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ON SOME PROPERTIES OF CONTINUOUS *G*-FRAMES AND RIESZ-TYPE CONTINUOUS *G*-FRAMES

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In this paper we provide some necessary and sufficient conditions under which, a family of bounded operators is a continuous g-frame (Riesz-type continuous g-frame). Also, we study stability of duals of continuous g-frames.

Key words : Continuous frame; continuous g-frame; Riesz-type continuous g-frame.

1. INTRODUCTION

In 1952, the concept of frames for Hilbert spaces was defined by Duffin and Schaeffer [4]. Frames are important tools in the signal processing, image processing, data compression, etc. In 1993, Ali *et al.*, [2] developed the notion of ordinary frame to a family indexed by a measurable space which are known as continuous frames.

Definition 1.1 — Let $\mathcal{H}$ be a complex Hilbert space and (Ω, μ) be a measure space. A mapping $F : \Omega \to \mathcal{H}$ is called a continuous frame with respect to (Ω, μ), if

(i) F is weakly-measurable, i.e., for all $f \in \mathcal{H}$, $\omega \to \langle f, F(\omega)\rangle$ is a measurable function on Ω,

(ii) there exist constants $A_F, B_F > 0$ such that

$$A_F\|f\|^2 \leq \int_\Omega |\langle f, F(\omega)\rangle|^2 \, d\mu(\omega) \leq B_F\|f\|^2, \quad f \in \mathcal{H}.$$

In 2006, g-frames or generalized frames introduced by Sun [6] and Abdollahpour and Faroughi [1] introduced and investigated continuous g-frames and Riesz-type continuous g-frames. In the rest of this paper we assume that $\mathcal{H}$ is a complex Hilbert space and (Ω, μ) is a measure space with positive measure μ and $\{\mathcal{K}_\omega : \omega \in \Omega\}$ is a family of Hilbert spaces. Now, we summarize some facts about continuous g-frames from [1].

We say that $F \in \prod_{\omega\in\Omega} \mathcal{K}_\omega$ is strongly measurable if F as a mapping of Ω to $\bigoplus_{\omega\in\Omega} \mathcal{K}_\omega$ is measurable, where

$$\prod_{\omega\in\Omega} \mathcal{K}_\omega = \left\{ f : \Omega \to \bigcup_{\omega\in\Omega} \mathcal{K}_\omega : f(\omega) \in \mathcal{K}_\omega \right\}.$$

Definition 1.2 — We say that $\Lambda = \{\Lambda_\omega \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$ is a continuous g-frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_\omega : \omega \in \Omega\}$ if

(i) for each $f \in \mathcal{H}$, $\{\Lambda_\omega f : \omega \in \Omega\}$ is strongly measurable,

(ii) there are two constants $0 < A_\Lambda \leq B_\Lambda < \infty$ such that

$$A_\Lambda \|f\|^2 \leq \int_\Omega \|\Lambda_\omega(f)\|^2 d\mu(\omega) \leq B_\Lambda \|f\|^2, \ f \in \mathcal{H}. \tag{1.1}$$

We call A_Λ, B_Λ the lower and upper continuous g-frame bounds, respectively. Λ is called a tight continuous g-frame if $A_\Lambda = B_\Lambda$, and a Parseval continuous g-frame if $A_\Lambda = B_\Lambda = 1$. If for each $\omega \in \Omega$, $\mathcal{K} = \mathcal{K}_\omega$, then Λ is called a continuous g-frame for $\mathcal{H}$ with respect to $\mathcal{K}$. A family $\Lambda = \{\Lambda_\omega \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$ is called a continuous g-Bessel family for $\mathcal{H}$ with respect to $\{\mathcal{K}_\omega : \omega \in \Omega\}$ if the right hand inequality in (1.1) holds for all $f \in \mathcal{H}$. In this case, B_Λ is called the Bessel constant.

If there is no confusion, we use continuous g-frame (continuous g-Bessel family) instead of continuous g-frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_\omega : \omega \in \Omega\}$ (continuous g-Bessel family for $\mathcal{H}$ with respect to $\{\mathcal{K}_\omega : \omega \in \Omega\}$).

Proposition 1.3 [1] — Let $\Lambda = \{\Lambda_\omega \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$ be a continuous g-frame. There exists a unique positive and invertible operator $S_\Lambda : \mathcal{H} \to \mathcal{H}$ such that for each $f, g \in \mathcal{H}$

$$\langle S_\Lambda f, g \rangle = \int_\Omega \langle f, \Lambda_\omega^* \Lambda_\omega g \rangle d\mu(\omega), \quad f, g \in \mathcal{H},$$

and $A_\Lambda I \leq S_\Lambda \leq B_\Lambda I$.

The operator S_Λ in Proposition 1.3 is called the continuous g-frame operator of Λ. Also, we have

$$\begin{aligned} \langle f, g \rangle &= \int_\Omega \langle S_\Lambda^{-1} f, \Lambda_\omega^* \Lambda_\omega g \rangle \, d\mu(\omega) \\ &= \int_\Omega \langle f, \Lambda_\omega^* \Lambda_\omega S_\Lambda^{-1} g \rangle \, d\mu(\omega), \quad f, g \in \mathcal{H}. \end{aligned} \tag{1.2}$$

We consider the space

$$\widehat{\mathcal{K}} = \left\{ F \in \prod_{\omega\in\Omega} \mathcal{K}_\omega : \text{F is strongly measurable, } \int_\Omega \|F(\omega)\|^2 d\mu(\omega) < \infty \right\}.$$

It is clear that $\widehat{\mathcal{K}}$ is a Hilbert space with point wise operations and the inner product given by

$$\langle F, G\rangle = \int_\Omega \langle F(\omega), G(\omega)\rangle d\mu(\omega).$$

Proposition 1.4 [1] — Let $\Lambda = \{\Lambda_\omega \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$ be a continuous g-Bessel family. Then the mapping $T_\Lambda : \widehat{\mathcal{K}} \to \mathcal{H}$ defined by

$$\langle T_\Lambda F, g\rangle = \int_\Omega \langle \Lambda_\omega^* F(\omega), g\rangle d\mu(\omega),\ F \in \widehat{\mathcal{K}},\ g \in \mathcal{H}, \tag{1.3}$$

is linear and bounded with $\|T_\Lambda\| \leq \sqrt{B_\Lambda}$. Also, for each $g \in \mathcal{H}$ we have

$$T_\Lambda^*(g)(\omega) = \Lambda_\omega g,\ \omega \in \Omega.$$

***Theorem* 1.5** [1] — *Let (Ω, μ) be a measure space, where μ is σ-finite. Suppose that $\Lambda = \{\Lambda_\omega \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$ is a family of operators such that $\{\Lambda_\omega f : \omega \in \Omega\}$ is strongly measurable, for each $f \in \mathcal{H}$. Then Λ is a continuous g-frame if and only if the operator $T_\Lambda : \widehat{\mathcal{K}} \to \mathcal{H}$ defined by (1.3) is a bounded and onto operator.*

The operators T_Λ and T_Λ^* in Theorem 1.8 are called synthesis and analysis operators of Λ, respectively.

