentially lower semicontinuous, that is, $u_n \rightharpoonup u$ in E implies $\Psi(u) \leq \liminf \Psi(u_n)$;

(iii) $\Psi' : (E, \mathcal{T}_w) \to (E^*, \mathcal{T}_{w^*})$ is sequentially continuous;

(iv) $\nu : E \to \mathbb{R}, \nu(u) = \|u\|^2$, is C^1 and $\nu' : (E, \mathcal{T}_w) \to (E^*, \mathcal{T}_{w^*})$ is sequentially continuous.

Then Φ satisfies (Φ_0).

To prove our main result, we will make use of the associated limit problem. Precisely, we will consider the following periodic problem

$$\begin{cases} -\Delta u + V(x)u = a|u|^{p-2}u, \\ u \in H^1(\mathbb{R}^3) \end{cases} \tag{3.11}$$

where $V(x)$ satisfies assumption (V), and a is given in assumption (F). Similar to the previous variational setting, we know that the solutions of (3.11) are critical points of the following functional defined by

$$\begin{aligned} \Phi_\infty(u) &= \frac{1}{2}(\|u^+\|^2 - \|u^-\|^2) - \frac{1}{p} a \int_{\mathbb{R}^3} |u|^p \\ &= \frac{1}{2}(\|u^+\|^2 - \|u^-\|^2) - \Psi_\infty(u), \ \ u \in E, \end{aligned}$$

where $\Psi_\infty(u) = \frac{1}{p}a|u|_p^p$.

Let $\mathcal{K}_\infty := \{u \in E : \Phi'_\infty(u) = 0\}$ be critical set, $\tilde{C} := \inf\{\Phi_\infty(u), u \in \mathcal{K}_\infty \backslash \{0\}\}$ be the least energy, $\tilde{S} := \{u \in \mathcal{K}_\infty : \Phi_\infty(u) = \tilde{C}\}$ be least energy solution set. It is well known that there exist some works for problem (3.11) about the study of existence of ground state solutions, see [30, 35, 36, 38, 42]. The following lemma can be found in [35] (see also [30]).

Lemma 3.5 — $\mathcal{K}_\infty \neq \emptyset$, $\tilde{C} > 0$. Moreover, $\tilde{C}$ is achieved, that is, $\tilde{S} \neq \emptyset$.

Following Ackermann [1] and Ding and Lee [17] (see also [18, 19]), for fixed $u \in E^+$, we introduce the functional $F_u : E^- \to \mathbb{R}$ by

$$F_u(v) = \Phi_\infty(u+v) = \frac{1}{2}(\|u\|^2 - \|v\|^2) - \Psi_\infty(u+v).$$

For any $v, \eta \in E^-$, we have

$$F''_u(v)[\eta,\eta] = -\|\eta\|^2 - \Psi''_\infty(u+v)[\eta,\eta] \leq -\|\eta\|^2,$$

which implies $F_u(\cdot)$ is strictly concave. Moreover,

$$F_u(v) \leq \frac{1}{2}(\|u\|^2 - \|v\|^2) \to -\infty, \text{ as } \|v\| \to \infty.$$

Plainly, F_u is weakly sequentially upper semicontinuous. Therefore, there is a unique $h_\infty(u) \in E^-$ such that

$$F_u(h_\infty(u)) = \max_{v \in E^-} F_u(v).$$

As Lemma 5.6 in [1] or Lemma 3.5 in [17], the map $h_\infty : E^+ \to E^-$ has following properties:

(1) h_∞ is $\mathbb{R}^3$-invariant, i.e., $h_\infty(k * u) = h_\infty(u)$ where $(k * u)(x) = u(x+k)$ for all $k \in \mathbb{R}^3$;

(2) h_∞ is bounded map in $C^1(E^+, E^-)$, and $h_\infty(0) = 0$;

(3) If $u_n \rightharpoonup u$ in E^+, the $h_\infty(u_n) - h_\infty(u_n - u) \to h_\infty(u)$ and $h_\infty(u_n) \rightharpoonup h_\infty(u)$.

