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GENERALIZATIONS OF THE AREA THEOREM FOR MEROMORPHIC UNIVALENT FUNCTIONS WITH NONZERO POLE[1]

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In this article, we consider meromorphic univalent functions f in the unit disc of the complex plane having a simple pole at $z = \alpha \in (0, 1)$ with nonzero residue b at $z = \alpha$. In 1969, P.N. Chichra proved an area theorem for such functions. In this note, we generalize this theorem and prove an interesting consequence of this result.

Key words : Univalent functions; meromorphic fuctions.

1. Introduction

Let $\mathbb{C}$ denote the complex plane and $\widehat{\mathbb{C}}$ denote the extended complex plane $\mathbb{C} \cup \{\infty\}$. We shall use the following notations: $\mathbb{D} = \{z : |z| < 1\}$, $\overline{\mathbb{D}} = \{z : |z| \leq 1\}$ and $N_\delta(\overline{\mathbb{D}}) = \{z : |z| < 1 + \delta\}$, $\delta > 0$. Let f be a meromorphic and univalent function in the unit disk $\mathbb{D}$ having a simple pole at $z = \alpha \in [0, 1)$ with residue b. Since $f - b/(z - \alpha)$ is analytic in $\mathbb{D}$, one has an expansion for f of the form

$$f(z) = \frac{b}{z - \alpha} + h(z) \tag{1.1}$$

where h is analytic in $\mathbb{D}$ with the Taylor expansion $h(z) = \sum_{n=0}^{\infty} a_n z^n$, $z \in \mathbb{D}$. We denote the class of such functions f by $\Sigma(\alpha)$. In 1914, Gronwall proved the classical area theorem for functions in $\Sigma(0)$ having the Laurent expansion (1.1) with $\alpha = 0$ in $\mathbb{D}$ can be restated as:

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***Theorem* A** — *If* $\phi(z) = \sum_{n=0}^{\infty} b_n z^n$*,* $b_0 \neq 0$ *is an analytic function in* $\mathbb{D}$ *such that the meromorphic function* ϕ/z *is one-one in* $\mathbb{D}$*, then*

$$\sum_{n=1}^{\infty} \frac{n^2|b_n|^2}{n+1} \leq \sum_{n=0}^{\infty} \frac{|b_n|^2}{n+1}$$

or, equivalently,

$$\int_{\mathbb{D}} |z\phi'(z)|^2 \, dm(z) \leq \int_{\mathbb{D}} |\phi(z)|^2 \, dm(z),$$

where $dm(z)(= dx\,dy)$ *denotes the area measure on* $\mathbb{D}$*.*

In 2011, Pavlović *et al.* [3, Theorem 2] generalized the afore-mentioned Gronwall's result in the following form:

***Theorem* B** — *Let* $p > 0$*. If* ϕ *is a function as in Theorem A, then we have*

$$\int_{\mathbb{D}} |z|^p |\phi(z)|^{p-2} |\phi'(z)|^2 \, dm(z) \leq \int_{\mathbb{D}} |z|^{p-2} |\phi(z)|^p \, dm(z).$$

Note that, the particular case of $p = 2$ in the above theorem is nothing but the Gronwall's area theorem. In 1969, the Gronwall's area theorem was extended by Chichra [1, Lemma, p. 317] for the functions in $\Sigma(\alpha)$, $\alpha \in (0, 1)$, as follows:

***Theorem* C** — *Let* $h(z) = \sum_{n=0}^{\infty} a_n z^n$ *be an analytic function in* $\mathbb{D}$*. If* $b \in \mathbb{C} \setminus \{0\}$ *is such that the meromorphic function* $b/(z - \alpha) + h$*,* $\alpha \in (0, 1)$*, is univalent in* $\mathbb{D}$*, then*

$$\int_{\mathbb{D}} |h'(z)|^2 dm(z) = \pi \sum_{n=1}^{\infty} n|a_n|^2 \leq \frac{\pi|b|^2}{(1 - \alpha^2)^2}. \tag{1.2}$$

The above theorem can also be expressed as:

***Theorem* D** — *Let* ϕ *be an analytic function in* $\mathbb{D}$ *such that the function*

$$\frac{\phi}{z - \alpha} = \frac{b}{z - \alpha} + h$$

is univalent in $\mathbb{D}$ *where* h *is analytic in* $\mathbb{D}$ *with the expansion* $h(z) = \sum_{n=0}^{\infty} a_n z^n$*. Then we have*

$$\int_{\mathbb{D}} \left| \left(\frac{\phi(z) - b}{z - \alpha} \right)' \right|^2 \, dm(z) \leq \frac{\pi|b|^2}{(1 - \alpha^2)^2}.$$

In this article, we wish to obtain generalizations of Theorem C and Theorem D. Further, we obtain an interesting consequence of these results. The idea of the proof of the main theorem (Theorem 1) in the next Section is borrowed from [3].

2. RESULTS

The proof of our main Theorem is based on a result due to Prawitz [4]. We give a proof of a similar result by slightly modifying it (compare [2]), as we need this to establish our main Theorem. We present this modified result below:

Lemma 1 — Let $f : \mathbb{D} \to \mathbb{C}$ be an analytic univalent function such that $f(\alpha) = 0$ where $\alpha \in (0,1)$. For $r \in (\alpha, 1)$, let

$$J_p(r) = J_p(r,f) = \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^{-p}\, d\theta, \quad p > 0.$$

Then $J_p'(r) < 0$, for $r \in (\alpha, 1)$.

PROOF : We first observe that

$$\begin{aligned} 2\pi J_p'(r) &= -p\int_0^{2\pi} |f(re^{i\theta})|^{-p-2}\mathrm{Re}\left(\overline{f(re^{i\theta})}f'(re^{i\theta})e^{i\theta}\right)\, d\theta \\ &= (-p/r)\,\mathrm{Im}\left(\int_{|\zeta|=r} |f(\zeta)|^{-p-2}\overline{f(\zeta)}f'(\zeta)\, d\zeta\right) \\ &= (-p/r)\,\mathrm{Im}\left(\int_{\Gamma_r} |w|^{-p-2}\overline{w}\, dw\right), \quad w = f(\zeta) = u + iv \\ &= -(p/r)\int_{\Gamma_r} |w|^{-p-2}(udv - vdu), \end{aligned} \tag{2.1}$$

for all $r \in (\alpha, 1)$ where $\Gamma_r := f(|\zeta| = r)$. Here Γ_r orients *positively*. Applying the Green's theorem in the domain $\Omega_{r,R}$ bounded by Γ_r and the circle $|w| = R$ where $R > \max_{|z|=r} |f(z)|$ for $r \in (\alpha, 1)$, we have

$$\begin{aligned} &\int_{|w|=R} |w|^{-p-2}(udv - vdu) - \int_{\Gamma_r} |w|^{-p-2}(udv - vdu) \\ = &\iint_{\Omega_{r,R}} \left(\partial(|w|^{-p-2}u)/\partial u - \partial(-|w|^{-p-2}v)/\partial v\right)\, du\, dv \\ = & -p\iint_{\Omega_{r,R}} |w|^{-p-2}\, du\, dv. \end{aligned}$$

Now a little calculation reveals that

$$\int_{|w|=R} |w|^{-p-2}(udv - vdu) = 2\pi R^{-p}.$$

Hence the equation (2.1) takes the following form

$$J_p'(r) = -(p/r)R^{-p} - (p^2/2\pi r)\iint_{\Omega_{r,R}} |w|^{-p-2}\, du\, dv.$$

Now, letting $R \to \infty$, we have

$$J_p'(r) = -(p^2/2\pi r) \iint_{\Omega_r} |w|^{-p-2}\, du\, dv < 0,$$

where Ω_r is the exterior of the curve Γ_r. $\square$

Using the above Lemma, we prove our first result:

***Theorem* 1** — *Let $h(z) = \sum_{n=0}^{\infty} a_n z^n$ be a function analytic in $\mathbb{D}$ and $p > 0$. If $b \in \mathbb{C} \setminus \{0\}$ such that the function*

$$\frac{b}{z-\alpha} + h,\ \alpha \in (0,1)$$

is univalent and non-vanishing in $\mathbb{D}$, then we have

$$\int_{\mathbb{D}} \left| \frac{b\overline{z}}{1-\alpha\overline{z}} + h(z) \right|^{p-2} |h'(z)|^2\, dm(z) \le |b|^2 \int_{\mathbb{D}} \left| \frac{b\overline{z}}{1-\alpha\overline{z}} + h(z) \right|^{p-2} \frac{dm(z)}{|1-\alpha z|^4}. \tag{2.2}$$

PROOF : Let

$$g(z) = \frac{b}{z-\alpha} + h(z) \text{ and } f(z) = 1/g(z), \quad z \in \mathbb{D};$$

and

$$I_p(r) = I_p(r,g) = \frac{1}{2\pi} \int_0^{2\pi} |g(re^{i\theta})|^p\, d\theta = J_p(r,f), \quad p > 0, r \in (\alpha, 1).$$

Now an application of Lemma 1, to the function $w = f(z)$ (as f is analytic and univalent in $\mathbb{D}$ with $f(\alpha) = 0$), yields

$$\begin{aligned} J_p'(r,f) &= \frac{-p}{2\pi r} \int_{\Gamma_r} |w|^{-p-2}(udv - vdu),\ \ w = u + iv \\ &< 0, \end{aligned}$$

where Γ_r orients positively. We now apply a change of variable $w \to \frac{1}{w}$, and get

$$J_p'(r,f) = \frac{p}{2\pi r} \int_{\gamma_r} |w|^{p-2}(udv - vdu) < 0,$$

where γ_r denotes the curve $w = g(re^{it})$, $t \in [0, 2\pi]$, $r \in (\alpha, 1)$ that orients negatively. Now if we parameterize the curve γ_r by $w = F(e^{it})$ where F is an arbitrary function defined and smooth in $N_\delta(\overline{\mathbb{D}})$ such that $F(e^{it}) \equiv g(re^{it})$. We thus have

$$\begin{aligned} \int_{\gamma_r} |w|^{p-2}(udv - vdu) &= \operatorname{Im}\left(\int_{|\zeta|=1} |F(\zeta)|^{p-2}\overline{F(\zeta)}\, dF(\zeta) \right) \\ &= p\left(\int_{\mathbb{D}} |F(z)|^{p-2} J_F(z)\, dm(z) \right) \end{aligned}$$

where the Jacobian $J_F(z) = |\partial F/\partial z|^2 - |\partial F/\partial \bar{z}|^2$. Therefore we get

$$\int_{\mathbb{D}} |F(z)|^{p-2} J_F(z)\, dm(z) < 0. \tag{2.3}$$

Now if we take

$$F(z) = \frac{b\bar{z}}{r - \alpha\bar{z}} + \sum_{n=0}^{\infty} a_n r^n z^n,$$

then we have

$$\partial F/\partial z = \sum_{n=1}^{\infty} n a_n r^n z^{n-1} \quad \text{and} \quad \partial F/\partial \bar{z} = \frac{br}{(r - \alpha\bar{z})^2}.$$

After plugging in the above expressions in (2.3) and then letting $r \to 1-$, we get the desired inequality (2.2). □

Remark : Note that for the case $p = 2$ in the above theorem, we get from (2.2),

$$\pi \sum_{n=1}^{\infty} n|a_n|^2 = \int_{\mathbb{D}} |h'(z)|^2\, dm(z) \le |b|^2 \int_{\mathbb{D}} \frac{dm(z)}{|1 - \alpha z|^4} \le \frac{\pi|b|^2}{(1 - \alpha^2)^2}.$$

This is nothing but the Chichra's Area Theorem (compare Theorem C).