Definition 1.6 — Let $\Lambda = \{\Lambda_\omega \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$ and $\Theta = \{\Theta_\omega \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$ be two continuous g-frames such that

$$\langle f, g\rangle = \int_\Omega \langle f, \Theta_\omega^* \Lambda_\omega g\rangle d\mu(\omega),\ f, g \in \mathcal{H},$$

then Θ is called a dual continuous g-frame of Λ.

Let $\Lambda = \{\Lambda_\omega \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$ be a continuous g-frame. Then $\widetilde{\Lambda} = \{\Lambda_\omega S_\Lambda^{-1} \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$ is a continuous g-frame and by (1.2), $\widetilde{\Lambda}$ is a dual of Λ. we call $\widetilde{\Lambda}$ the canonical dual of Λ. One can always get a tight continuous g-frame from any continuous g-frame, in fact, if $\Lambda = \{\Lambda_\omega \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$ is a continuous g-frame for $\mathcal{H}$ then $\{\Lambda_\omega S_\Lambda^{-1/2} \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$ is a Parseval continuous g-frame.

Definition 1.7 — Let $\Lambda = \{\Lambda_\omega \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$ and $\Theta = \{\Theta_\omega \in B(\mathcal{K}, \mathcal{K}_\omega) : \omega \in \Omega\}$ be two continuous g-frames. Then Λ and Θ are said similar if there is an invertible operator $S : \mathcal{H} \to \mathcal{K}$ such that $\Theta_\omega S = \Lambda_\omega$ for a.e. $\omega \in \Omega$.

Two continuous g-Bessel families $\Lambda = \{\Lambda_\omega \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$ and $\Theta = \{\Theta_\omega \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$ are weakly equal, if for all $f \in \mathcal{H}$,

$$\Lambda_\omega f = \Theta_\omega f,\ a.e.\ \ \omega \in \Omega.$$

Proposition 1.8 [1] — Let $\Lambda = \{\Lambda_\omega \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$ and $\Theta = \{\Theta_\omega \in B(\mathcal{K}, \mathcal{K}_\omega) : \omega \in \Omega\}$ be two continuous g-frames. Then Λ and Θ are similar if and only if their analysis operators have the same ranges (weakly).

If the continuous g-frame $\Lambda = \{\Lambda_\omega \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$ have only one dual(weakly), i.e., every dual of Λ is weakly equal to the canonical dual of Λ, then Λ is called a Riesz-type continuous g-frame.

***Theorem* 1.9** — *Let $\Lambda = \{\Lambda_\omega \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$ be a continuous g-frame. Then Λ is a Riesz-type continuous g-frame if and only if $Range T_\Lambda^* = \widehat{\mathcal{K}}$.*

2. Some New Results for Continuous G-frames

In this section by generalizing some results of [5], we give some necessary and sufficient conditions that a family of bounded operators $\{\Lambda_\omega \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$ is a continuous g-frame (Riesz-type continuous g-frame). In this section until the end of Proposition 2.5, $\mathcal{K}_\omega$ is a finite dimensional Hilbert space with orthonormal basis $\{e_{\omega,j} : j \in J_\omega\}$ for all $\omega \in \Omega$.

Proposition 2.1 — Let $\{e_i\}_{i \in I}$ be an orthonormal basis for $\mathcal{H}$ and $\{e_{\omega j}\}_{j \in J_\omega}$ be an orthonormal basis for $\mathcal{K}_\omega$, for all $\omega \in \Omega$. Then the following are equivalent:

(i) $\Lambda = \{\Lambda_\omega \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$ is a Parseval continuous g-frame.

(ii) There exist an orthonormal set $\{\psi_i\}_{i \in I} \subseteq \widehat{\mathcal{K}}$ such that $\sum_{i \in I} \|\psi_i(\omega)\|^2 < \infty$ for a.e. $\omega \in \Omega$ and for all $f \in \mathcal{H}$,

$$\Lambda_\omega f = \sum_{i \in I} \langle f, e_i \rangle \psi_i(\omega), \quad a.e.\ \ \omega \in \Omega.$$

PROOF : $(i) \Rightarrow (ii)$ We consider $\psi_i = T_\Lambda^* e_i$, for all $i \in I$. Then

$$\|\psi_i\|^2 \ \ = \ \ \|T_\Lambda^* e_i\|^2 = \int_\Omega \|\Lambda_\omega e_i\|^2 d\mu(\omega) = \|e_i\|^2 = 1,$$

and

$$\langle \psi_i, \psi_j \rangle = \langle T_\Lambda^* e_i, T_\Lambda^* e_j \rangle = \int_\Omega \langle \Lambda_\omega e_i, \Lambda_\omega e_j \rangle d\mu(\omega) = \langle e_i, e_j \rangle = 0, \quad i \neq j.$$

On the other hand,

$$\begin{aligned} \sum_{i\in I} \|\psi_i(\omega)\|^2 &= \sum_{i\in I} \|\Lambda_\omega e_i\|^2 = \sum_{i\in I} \| \sum_{j\in J_\omega} \langle \Lambda_\omega e_i, e_{\omega j}\rangle e_{\omega j}\|^2 \\ &= \sum_{j\in J_\omega} \sum_{i\in I} |\langle e_i, \Lambda_\omega^* e_{\omega j}\rangle|^2 = \sum_{j\in J_\omega} \|\Lambda_\omega^* e_{\omega j}\|^2 < \infty. \end{aligned}$$

Furthermore, for any $f \in \mathcal{H}$,

$$\Lambda_\omega f = \sum_{i\in I} \langle f, e_i\rangle \Lambda_\omega e_i = \sum_{i\in I} \langle f, e_i\rangle \psi_i(\omega).$$