4. Proof of the Main Result

We are going to prove the main result. Let

$$\mathcal{K} := \{u \in E : \Phi'(u) = 0\}$$

be the critical set of Φ. Set

$$\hat{C} := \inf\{\Phi(u) : u \in \mathcal{K} \backslash \{0\}\} \text{ and } \hat{S} := \{u \in \mathcal{K} : \Phi(u) = \hat{C}\}.$$

To apply Theorem 3.3. Now, we will show the properties of Γ, Ψ.

Lemma 4.1 — Γ, Ψ are non-negative and weakly sequentially lower semicontinuous. Γ', Ψ' are weakly sequentially continuous.

PROOF : It is clear that Γ and Ψ are non-negative by the assumption (F) and Lemma 3.1. Ψ' is weakly sequentially continuous from the fact that E embeds continuously in L^s for all $s \in [2,6]$, and compactly in L^s_{loc} for all $s \in [2,6)$. Similar to the conclusion (3.2) of Lemma 3.2, we know that Γ' is weakly sequentially continuous. □

Now we discuss the linking structure of Φ.

Lemma 4.2 — There exists $r > 0$ and $\rho > 0$ such that $\Phi|_{B_r}(u) \geq 0$ and $\Phi|_{S_r}(u) \geq \rho$, where $B_r = \{u \in E^+ : \|u\| \leq r\}$ and $S_r = \{u \in E^+ : \|u\| = r\}$.

PROOF : For any $u \in E^+$, by (2.4) and Sobolev inequality, we have

$$\begin{aligned} \Phi(u) &= \frac{1}{2}\|u\|^2 - \Gamma(u) - \Psi(u) \\ &\geq \frac{1}{2}\|u\|^2 - C_2\|u\|^4 - C_3\|u\|^p. \end{aligned}$$

Since $p \in (2,6)$, choosing suitable $r > 0$ we see that the desired conclusion holds. □

Lemma 4.3 — There exists $R > 0$ such that, for any $e \in E^+$ with $\|e\| = 1$ and $E_e = E^- \oplus \mathbb{R}e$, $\Phi(u) < 0$ for all $u \in E_e \backslash B_R$.

PROOF : For any $u \in E_e$, that is $u = te + v$ for some $t \in \mathbb{R}$ and $v \in E^-$. By (2.1), we have

$$\begin{aligned} \Phi(u) &= \frac{1}{2}(|t|^2\|e\|^2 - \|v\|^2) - \Gamma(u) - \Psi(u) \\ &\leq \frac{1}{2}|t|^2 - \frac{1}{2}\|v\|^2 - \frac{a}{p}|te+v|_p^p \\ &\leq \frac{1}{2}|t|^2 - \frac{1}{2}\|v\|^2 - \frac{\beta_p a}{p}|t|^p. \end{aligned}$$

Since $p \in (2,6)$, choosing large $R > 0$ we see that the desired conclusion holds. □

Let h_∞ be the induced map from $E^+ \to E^-$ as in Section 3. From now on, we assume that $e_0 \in E^+$ such that $e_0 + h_\infty(e_0) \in \tilde{S}$. Set $E_{e_0} = E^- + \mathbb{R}e_0$.

Lemma 4.4 — We have

$$d := \sup\{\Phi(u) : u \in E_{e_0}\} < \tilde{C}.$$

PROOF : Observe that by Lemma 4.2 and the linking property we have $d \geq \rho$. Since $\Phi(u) \leq \Phi_\infty(u)$ for all $u = te_0 + v$, $v \in E^-, t \in \mathbb{R}$, and by Lemma 3.11 in Ding and Lee [17],

$$\Phi_\infty(u) = \Phi_\infty(te_0 + v) \leq \Phi_\infty(te_0 + h_\infty(te_0)) \leq \tilde{C}.$$