Next we prove a generalized version of Theorem D:

***Theorem* 2** — *Let $p > 0$. Let ϕ be an analytic function in $\mathbb{D}$ such that it satisfies the hypothesis of Theorem D. Then we have*

$$\int_{\mathbb{D}} \frac{|z\phi(z)|^{p-2}|z\phi'(z)|^2}{|1 - \alpha z|^p}\, dm(z) \le \int_{\mathbb{D}} \frac{|z|^{p-2}|\phi(z)|^p}{|1 - \alpha z|^{p+2}}\, dm(z). \tag{2.4}$$

PROOF : Following the notations of Theorem 1, we let

$$g(z) = \phi(z)/(z - \alpha),\ z \in \mathbb{D} \text{ and } F(z) = (\bar{z}\phi(rz))/(r - \alpha\bar{z}),\ z \in N_\delta(\overline{\mathbb{D}}).$$

We observe here that $g(re^{it}) \equiv F(e^{it})$, $t \in [0, 2\pi]$, $0 < r < 1$ and the Jacobian of the mapping F is given by

$$J_F(z) = |\partial F/\partial z|^2 - |\partial F/\partial \bar{z}|^2 = \left|\frac{rz\phi'(rz)}{r - \alpha z}\right|^2 - \frac{|r\phi(rz)|^2}{|r - \alpha z|^4}.$$

Now following the steps of the proof of the previous theorem, we obtain the desired result from the inequality (2.3). □

Next, as an application of the Theorem 2, we prove

***Theorem* 3** — *Let $p>0$. Let f be a meromorphic and univalent function in $\mathbb{D}$ with a simple pole at $z=\alpha\in(0,1)$ and $f(z)\neq 0$ for $z\in\mathbb{D}\setminus\{0\}$. Then we have*

$$\int_{\mathbb{D}} \frac{|zf(z)|^p|z-\alpha|^{p-2}}{|1-\alpha z|^p}\left|1+\frac{(z-\alpha)f'(z)}{f(z)}\right|^2 dm(z) \leq \int_{\mathbb{D}} \frac{|z|^{p-2}|(z-\alpha)f(z)|^p}{|1-\alpha z|^{p+2}}\, dm(z). \tag{2.5}$$

Equality holds in the above inequality for the function $f(z)=\frac{-z\alpha}{(z-\alpha)(1-\alpha z)}$.

PROOF : Following the notations of the Theorem 2, we let

$$g(z)=f(z)\ \text{ i.e. } \phi(z)=(z-\alpha)f(z), z\in\mathbb{D}.$$

Therefore, ϕ is analytic in $\mathbb{D}$ and $\phi'(z)=f(z)+(z-\alpha)f'(z),\ z\in\mathbb{D}$. Now we obtain the inequality stated in the Theorem by plugging in the above expressions of ϕ and ϕ' in the inequality (2.4). To establish the equality case, we observe that, whenever $f(z)=\frac{-z\alpha}{(z-\alpha)(1-\alpha z)}$, we get $\phi(z)=\frac{-\alpha z}{(1-\alpha z)}$. Therefore,

$$F(z)=\frac{\bar{z}\phi(z)}{1-\alpha\bar{z}}=\frac{-\alpha|z|^2}{|1-\alpha z|^2},\ z\in N_\delta(\overline{\mathbb{D}}).$$

Now it is clear from the proof of the Theorem 1 that the equality in the inequality (2.5) will be established if we can show that for the above function F,

$$\operatorname{Im}\left(\int_{|\zeta|=1}|F(\zeta)|^{p-2}\overline{F(\zeta)}\,dF(\zeta)\right)=0.$$

This will be true if we can show that

$$\operatorname{Im}\left(\overline{F(\zeta)}dF(\zeta)\right)=0,\ |\zeta|=1.$$

Since in this case F is real, the above assertion will be established if we can prove that $\operatorname{Im}(dF(\zeta))=0$ for $|\zeta|=1$. A little computation reveals that for $|\zeta|=1$,

$$\begin{aligned}\operatorname{Im}(dF(\zeta)) &= \operatorname{Im}(F_\zeta\, d\zeta+F_{\bar{\zeta}}\, d\bar{\zeta})\\ &= \frac{-\alpha}{|1-\alpha\zeta|^2}\operatorname{Im}\left(\frac{\bar{\zeta}d\zeta}{1-\alpha\zeta}+\frac{\zeta d\bar{\zeta}}{1-\alpha\bar{\zeta}}\right)\\ &= 0.\end{aligned}$$

This completes the proof of the theorem. □

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