Conversely, since $\{\psi_i\}_{i\in I} \subseteq \widehat{\mathcal{K}}$ is an orthonormal set then for any $f \in \mathcal{H}$

$$\begin{aligned} \int_\Omega \|\Lambda_\omega f\|^2 d\mu(\omega) &= \int_\Omega \Big\| \sum_{i\in I} \langle f, e_i\rangle \psi_i(\omega)\Big\|^2 d\mu(\omega) \\ &= \| \sum_{i\in I} \langle f, e_i\rangle \psi_i\|^2 \\ &= \sum_{i\in I} |\langle f, e_i\rangle|^2 = \|f\|^2. \end{aligned}$$

We recall that a Riesz basis for a Hilbert space $\mathcal{H}$ is a family of the form $\{Ue_i\}_{i\in I}$, where $\{e_i\}_{i\in I}$ is an orthonormal basis for $\mathcal{H}$ and $U : \mathcal{H} \to \mathcal{H}$ is a bounded bijective operator. □

***Theorem* 2.2** — *Let $\Lambda_\omega \in B(\mathcal{H}, \mathcal{K}_\omega)$ be given for all $\omega \in \Omega$. The following are equivalent:*

(i) $\Lambda = \{\Lambda_\omega \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$ *is a continuous g-frame.*

(ii) $\Lambda_\omega f = \sum_{i\in I} \langle f, e_i\rangle \psi_i(\omega)$ *for some orthonormal basis* $\{e_i\}_{i\in I}$ *of* $\mathcal{H}$ *and some family* $\{\psi_i\}_{i\in I} \subseteq \widehat{\mathcal{K}}$ *with the property that* $\{\psi_i\}_{i\in I}$ *is a Riesz basis for* $\overline{span}\{\psi_i\}_{i\in I}$ *and* $\sum_{i\in I} \|\psi_i(\omega)\|^2 < \infty$ *for a.e.* $\omega \in \Omega$.

(iii) $\Lambda_\omega f = \sum_{i\in I} \langle f, h_i\rangle \psi_i(\omega)$ *for some Riesz basis* $\{h_i\}_{i\in I}$ *of* $\mathcal{H}$ *and some orthonormal set* $\{\psi_i\}_{i\in I} \subseteq \widehat{\mathcal{K}}$ *and with the property that* $\sum_{i\in I} \|\psi_i(\omega)\|^2 < \infty$ *for a.e.* $\omega \in \Omega$.

(iv) $\Lambda_\omega f = \sum_{i\in I} \langle f, h_i\rangle \psi_i(\omega)$ *for some Riesz basis* $\{h_i\}_{i\in I}$ *of* $\mathcal{H}$ *and some family* $\{\psi_i\}_{i\in I} \subseteq \widehat{\mathcal{K}}$ *with the property that* $\{\psi_i\}_{i\in I}$ *is a Riesz basis for* $\overline{span}\{\psi_i\}_{i\in I}$ *and* $\sum_{i\in I} \|\psi_i(\omega)\|^2 < \infty$ *for a.e.* $\omega \in \Omega$.

PROOF : $(i) \Rightarrow (ii)$ Let $\{e_i\}_{i\in I}$ be an orthonormal basis of $\mathcal{H}$ and $\psi_i = T^*_\Lambda e_i$ for all $i \in I$. Then for any finite sequence of scalars $\{c_k\}$, we have

$$\begin{aligned} \|\sum c_k\psi_k\|^2 = \|\sum c_k T^*_\Lambda e_k\|^2 &= \|T^*_\Lambda\big(\sum c_k e_k\big)\|^2 \\ &= \int_\Omega \|\Lambda_\omega\big(\sum c_k e_k\big)\|^2 d\mu(\omega), \end{aligned}$$

So

$$\begin{aligned} A_\Lambda \sum |c_k|^2 = A_\Lambda \|\sum c_k e_k\|^2 &\le \|\sum c_k\psi_k\|^2 \\ &\le B_\Lambda \|\sum c_k e_k\|^2 = B_\Lambda \sum |c_k|^2. \end{aligned}$$

Therefore, by [3, Theorem 3.6.6], $\{\psi_i\}_{i\in I}$ is a Riesz basis for $\overline{span}\{\psi_i\}_{i\in I}$. We also have $\Lambda_\omega f = \sum_{i\in I}\langle f, e_i\rangle\psi_i(\omega)$ and $\sum_{i\in I}\|\psi_i(\omega)\|^2 < \infty$.

$(ii) \Rightarrow (i)$ Since $\{\psi_i\}_{i\in I} \subseteq \widehat{\mathcal{K}}$ is a Riesz basis for $\overline{span}\{\psi_i\}_{i\in I}$, there exist $B \ge A > 0$ such that

$$\begin{aligned} A\|f\|^2 = A\sum_{i\in I}|\langle f, e_i\rangle|^2 &\le \|\sum_{i\in I}\langle f, e_i\rangle\psi_i\|^2 \\ &\le B\sum_{i\in I}|\langle f, e_i\rangle|^2 = B\|f\|^2. \end{aligned}$$

Thus

$$A\|f\|^2 \le \int_\Omega \|\Lambda_\omega f\|^2 d\mu(\omega) = \|\sum_{i\in I}\langle f, e_i\rangle\psi_i\|^2 \le B\|f\|^2.$$

$(i) \Rightarrow (iii)$ Let $\{e_i\}_{i\in I}$ be an orthonormal basis of $\mathcal{H}$. By Proposition 2.1, there exists an orthonormal set $\{\psi_i\}_{i\in I}$ in $\widehat{\mathcal{K}}$ such that $\sum_{i\in I}\|\psi_i(\omega)\|^2 < \infty$ for a.e. $\omega \in \Omega$, and for all $g \in \mathcal{H}$

$$\Lambda_\omega S_\Lambda^{-1/2} g = \sum_{i\in I}\langle g, e_i\rangle\psi_i(\omega), \quad a.e.\ \omega \in \Omega.$$

For any $f \in \mathcal{H}$ there exist $g \in \mathcal{H}$ such that $S_\Lambda^{-1/2} g = f$. Thus

$$\Lambda_\omega f = \Lambda_\omega S_\Lambda^{-1/2} g = \sum_{i\in I}\langle S_\Lambda^{1/2} f, e_i\rangle\psi_i(\omega) = \sum_{i\in I}\langle f, S_\Lambda^{1/2} e_i\rangle\psi_i(\omega).$$

It is sufficient to take $h_i = S_\Lambda^{1/2} e_i$, for all $i \in I$.