Hence $d \leq \tilde{C}$. Next we prove that $d < \tilde{C}$. Assume by contradiction that $d = \tilde{C}$. Let $u_n = v_n + t_n e_0 \in E_{e_0}$ be such that $d - \frac{1}{n} \leq \Phi(u_n) \to d$. It follows from Lemma 4.3 that $\{u_n\}$ is bounded. Hence we can assume that, up to a subsequence, $u_n \rightharpoonup u$ in E_{e_0} with $v_n \rightharpoonup v \in E^-$ and $t_n \to t$. It is clear that $t \neq 0$. In fact, if $t = 0$, then

$$d - \frac{1}{n} \leq \Phi(v_n + t_n e_0) \leq \Phi_\infty(v_n + t_n e_0) \leq \Phi_\infty(t_n e_0 + h_\infty(t_n e_0)) \leq \tilde{C},$$

which implies that $\tilde{C} = 0$. Hence

$$\begin{aligned} d - \frac{1}{n} \leq \Phi(u_n) &\leq \Phi_\infty(u_n) - \frac{1}{p}\int_{\mathbb{R}^3}(q(x) - a)|u_n|^p dx \\ &\leq \tilde{C} - \frac{1}{p}\int_{\mathbb{R}^3}(q(x) - a)|u|^p dx. \end{aligned}$$

Taking the limit yields

$$\tilde{C} \leq \tilde{C} - \frac{1}{p}\int_{\mathbb{R}^3}(q(x) - a)|u|^p dx$$

which implies that $u = 0$ since $q(x) > a$ as in condition (F), a contradiction. □

Set

$$Q := \{u = v + te_0 : v \in E^-, t \geq 0, \|u\| \leq R\}.$$

As a consequence of Lemma 4.4, we have the following

Lemma 4.5 — $\sup \Phi(Q) < \tilde{C}$.

We now turn to the analysis on $(C)_c$-sequences including the boundness and the compactness. Firstly, we have

Lemma 4.6 — Under the assumptions of Theorem 1.1. Then any $(C)_c$-sequences of Φ is bounded.

PROOF : Let $\{u_n\} \subset E$ be such that

$$\Phi(u_n) \to c \text{ and } (1 + \|u_n\|)\Phi'(u_n) \to 0, \tag{4.1}$$

and

$$\frac{1}{2}\Phi'(u_n)u_n = \frac{1}{2}(\|u_n^+\|^2 - \|u_n^-\|^2) - 2\Gamma(u_n) - \frac{p}{2}\Psi(u_n). \tag{4.2}$$

By (4.1) and (4.2), there is constant $C_4 > 0$ such that we have

$$C_4 \geq \Phi(u_n) - \frac{1}{2}\Phi'(u_n)u_n = \Gamma(u_n) + (\frac{p}{2} - 1)\Psi(u_n) \geq 0. \tag{4.3}$$

Arguing indirectly, assume that up to a subsequence $\|u_n\| \to \infty$. Set $v_n = \frac{u_n}{\|u_n\|}$. Then $\|v_n\| = 1$ and $|v_n|_s \leq c_s\|v_n\| = c_s$ for $s \in [2, 6)$. Hence, we can assume that up to a subsequence $v_n \rightharpoonup v$. Moreover, by (4.3),

$$\frac{\Psi(u_n)}{\|u_n\|} \to 0. \tag{4.4}$$

Then, by (4.4)

$$\|u_n\|^{p-1}|v_n|_p^p \to 0. \tag{4.5}$$

Similarly, we have

$$\frac{\Gamma(u_n)}{\|u_n\|} \to 0. \tag{4.6}$$

On the other hand,

$$\Phi'(u_n)(u_n^+ - u_n^-) = \|u_n\|^2 \left(1 - \frac{\Gamma'(u_n)(u_n^+ - u_n^-) + \Psi'(u_n)(u_n^+ - u_n^-)}{\|u_n\|^2}\right). \tag{4.7}$$

By Hölder inequality, we have

$$\begin{aligned} \left|\Psi'(u_n)(u_n^+ - u_n^-)\right| &= \left|\int_{\mathbb{R}^3} q(x)|u_n|^{p-2}u_n(u_n^+ - u_n^-)dx\right| \\ &\leq C_5 \int_{\mathbb{R}^3} |u_n|^{p-1}|u_n^+ - u_n^-|dx \\ &\leq C_6|u_n|_p^{p-1}|u_n^+ - u_n^-|_p. \end{aligned} \tag{4.8}$$