$(iii) \Rightarrow (i)$ Let $\{e_i\}_{i\in I}$ be an orthonormal basis for $\mathcal{H}$ and $U \in B(\mathcal{H})$ be a bijective operator and

$h_i = Ue_i$ for all $i \in I$. Then for any $f \in \mathcal{H}$ we have

$$\begin{aligned}\int_\Omega \|\Lambda_\omega f\|^2 d\mu(\omega) &= \int_\Omega \|\sum_{i\in I}\langle f, h_i\rangle \psi_i(\omega)\|^2 d\mu(\omega) \\ &= \|\sum_{i\in I}\langle f, h_i\rangle \psi_i\|^2 \\ &= \sum_{i\in I} |\langle f, h_i\rangle|^2 = \sum_{i\in I} |\langle U^* f, e_i\rangle|^2.\end{aligned}$$

So

$$\|U^{-1}\|^{-2}\|f\|^2 \le \int_\Omega \|\Lambda_\omega f\|^2 d\mu(\omega) = \sum_{i\in I} |\langle f, h_i\rangle|^2 \le \|U\|^2\|f\|^2, \quad f \in \mathcal{H}.$$

$(iv) \Rightarrow (i)$ There exist $C, D > 0$ such that

$$C\|f\|^2 \le \sum_{i\in I} |\langle f, h_i\rangle|^2 \le D\|f\|^2, \quad f \in \mathcal{H}.$$

On the other hand, let $B \ge A > 0$ be the bounds of Riesz basis $\{\psi_i\}_{i\in I}$. Then

$$\begin{aligned}AC\|f\|^2 \le A\sum_{i\in I} |\langle f, h_i\rangle|^2 &\le \|\sum_{i\in I}\langle f, h_i\rangle \psi_i\|^2 \\ &= \int_\Omega \|\sum_{i\in I}\langle f, h_i\rangle \psi_i(\omega)\|^2 d\mu(\omega) \\ &\le B\sum_{i\in I} |\langle f, h_i\rangle|^2 \le BD\|f\|^2.\end{aligned}$$

for all $f \in \mathcal{H}$. According to the $(i) \Rightarrow (ii)$ the implication $(i) \Rightarrow (iv)$ is obvious. □

Proposition 2.3 — The following are equivalent:

(i) There exists a Riesz-type continuous g-frame $\Lambda = \{\Lambda_\omega \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$ for $\mathcal{H}$.

(ii) There exist an orthonormal basis $\{\psi_i\}_{i\in I}$ for $\widehat{\mathcal{K}}$ such that

$$\sum_{i\in I} \|\psi_i(\omega)\|^2 < \infty, \quad a.e.\ \omega \in \Omega.$$

(iii) For every orthonormal basis $\{\phi_i\}_{i\in I}$ for $\widehat{\mathcal{K}}$ we have

$$\sum_{i\in I} \|\phi_i(\omega)\|^2 < \infty, \quad a.e.\ \omega \in \Omega.$$

PROOF : $(i) \Rightarrow (ii)$ By Theorem 2.2, $\Lambda_\omega f = \sum_{i\in I}\langle f, h_i\rangle \psi_i(\omega)$ for some Riesz basis $\{h_i\}_{i\in I}$ of $\mathcal{H}$ and some orthonormal set $\{\psi_i\}_{i\in I} \subseteq \widehat{\mathcal{K}}$ with the property that $\sum_{i\in I}\|\psi_i(\omega)\|^2 < \infty$ for a.e. $\omega \in \Omega$. By Theorem 1.9, for any ψ in $\widehat{\mathcal{K}}$ there exists $f \in \mathcal{H}$ such that $T^*_\Lambda f = \psi$. Thus

$$\psi = \sum_{i\in I}\langle f, h_i\rangle \psi_i.$$

So, $\overline{span}\{\psi_i\}_{i\in I} = \widehat{\mathcal{K}}$, therefore $\{\psi_i\}_{i\in I}$ is an orthonormal basis for $\widehat{\mathcal{K}}$.
$(ii) \Rightarrow (i)$ Let us define $\Lambda_\omega \in B(\mathcal{H}, \mathcal{K}_\omega)$ by $\Lambda_\omega f = \sum_{i\in I}\langle f, e_i\rangle \psi_i(\omega)$ for all $\omega \in \Omega$, where $\{e_i\}_{i\in I}$ is an orthonormal basis for $\mathcal{H}$. By Proposition 2.1, $\Lambda = \{\Lambda_\omega \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$ is a Parseval continuous g-frame. For any ψ in $\widehat{\mathcal{K}}$ we have $\psi = \sum_{i\in I}\langle \psi, \psi_i\rangle \psi_i$. So we have

$$\begin{aligned} T^*_\Lambda\Big(\sum_{j\in I}\langle \psi, \psi_j\rangle e_j\Big)(\omega) &= \Lambda_\omega\Big(\sum_{j\in I}\langle \psi, \psi_j\rangle e_j\Big) \\ &= \sum_{i\in I}\big\langle \sum_{j\in I}\langle \psi, \psi_j\rangle e_j, e_i\big\rangle \psi_i(\omega) \\ &= \sum_{i\in I}\langle \psi, \psi_i\rangle \psi_i(\omega) = \psi(\omega), \end{aligned}$$

for all $\omega \in \Omega$. This means that T_Λ is surjective and by Theorem 1.9, the proof is complete.
$(iii) \Rightarrow (ii)$ It is clear.
$(i) \Rightarrow (iii)$ By assumption, $\Gamma = \{\Lambda_\omega S_\Lambda^{-1/2} : \omega \in \Omega\}$ is a Parseval continuous g-frame and $Range T^*_\Gamma = Range T^*_\Lambda = \widehat{\mathcal{K}}$. Therefore, T^*_Γ is a surjective isometry and so T^*_Γ is unitary. Let $\{\phi_i\}_{i\in I}$ be an orthonormal basis for $\widehat{\mathcal{K}}$. Let us consider $e_i = T_\Gamma \phi_i$, for all $i \in I$. Then, $\{e_i\}_{i\in I}$ is an orthonormal basis for $\mathcal{H}$ and

$$\begin{aligned} \sum_{i\in I}\|\phi_i(\omega)\|^2 &= \sum_{i\in I}\|\sum_{j\in J_\omega}\langle \phi_i(\omega), e_{\omega j}\rangle e_{\omega j}\|^2 \\ &= \sum_{i\in I}\sum_{j\in J_\omega}|\langle \phi_i(\omega), e_{\omega j}\rangle|^2 \\ &= \sum_{i\in I}\sum_{j\in J_\omega}|\langle \Lambda_\omega S_\Lambda^{-1/2} e_i,\ e_{\omega j}\rangle|^2 \\ &= \sum_{j\in J_\omega}\sum_{i\in I}|\langle e_i, S_\Lambda^{-1/2}\Lambda^*_\omega e_{\omega j}\rangle|^2 \\ &= \sum_{j\in J_\omega}\|S_\Lambda^{-1/2}\Lambda^*_\omega e_{\omega j}\|^2 < \infty, \end{aligned}$$

for a.e. $\omega \in \Omega$.

Corollary 2.4 — Let M be a closed subspace of $\widehat{\mathcal{K}}$. Then the following are equivalent:

(i) There exists a continuous g-frame $\Lambda = \{\Lambda_\omega \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$ for $\mathcal{H}$ and $RangeT^*_\Lambda = M$.

(ii) There exist an orthonormal basis $\{\psi_i\}_{i\in I}$ for M with the property that $\sum_{i\in I} \|\psi_i(\omega)\|^2 < \infty$, for a.e. $\omega \in \Omega$.

(iii) Every orthonormal basis $\{\phi_i\}_{i\in I}$ for M satisfies $\sum_{i\in I} \|\phi_i(\omega)\|^2 < \infty$, for a.e. $\omega \in \Omega$.

PROOF : $(i) \Rightarrow (ii)$ Since Λ is a continuous g-frame, $\Gamma = \{\Lambda_\omega S_\Lambda^{-1/2} \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$ is a Parseval continuous g-frame. Let $\{e_i\}_{i\in I}$ be an orthonormal basis for $\mathcal{H}$. If we take $\psi_i = T^*_\Gamma e_i$, for all $i \in I$ then by Proposition 2.1, $\{\psi_i\}_{i\in I}$ is an orthonormal set of M with the property that $\sum_{i\in I} \|\psi_i(\omega)\|^2 < \infty$ for a.e. $\omega \in \Omega$ and

$$\Lambda_\omega S_\Lambda^{-1/2} g = \sum_{i\in I} \langle g, e_i\rangle \psi_i(\omega), \quad a.e.\ \omega \in \Omega, \quad g \in \mathcal{H}.$$

For any $\psi \in M$ there exists $f \in \mathcal{H}$ such that $T^*_\Lambda f = \psi$. Let $g = S_\Lambda^{1/2} f$, then

$$\psi(\omega) = \Lambda_\omega f = \Lambda_\omega S_\Lambda^{-1/2} g = \sum_{i\in I} \langle g, e_i\rangle \psi_i(\omega).$$

$(ii) \Rightarrow (i)$ Let $\{e_i\}_{i\in I}$ be an orthonormal basis for $\mathcal{H}$. We define bounded operator

$$\Lambda_\omega : \mathcal{H} \to \mathcal{K}_\omega, \quad \Lambda_\omega f = \sum_{i\in I} \langle f, e_i\rangle \psi_i(\omega),$$

for a.e. $\omega \in \Omega$. Then by Proposition 2.1, $\Lambda = \{\Lambda_\omega \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$ is a parseval continuous g-frame. If $\psi \in M$, then $\psi = \sum_{i\in I} \langle \psi, \psi_i\rangle \psi_i$ and

$$T^*_\Lambda\Big(\sum_{i\in I} \langle \psi, \psi_i\rangle e_i\Big)(\omega) = \psi(\omega).$$

Thus, the proof is complete.

$(iii) \Rightarrow (ii)$ It is clear.

$(i) \Rightarrow (iii)$ Let $\{\phi_i\}_{i\in I}$ be an orthonormal basis for M. Let $\Gamma = \{\Lambda_\omega S_\Lambda^{-1/2} \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$, then $T_\Gamma T^*_\Gamma = T^*_\Gamma T_\Gamma = I$. If we take $e_i = T_\Gamma \phi_i$, for all $i \in I$ then $\{e_i\}_{i\in I}$ is an orthonormal basis for $\mathcal{H}$ and

$$\sum_{i\in I} \|\phi_i(\omega)\|^2 = \sum_{j\in j_\omega} \|S_\Lambda^{-1/2} \Lambda^*_\omega e_{\omega_j}\|^2 < \infty.$$

Proposition 2.5 — Let $\Lambda = \{\Lambda_\omega \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$ and $\Theta = \{\Theta_\omega \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$ be two continuous g-frames with representation $\Lambda_\omega f = \sum_{i\in I}\langle f, e_i\rangle\psi_i(\omega)$ and $\Theta_\omega f = \sum_{i\in I}\langle f, f_i\rangle\phi_i(\omega)$ for some $\{\phi_i\}_{i\in I}$ and $\{\psi_i\}_{i\in I}$ in $\widehat{\mathcal{K}}$, and for some orthonormal bases $\{e_i\}_{i\in I}$ and $\{f_i\}_{i\in I}$ for $\mathcal{H}$. Then Λ is a dual of Θ if and only if

$$\langle\psi_i, \phi_j\rangle = \langle e_i, f_j\rangle, \quad i, j \in I.$$

In particular, if $e_i = f_i$, for all $i \in I$, then Λ is a dual of Θ if and only if $\{\psi_i\}_{i\in I}$ and $\{\phi_i\}_{i\in I}$ are biorthogonal.

PROOF : For any $f, g \in \mathcal{H}$ we have

$$\langle f, g\rangle = \sum_{i,j\in I}\langle f, e_i\rangle\langle f_j, g\rangle\langle e_i, f_j\rangle,$$

and

$$\begin{aligned}\int_\Omega \langle f, \Lambda_\omega^*\Theta_\omega g\rangle d\mu_\omega &= \int_\Omega \sum_{i,j\in I}\langle f, e_i\rangle\langle f_j, g\rangle\langle\psi_i(\omega), \phi_j(\omega)\rangle d\mu(\omega) \\ &= \sum_{i,j\in I}\langle f, e_i\rangle\langle f_j, g\rangle\langle\psi_i, \phi_j\rangle.\end{aligned}$$

Thus, the proof is completed. □

In the rest of this section we intend to generalize some results of [7] to continuous g-frame.