Then by (4.5) and (4.8), we have

$$\frac{\Psi'(u_n)(u_n^+ - u_n^-)}{\|u_n\|^2} \to 0. \tag{4.9}$$

Observe that the functional Γ, which is the unique non-local term in Φ, satisfies the conditions in Ackermann [2]. Hence, by Lemma 3.6 in [2] and (4.6), we have

$$\begin{aligned} \left|\frac{\Gamma'(u_n)(u_n^+ - u_n^-)}{\|u_n\|^2}\right| &\leq \frac{\|\Gamma'(u_n)\|_{E^*}\|u_n^+ - u_n^-\|}{\|u_n\|^2} \\ &\leq C_7\left|\frac{(\sqrt{\Gamma'(u_n)u_n} + \Gamma'(u_n)u_n)\|u_n^+ - u_n^-\|}{\|u_n\|^2}\right| \\ &\leq C_8\left|\frac{\sqrt{\Gamma'(u_n)u_n} + \Gamma'(u_n)u_n}{\|u_n\|}\right| \\ &= C_8\left(\frac{1}{\sqrt{\|u_n\|}}\sqrt{\frac{4\Gamma(u_n)}{\|u_n\|}} + \frac{4\Gamma(u_n)}{\|u_n\|}\right) \to 0. \end{aligned} \tag{4.10}$$

Therefore, from (4.7), (4.9) and (4.10), we have $\Phi'(u_n)(u_n^+ - u_n^-) \to \infty$. It is a contradiction. Hence $\{u_n\}$ is bounded in E. □

In what follows, let $\{u_n\}$ be a $(C)_c$-sequence of Φ. By Lemma 4.6, it is bounded, hence, up to a subsequence, $u_n \rightharpoonup u$ in E. It is obvious that u is a critical point of Φ. In order to establish compactness condition, we need to prove some results.

Lemma 4.7 — Suppose that $\{u_n\}$ does not converge to u strongly in E, then $u_n^1 := u_n - u$ is a $(PS)_{c_0}$-sequence for Φ_∞ with $c_0 = c - \Phi(u) \geq 0$ and $u_n^1 \rightharpoonup 0$.

PROOF : Since $u_n \rightharpoonup u$, we have $u_n^1 \rightharpoonup 0$. We can assume that $u_n^+ \rightharpoonup u^+$ (resp.$u_n^- \rightharpoonup u^-$) and $u_n^{1+} := u_n^+ - u^+ \rightharpoonup 0$ (resp. $u_n^{1-} := u_n^+ - u^+ \rightharpoonup 0$). Then by direct computation we have

$$\|u_n^+\|^2 = \|u_n^{1+} + u^+\|^2 = \|u_n^{1+}\|^2 + \|u^+\|^2 + o(1) \tag{4.11}$$

and

$$\|u_n^-\|^2 = \|u_n^{1-} + u^-\|^2 = \|u_n^{1-}\|^2 + \|u^-\|^2 + o(1). \tag{4.12}$$

According to Brezis-Lieb lemma in [40], we have

$$|u_n|_p^p = |u|_p^p + |u_n^1|_p^p + o(1). \tag{4.13}$$

Now, let us show that

$$(q(x) - a)|u_n^1|^{p-2}u_n^1 \to 0 \text{ in } E^*. \tag{4.14}$$

Indeed, for any $\varphi \in E$,

$$\begin{aligned}&\int_{\mathbb{R}^3}(q(x)-a)|u_n^1|^{p-2}u_n^1\varphi dx\\ &= \int_{|x|\leq R}(q(x)-a)|u_n^1|^{p-2}u_n^1\varphi dx + \int_{|x|\geq R}(q(x)-a)|u_n^1|^{p-2}u_n^1\varphi dx.\end{aligned}$$

Since $E \hookrightarrow L_{loc}^p$ compactly, $u_n^1 \to 0$ in L_{loc}^p for $p \in [2,6)$. Hence, for any $\varepsilon > 0$, we have

$$\left|\int_{|x|\leq R}(q(x)-a)|u_n^1|^{p-2}u_n^1\varphi dx\right| \leq C_9|u_n^1|_p^{p-1}|\varphi|_p \leq \varepsilon\|\varphi\|.$$

In virtue of assumption (F) we know when $R > 0$ is large enough, $|q(x) - a| \leq \varepsilon$ for $|x| \geq R$. By the boundness of u_n^1 in E, there holds

$$\left|\int_{|x|\geq R}(q(x)-a)|u_n^1|^{p-2}u_n^1\varphi dx\right| \leq \varepsilon|u_n^1|_p^{p-1}|\varphi|_p \leq \varepsilon C_{10}\|\varphi\|.$$

Thus

$$\int_{\mathbb{R}^3}(q(x)-a)|u_n^1|^{p-2}u_n^1\varphi dx\to 0.$$

Hence (4.14) holds.