Lemma 2.6 — Let (Ω, μ) be a measure space. Let $\Lambda = \{\Lambda_\omega \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$ and $\Theta = \{\Theta_\omega \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$ be two Parseval continuous g-frames. Then $Range(T_\Theta^*) \subseteq Range(T_\Lambda^*)$ if and only if there exists an isometry $U : \mathcal{H} \to \mathcal{H}$ such that $\Theta_\omega = \Lambda_\omega U$ for a.e. $\omega \in \Omega$.

PROOF : Let $Range(T_\Theta^*) \subseteq Range(T_\Lambda^*)$. $RangeT_\Lambda^*$ is a closed subspace of $\widehat{\mathcal{K}}$. We take $P = T_\Lambda^* T_\Lambda$. So, $Range(P) = Range(T_\Lambda^*)$. Since Λ is a Parseval continuous g-frame, then P is an orthogonal projection from $\widehat{\mathcal{K}}$ onto $Range(T_\Lambda^*)$. We consider $U = T_\Lambda T_\Theta^*$. Then

$$\|Uf\|^2 = \langle U^*Uf, f\rangle = \langle T_\Theta P T_\Theta^* f, f\rangle = \|f\|^2, \quad f \in \mathcal{H}.$$

and

$$\langle U^* f, g\rangle = \langle T_\Theta T_\Lambda^* f, g\rangle = \int_\Omega \langle\Lambda_\omega f, \Theta_\omega g\rangle d\mu(\omega), \quad f, g \in \mathcal{H}.$$

Thus

$$\begin{aligned}\int_{\Omega}\|(\Lambda_{\omega}U-\Theta_{\omega})f\|^{2}d\mu(\omega) &= \int_{\Omega}\|\Lambda_{\omega}U(f)\|^{2}d\mu(\omega)-\int_{\Omega}\langle\Lambda_{\omega}U(f),\Theta_{\omega}f\rangle d\mu(\omega)\\ &\quad -\int_{\Omega}\langle\Theta_{\omega}f,\Lambda_{\omega}U(f)\rangle d\mu(\omega)+\int_{\Omega}\|\Theta_{\omega}f\|^{2}d\mu(\omega)\\ &= \|Uf\|^{2}-\langle U^{*}Uf,f\rangle-\langle Uf,Uf\rangle+\|f\|^{2}\\ &= \|Uf\|^{2}-\|Uf\|^{2}-\|Uf\|^{2}+\|f\|^{2}=0.\end{aligned}$$

Hence, $\Lambda_{\omega}Uf=\Theta_{\omega}f$ for a.e. $\omega\in\Omega$ and $f\in\mathcal{H}$. The other implication is clear. □

Proposition 2.7 — Let (Ω,μ) be a measure space. Let $\Lambda=\{\Lambda_{\omega}\in B(\mathcal{H},\mathcal{K}_{\omega}):\omega\in\Omega\}$ and $\Theta=\{\Theta_{\omega}\in B(\mathcal{H},\mathcal{K}_{\omega}):\omega\in\Omega\}$ be two Parseval continuous g-frames and Λ be a Riesz-type continuous g-frame. Then Θ is Riesz-type if and only if there exists an unitary operator $U:\mathcal{H}\to\mathcal{K}$ such that $\Theta_{\omega}U=\Lambda_{\omega}$ for a.e. $\omega\in\Omega$.

PROOF : Let Θ be Riesz-type, then $Range(T_{\Lambda}^{*})=Range(T_{\Theta}^{*})=\widehat{\mathcal{K}}$. By Lemma 2.6, there is a bounded operator U on $\mathcal{H}$ such that $U^{*}U=I_{\mathcal{H}}$ and $\Theta_{\omega}=\Lambda_{\omega}U$ for a.e. $\omega\in\Omega$ and hence $T_{\Theta}^{*}=T_{\Lambda}^{*}U$. Since both T_{Λ}^{*} and T_{Θ}^{*} are invertible, it follows that U is a unitary operator. For the other implication, let $U:\mathcal{H}\to\mathcal{H}$ be a unitary linear operator such that $\Theta_{\omega}=\Lambda_{\omega}U$, then $T_{\Theta}^{*}=T_{\Lambda}^{*}U$ and so, $RangeT_{\Theta}^{*}=\widehat{\mathcal{K}}$. By Theorem 1.9, Θ is Riesz-type. □

Proposition 2.8 — Let $\Lambda=\{\Lambda_{\omega}\in B(\mathcal{H},\mathcal{K}_{\omega}):\omega\in\Omega\}$ be a Riesz-type continuous g-frame and $\Theta=\{\Theta_{\omega}\in B(\mathcal{H},\mathcal{K}_{\omega}):\omega\in\Omega\}$ be a continuous g-frame. Then Θ is a Riesz-type continuous g-frame if and only if Λ and Θ are similar.

PROOF : It is obvious by Proposition 1.8 and Theorem 1.9. □

***Theorem* 2.9** — *Let $\Lambda=\{\Lambda_{\omega}\in B(\mathcal{H},\mathcal{K}_{\omega}):\omega\in\Omega\}$ be a Riesz-type continuous g-frame and $\Theta=\{\Theta_{\omega}\in B(\mathcal{H},\mathcal{K}_{\omega}):\omega\in\Omega\}$ be a continuous g-frame. Then the following conditions are equivalent:*

(i) *Θ is a Riesz-type continuous g-frame.*

(ii) *There exists a constant $M>0$ such that for all $\phi\in\widehat{\mathcal{K}}$*

$$\|T_{\Lambda}\phi-T_{\Theta}\phi\|^{2}\leq M.min\{\|T_{\Lambda}\phi\|^{2},\|T_{\Theta}\phi\|^{2}\}. \tag{2.1}$$

PROOF : $(i) \Rightarrow (ii)$ By Proposition 2.8, there is an invertible bounded operator $U : \mathcal{H} \to \mathcal{H}$ such that $\Theta_\omega = \Lambda_\omega U$ for a.e. $\omega \in \Omega$. For any $g \in \mathcal{H}$ and $\phi \in \widehat{\mathcal{K}}$ we have

$$\begin{aligned}\langle U^* T_\Lambda \phi, g\rangle &= \langle T_\Lambda \phi, Ug\rangle = \int_\Omega \langle U^* \Lambda_\omega^* \phi(\omega), g\rangle d\mu(\omega) \\ &= \int_\Omega \langle \Theta_\omega^* \phi(\omega), g\rangle d\mu(\omega) = \langle T_\Theta \phi, g\rangle.\end{aligned}$$

Then $T_\Theta = U^* T_\Lambda$ and so

$$\begin{aligned}\|T_\Lambda \phi - T_\Theta \phi\|^2 = \|T_\Lambda \phi - U^* T_\Lambda \phi\|^2 &\leq \|I - U^*\|^2 \|T_\Lambda \phi\|^2 \\ &\leq (1 + \|U\|)^2 \|T_\Lambda \phi\|^2.\end{aligned}$$

Similarity

$$\|T_\Lambda \phi - T_\Theta \phi\|^2 \quad \leq \quad (1 + \|U^{-1}\|)^2 \|T_\Theta \phi\|^2.$$

Then we have

$$\|T_\Lambda \phi - T_\Theta \phi\|^2 \leq M.min\{\|T_\Lambda \phi\|^2, \|T_\Theta \phi\|^2\}$$

where $M = max\{(1 + \|U\|)^2, (1 + \|U^{-1}\|)^2\}$.