Similarly, we also have

$$\int_{\mathbb{R}^3}(q(x)-a)|u_n^1|^p dx\to 0. \tag{4.15}$$

Moreover, by using Lemma 8.1 in [40], we have

$$|u_n|^{p-2}u_n=|u|^{p-2}u+|u_n^1|^{p-2}u_n^1+o(1)\ \text{ in } E^*. \tag{4.16}$$

Therefore, by (3.1)-(3.3) and (4.11)-(4.15), we obtain

$$\begin{aligned}
\Phi(u_n) &= \frac{1}{2}(\|u_n^+\|^2-\|u_n^-\|^2)-\frac{1}{4}\int_{\mathbb{R}^3}K(x)\phi_{u_n}u_n^2-\frac{1}{p}\int_{\mathbb{R}^3}q(x)|u_n|^p\\
&= \frac{1}{2}(\|u^+\|^2-\|u^-\|^2)-\frac{1}{4}\int_{\mathbb{R}^3}K(x)\phi_u u^2-\frac{1}{p}\int_{\mathbb{R}^3}q(x)|u|^p\\
&\quad +\frac{1}{2}(\|u_n^{1+}\|^2-\|u_n^{1-}\|^2)-\frac{1}{p}\int_{\mathbb{R}^3}a|u_n^1|^p+o(1)\\
&= \Phi(u)+\Phi_\infty(u_n^1)+o(1),
\end{aligned}$$

and for all $\varphi\in E$,

$$\begin{aligned}
o(1) &= (\Phi'(u_n),\varphi)\\
&= (u_n^+,\varphi)-(u_n^-,\varphi)-\int_{\mathbb{R}^3}K(x)\phi_{u_n}u_n\varphi dx-\int_{\mathbb{R}^3}q(x)|u_n|^{p-2}u_n\varphi dx\\
&= (u^+,\varphi)-(u^-,\varphi)-\int_{\mathbb{R}^3}K(x)\phi_u u\varphi dx-\int_{\mathbb{R}^3}q(x)|u|^{p-2}u\varphi dx\\
&\quad +(u_n^{1+},\varphi)-(u_n^{1-},\varphi)-\int_{\mathbb{R}^3}a|u_n^1|^{p-2}u_n^1\varphi dx+o(1)\\
&= (\Phi'(u),\varphi)+(\Phi'_\infty(u_n^1),\varphi)+o(1)\\
&= (\Phi'_\infty(u_n^1),\varphi)+o(1).
\end{aligned}$$

Hence

$$\Phi_\infty(u_n^1)=c-\Phi(u)+o(1)$$

and

$$\Phi'_\infty(u_n^1)=o(1)\ \text{ in } E^*. \tag{4.17}$$

Furthermore, it holds that

$$\Phi_\infty(u_n^1) = \frac{1}{2}\Phi'_\infty(u_n^1)u_n^1 = (\frac{p}{2} - 1)\Psi_\infty(u_n^1) \geq 0. \tag{4.18}$$

Thus we have proved the conclusion. □

Lemma 4.8 — Under the assumptions of Lemma 4.7. There exist a sequence $k_n^1 \in \mathbb{R}^3$ with $|k_n^1| \to \infty$ and a critical point $u^1 \neq 0$ of Φ_∞ satisfying $k_n^1 * u_n^1 \rightharpoonup u^1$ and

$$\Phi_\infty(k_n^1 * u_n^1 - u^1) \to c - \Phi(u) - \Phi_\infty(u^1) \geq 0. \tag{4.19}$$

PROOF : Observe that

$$(\Phi'_\infty(u_n^1), u_n^{1+} - u_n^{1-}) = \|u_n^1\|^2 - a\int_{\mathbb{R}^3} |u_n^1|^{p-2}u_n^1(u_n^{1+} - u_n^{1-})dx. \tag{4.20}$$

By a direct computation, we obtain

$$\begin{aligned} o(1) &= (\Phi'(u_n), u_n^+ - u_n^-) \\ &= (\Phi'(u), u^+ - u^-) + (\Phi'_\infty(u_n^1), u_n^{1+} - u_n^{1-}) + o(1) \\ &= (\Phi'_\infty(u_n^1), u_n^{1+} - u_n^{1-}) + o(1). \end{aligned} \tag{4.21}$$

Setting

$$\delta := \limsup_{n\to\infty}\left(\sup_{y\in\mathbb{R}^3}\int_{B_1(y)} |u_n^1|^p dx\right),$$

we have $\delta > 0$. Actually, if $\delta = 0$ would be true, then by Lemma 2.1 in [40], $u_n^1 \to 0$ in L^p for $p \in (2,6)$. Thus by (4.20) and (4.21) and Hölder inequality, $u_n^1 \to 0$ in E. It is a contradiction. Then we may assume the existence of $\{k_n^1\} \subset \mathbb{R}^3$ such that

$$\int_{B_1(k_n^1)} |u_n^1|^p dx > \frac{\delta}{2}.$$

Let us now consider $k_n^1 * u_n^1$. We may assume that $k_n^1 * u_n^1 \rightharpoonup u^1$ in E. Therefore, $k_n^1 * u_n^1 \to u^1$ a.e. in $\mathbb{R}^3$. Since

$$\int_{B_1(0)} |k_n^1 * u_n^1|^p dx > \frac{\delta}{2},$$

from the Rellich theorem it follows that

$$\int_{B_1(0)} |u^1|^p dx > \frac{\delta}{2},$$

thus $u^1 \neq 0$. Since $u_n^1 \rightharpoonup 0$ in E, $\{k_n^1\}$ must be unbounded, and up to a subsequence, we can assume $|k_n^1| \to +\infty$. Furthermore, (4.17) implies $\Phi'_\infty(u^1) = 0$, and similar argument as Lemma 4.7 shows

$$\begin{aligned} \Phi_\infty(u^1) + \Phi_\infty(k_n^1 * u_n^1 - u^1) + o(1) &= \Phi_\infty(k_n^1 * u_n^1) = \Phi_\infty(u_n^1) \\ &= \Phi(u_n) - \Phi(u) + o(1) \\ &= c - \Phi(u) + o(1) \end{aligned} \tag{4.22}$$

and

$$\Phi'_\infty(k_n^1 * u_n^1 - u^1) = o(1) \text{ in } E^*. \tag{4.23}$$

By (4.22), (4.23) and similar computation as in (4.18) yield (4.19). The proof is complete. □

With these preparations, we have the following compactness lemma.

Lemma 4.9 — Either

(i) $u_n \to u$, or

(ii) $c \geq \tilde{C}$ and there a positive integer m, points $u^1, \cdots, u^m \in \mathcal{K}_\infty$, a subsequence denoted again by $\{u_n\}$, and sequence $\{k_n^i\} \subset \mathbb{R}^3$, such that, as $n \to \infty$,

$$\left\| u_n - u - \sum_{i=1}^{m} (k_n^i * u^i) \right\| \to 0,$$

$$|k_n^i| \to \infty, \ \ |k_n^i - k_n^j| \to \infty, \ \ \text{if } i \neq j$$

and

$$\Phi(u) + \sum_{i=1}^{m} \Phi_\infty(u^i) = c.$$

PROOF : Suppose that conclusion (i) is false, then $\{u_n\}$ does not converge to u strongly in E. Therefore, we have the results in Lemma 4.7 and Lemma 4.8. Note that $c \geq \tilde{C}$ since $\Phi(u) \geq 0$ and $\Phi_\infty(u^1) \geq \tilde{C}$. If $k_n^1 * u_n^1 \to u^1$, then we are done. Otherwise, repeating the argument as in Lemma 4.7 and Lemma 4.8, after at most finitely many steps we can finish the proof since $\Phi_\infty(u^i) \geq \tilde{C} > 0$. □

Lemma 4.10 — If $c < \tilde{C}$, then Φ satisfies the $(C)_c$-condition.