$(ii) \Rightarrow (i)$ Since T_Λ is invertible, then for any $f \in \mathcal{H}$ there exists an unique $\phi \in \widehat{\mathcal{K}}$ such that $T_\Lambda \phi = f$. Let us consider well defined bounded operator $U : \mathcal{H} \to \mathcal{H}$ by

$$Uf = T_\Theta \phi, \quad f \in \mathcal{H}.$$

By the inequality (2.1), U is injective. On the other hand, since T_Θ is surjective, then U is surjective. So, U is invertible. For any $\phi \in \widehat{\mathcal{K}}$ and $g \in \mathcal{H}$ we have

$$\begin{aligned}\langle \phi, \{(\Theta_\omega - \Lambda_\omega U^*)g\}_{\omega\in\Omega}\rangle &= \int_\Omega \langle \phi(\omega), (\Theta_\omega - \Lambda_\omega U^*)g\rangle \, d\mu(\omega) \\ &= \int_\Omega \langle \Theta_\omega^* \phi(\omega), g\rangle d\mu(\omega) - \int_\Omega \langle \Lambda_\omega^* \phi(\omega), U^* g\rangle d\mu(\omega) \\ &= \langle T_\Theta \phi, g\rangle - \langle T_\Lambda \phi, U^* g\rangle \\ &= \langle U T_\Lambda \phi, g\rangle - \langle U T_\Lambda \phi, g\rangle = 0\end{aligned}$$

Thus, $\Theta_\omega = \Lambda_\omega U^*$ for a.e. $\omega \in \Omega$. Therefore, by Proposition 2.8, Θ is Riesz-type. $\square$

3. STABILITY OF DUALS OF CONTINUOUS G-FRAMES

In this section we study stability of duals of continuous g-frames.

Lemma 3.1 — $\Lambda = \{\Lambda_\omega \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$ be a continuous g-frame and $\Theta = \{\Theta_\omega \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$ is a dual of Λ. Then there exists a operator $S \in B(\mathcal{H}, \widehat{\mathcal{K}})$ such that $T_\Lambda S = 0$ and $\Theta_\omega f = (Sf)(\omega) + \Lambda_\omega S_\Lambda^{-1} f$, for all $\omega \in \Omega$ and $f \in \mathcal{H}$.

PROOF : Let us consider $S : \mathcal{H} \to \widehat{\mathcal{K}}$, by

$$(Sf)(\omega) = \Theta_\omega f - \Lambda_\omega S_\Lambda^{-1} f, \quad f \in \mathcal{H}, \omega \in \Omega.$$

Then for all $f \in \mathcal{H}$,

$$\begin{aligned}
\|Sf\| &= (\int_\Omega \|(Sf)(\omega)\|^2 d\mu(\omega))^{1/2} \\
&\leq (\int_\Omega \|\Theta_\omega f\|^2 d\mu(\omega))^{1/2} + (\int_\Omega \|\Lambda_\omega S_\Lambda^{-1} f\|^2 d\mu(\omega))^{1/2} \\
&\leq (\sqrt{B_\Theta} + \frac{1}{\sqrt{A_\Lambda}})\|f\|.
\end{aligned}$$

Thus, S is bounded. On the other hand, for any $f, g \in \mathcal{H}$,

$$\begin{aligned}
\langle T_\Lambda S f, g\rangle &= \int_\Omega \langle \Lambda_\omega^*(Sf)(\omega), g\rangle d\mu(\omega) \\
&= \int_\Omega \langle \Lambda_\omega^* \Theta_\omega f, g\rangle d\mu(\omega) - \int_\Omega \langle \Lambda_\omega^* \Lambda_\omega S_\Lambda^{-1} f, g\rangle d\mu(\omega) \\
&= \langle f, g\rangle - \langle f, g\rangle = 0.
\end{aligned}$$

So, $T_\Lambda S = 0$. □

***Theorem* 3.2** — *Let $\Lambda = \{\Lambda_\omega \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$ and $\Theta = \{\Theta_\omega \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$ be two continuous g-frames. Also, let $\hat{\Lambda} = \{\hat{\Lambda}_\omega \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$ be a fixed dual for Λ. If $\Lambda - \Theta$ is a continuous g-Bessel family with sufficiently small bound $\varepsilon > 0$, then there exists a dual $\hat{\Theta}$ for Θ such that $\hat{\Lambda} - \hat{\Theta}$ is also continuous g-Bessel.*

PROOF : By Lemma 3.1, there exist $S \in B(\mathcal{H}, \widehat{\mathcal{K}})$ such that $T_\Lambda S = 0$ and

$$\hat{\Lambda}_\omega f = (Sf)(\omega) + \Lambda_\omega S_\Lambda^{-1} f, \quad f \in \mathcal{H}, \omega \in \Omega.$$

Let $M = \{M_\omega \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$ such that

$$M_\omega f = (Sf)(\omega) + \Theta_\omega S_\Theta^{-1} f, \quad f \in \mathcal{H}.$$

It is easy to see that M is a continuous g-Bessel family with bound $(\frac{1}{\sqrt{A_\Theta}} + \|S\|)^2$. For $f, g \in \mathcal{H}$ we have

$$\begin{aligned}\langle f - T_\Theta T_M^* f, g\rangle &= \langle f, g\rangle - \int_\Omega \langle \Theta_\omega^*(T_M^* f)(\omega), g\rangle d\mu(\omega)\\ &= \langle f, g\rangle - \int_\Omega \langle \Theta_\omega^* M_\omega f, g\rangle d\mu(\omega)\\ &= \langle f, g\rangle - \int_\Omega \langle \Theta_\omega^*(Sf)(\omega), g\rangle d\mu(\omega) - \int_\Omega \langle \Theta_\omega^* \Theta_\omega S_\Theta^{-1} f, g\rangle d\mu(\omega)\\ &= \langle f, g\rangle - \langle T_\Theta Sf, g\rangle - \langle f, g\rangle\\ &= -\langle T_\Theta Sf, g\rangle.\end{aligned}$$

Thus, $f - T_\Theta T_M^* f = -T_\Theta Sf$, for all $f \in \mathcal{H}$. Hence

$$\begin{aligned}\|f - T_\Theta T_M^* f\| = \|T_\Theta Sf\| &= \|T_\Theta Sf - T_\Lambda Sf\|\\ &\leq \|T_\Theta - T_\Lambda\|\|S\|\|f\|\\ &\leq \sqrt{\varepsilon}\|S\|\|f\|,\end{aligned}$$

Therefore, for all $f \in \mathcal{H}$, $\|I_H - T_\Theta T_M^*\| \leq \sqrt{\varepsilon}\|S\|$ and thus, $T_\Theta T_M^*$ is invertible because ε is sufficiently small. Hence, $\hat{\Theta} = \{\hat{\Theta}_\omega \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$, $\hat{\Theta}_\omega = M_\omega (T_\Theta T_M^*)^{-1}$ is a dual for Θ. Because

$$\begin{aligned}\int_\Omega \langle \hat{\Theta}_\omega f, \Theta_\omega g\rangle d\mu(\omega) &= \int_\Omega \Big\langle (T_M^*(T_\Theta T_M^*)^{-1} f)(\omega), \Theta_\omega g\Big\rangle d\mu(\omega)\\ &= \int_\Omega \langle T_\Theta T_M^* (T_\Theta T_M^*)^{-1} f, g\rangle d\mu(\omega)\\ &= \langle f, g\rangle.\end{aligned}$$

And we have

$$\begin{aligned}\|S_\Lambda - S_\Theta\| &= \|T_\Lambda T_\Lambda^* - T_\Lambda T_\Theta^* + T_\Lambda T_\Theta^* - T_\Theta T_\Theta^*\|\\ &\leq \|T_\Lambda - T_\Theta\|(\|T_\Lambda\| + \|T_\Theta\|)\\ &\leq \sqrt{\varepsilon}(\sqrt{B_\Lambda} + \sqrt{B_\Theta}).\end{aligned} \tag{3.1}$$

And by $\widetilde{\Lambda}(f) = (Sf)(\omega) + \Lambda_\omega S_\Lambda^{-1} f$ and $M_\omega(f) = (Sf)(\omega) + \Theta_\omega S_\Theta^{-1} f$, we have

$$\begin{aligned}\langle T_{\hat{\Lambda}}\phi - T_M\phi, f\rangle &= \int_\Omega \langle \phi(\omega), \hat{\Lambda}_\omega f\rangle d\mu(\omega) - \int_\Omega \langle \phi(\omega), M_\omega f\rangle d\mu(\omega)\\ &= \int_\Omega \langle \phi(\omega), \Lambda_\omega S_\Lambda^{-1} f - \Theta_\omega S_\Theta^{-1} f\rangle d\mu(\omega)\\ &= \langle T_\Lambda \phi, S_\Lambda^{-1} f\rangle - \langle T_\Theta \phi, S_\Theta^{-1} f\rangle\\ &= \langle T_\Lambda \phi, (S_\Lambda^{-1} - S_\Theta^{-1}) f\rangle + \langle (T_\Lambda - T_\Theta)\phi, S_\Theta^{-1} f\rangle,\end{aligned}$$

Hence, by inequality (3.1) we have

$$\begin{aligned}
&\|T_{\hat{\Lambda}}\phi - T_M\phi\| \\
&\leq \|T_\Lambda\phi\|\|S_\Lambda^{-1} - S_\Theta^{-1}\| + \|T_\Lambda - T_\Theta\|\|\phi\|\|S_\Theta^{-1}\| \\
&\leq \|T_\Lambda\|\|\phi\|\|S_\Lambda^{-1}\|\|S_\Theta - S_\Lambda\|\|S_\Theta^{-1}\| + \frac{1}{A_\Theta}\sqrt{\varepsilon}\|\phi\| \\
&\leq \frac{1}{A_\Lambda}\frac{1}{A_\Theta}\sqrt{\varepsilon}\sqrt{B_\Lambda}(\sqrt{B_\Lambda} + \sqrt{B_\Theta})\|\phi\| + \frac{1}{A_\Theta}\sqrt{\varepsilon}\|\phi\| \\
&= \frac{\sqrt{\varepsilon}(A_\Lambda + B_\Lambda + \sqrt{B_\Lambda B_\Theta})}{A_\Lambda A_\Theta}\|\phi\|.
\end{aligned} \tag{3.2}$$

If we take $T = (T_\Theta T_M^*)^{-1}$, then

$$\|T\| \leq \frac{1}{1 - \|I_H - T^{-1}\|} \leq \frac{1}{1 - \sqrt{\varepsilon}S},$$

and so

$$\|I_H - T\| \leq \|T\|\|I_H - T^{-1}\| \leq \frac{\sqrt{\varepsilon}\|S\|}{1 - \sqrt{\varepsilon}\|S\|}. \tag{3.3}$$

Consequently, by inequalities (3.2) and (3.3)

$$\begin{aligned}
\|T_{\hat{\Lambda}}\phi - T_{\hat{\Theta}}\phi\| &= sup_{\|f\|=1}|\langle T_{\hat{\Lambda}}\phi - T_{\hat{\Theta}}\phi, f\rangle| \\
&= sup_{\|f\|=1}|\langle (I_H - T^*)T_{\hat{\Lambda}}\phi + T^*(T_{\hat{\Lambda}} - T_M)\phi, f\rangle| \\
&\leq \|I_H - T^*\|\|T_{\hat{\Lambda}}\|\|\phi\| + \|T^*\|\|(T_{\hat{\Lambda}} - T_M)\phi\| \\
&\leq \frac{\sqrt{\varepsilon}\|\phi\|}{1 - \sqrt{\varepsilon}\|S\|}(\|S\|\sqrt{B_{\hat{\Lambda}}} + \frac{A_\Lambda + B_\Lambda + \sqrt{B_\Lambda B_\Theta})}{A_\Lambda A_\Theta}).\square
\end{aligned}$$

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