PROOF : It is a straight consequence of Lemma 4.9. □

PROOF OF THEOREM 1.1 : Observe that, the combination of Lemma 3.4 and Lemma 4.1 implies that Φ verifies (Φ_0). It is clear that Φ checks (Φ_1) because of the form (2.6). Lemma 4.2 is nothing but (Φ_2). Lemma 4.3 shows that the linking condition of Theorem 3.4 is satisfied. These, together with Lemma 4.5 yield a $(C)_c$ sequence $\{u_n\}$ with $c < \tilde{C}$ for Φ. Hence, by virtue of Lemma 4.10, $u_n \to u$ such that $\Phi'(u) = 0$ and $\Phi(u) \geq \rho$. Therefore, $\mathcal{K}\backslash\{0\} \neq \emptyset$.

In what following, we prove $\hat{C} > 0$. Indeed, assume by contradiction that $\hat{C} = 0$. Then there exists $\{u_n\} \subset \mathcal{K}\backslash\{0\}$ such that $\Phi(u_n) \to 0$. Then, by Lemma 4.6, $\{u_n\}$ is bounded. We can assume that, up to a subsequence, $u_n \rightharpoonup u \in \mathcal{K}$. Then

$$\Phi(u_n) = \Gamma(u_n) + (\frac{p}{2} - 1)\Psi(u_n) \to 0.$$

Since $\Gamma(u_n) \geq 0$ and $\Psi(u_n) \geq 0$, we have

$$\Gamma(u_n) \to 0 \ \text{ and } \ \Psi(u_n) \to 0. \tag{4.24}$$

Note that $q(x) \geq a$. Hence $\Psi(u_n) \to 0$ implies $|u_n|_p \to 0$. Since $\Phi'(u_n)(u_n^+ - u_n^-) = 0$ and

$$\Phi'(u_n)(u_n^+ - u_n^-) = \|u_n\|^2 - \Gamma'(u_n)(u_n^+ - u_n^-) - \Psi'(u_n)(u_n^+ - u_n^-),$$

we have

$$\|u_n\|^2 = \Gamma'(u_n)(u_n^+ - u_n^-) + \Psi'(u_n)(u_n^+ - u_n^-). \tag{4.25}$$

By assumption (K) and Hölder inequality,

$$\begin{aligned} |\Psi'(u_n)(u_n^+ - u_n^-)| &= \left| \int_{\mathbb{R}^3} q(x)|u_n|^{p-2}u_n(u_n^+ - u_n^-) \right| \\ &\leq C_{11}|u_n|_p^{p-1}|u_n^+ - u_n^-|_p \to 0. \end{aligned} \tag{4.26}$$

By (4.24) and Lemma 3.6 in [2], we have

$$\begin{aligned} \left|\Gamma'(u_n)(u_n^+ - u_n^-)\right| &= \|\Gamma'(u_n)\|_{E^*}\|u_n^+ - u_n^-\| \\ &\leq C_{12}\left(\sqrt{\Gamma'(u_n)u_n} + \Gamma'(u_n)u_n\right)\|u_n\| \\ &= C_{12}\left(\sqrt{4\Gamma(u_n)} + 4\Gamma(u_n)\right)\|u_n\| \to 0. \end{aligned} \tag{4.27}$$

Here we used the fact that $\{u_n\}$ is bounded. Hence (4.25)-(4.27) imply $\|u_n\| \to 0$. Furthermore, by Sobolev embedding inequality, we have

$$\|u_n\|^2 \leq C_{14}\|u_n\|^p + C_{15}\|u_n\|^4 = C_{16}(\|u_n\|^{p-2} + \|u_n\|^2)\|u_n\|^2.$$

Hence, $1 \leq o(1)$, a contradiction.

Now, we show that $\hat{C}$ is achieved. The above argument implies $\hat{C} > 0$, then there exists $\{u_n\}$ such that $\Phi(u_n) \to \hat{C}$, $\Phi'(u_n) \to 0$. Since $\hat{C} < \tilde{C}$, we have $u_n \to u$ in E with $\Phi(u) = \hat{C}$, $\Phi'(u) = 0$, hence $\hat{S} \neq \emptyset$. The proof is complete. □

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