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WELL-POSEDNESS OF KORTEWEG-DE VRIES-BURGERS EQUATION ON A FINITE DOMAIN[1]

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In this paper,we consider the Korteweg-de Vries-Burgers equation on a finite domain with initial value and nonhomogeneous boundary conditions. This particular problem arises in the theory of ferroelectricity. We first get the local well-posedness of the problem, and then under the help of the local result, we use nonlinear interpolation to have the global well-posedness of the problem.

Key words : Well-posedness; Korteweg-de Vries-Burgers equation; nonhomogeneous boundary; semigroup; nonlinear interpolation.

1. Introduction

The Korteweg-de Vries-Burgers equation is one of the typical examples incorporating the effects of dispersion, dissipation and nonlinearity. This model provides a discription of the propagation of waves on an elastic tube filled with a viscous fluid [1], and also describes the phenomenon of containing turbulence [2] and gas bubbles [3]. Anzayko *et al*. have investigated the ferroelectricity problem of the Korteweg-de Vries-Burgers equation [4, 5]. Hayashi *et al*., [6] gave the local existence of the solution of the initial-boudnary value problem of KDV-B equations on half-line, and they also showed global existence and asymptotic behavior in time of solutions under sufficient conditions accordingly. Guo and Wang also have proved the global well-posedness and inviscid limit for the KDV-B equation [7]. Molinet and Vento have got the sharp ill-posedness and well-posedness result for the KDV-B equation on the one dimension torus $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ [8].

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In this paper, we study a class of initial-boundary value problem (IBVP) for the Korteweg-de Vries-Burgers (KDV-B) equations posed on a finite domain [0,1] with nonhomogeneous boundary conditions.

$$\begin{cases} u_t + u_x - u_{xx} + u_{xxx} + uu_x = 0, \quad u(x,0) = \phi(x) \quad \text{for} \quad x \in [0,1], \\ u(0,t) = h_1(t), \quad u(1,t) = h_2(t), \quad u_x(1,t) = h_3(t) \quad \text{for} \quad t \geq 0 \end{cases} \tag{1.1}$$

In order to describe our results precisely, we introduce the following notations:

If u is a smooth solution of eq. (1.1), then its initial data $u(x,0) = \phi(x)$ and its boundary values $h_j, j = 1,2,3$ must satisfy the following compatibility conditions:

$$\phi_k(0) = h_1^{(k)}(0), \quad \phi_k(1) = h_2^{(k)}(0), \quad \phi_k'(1) = h_3^{(k)}(0)$$

for $k = 0, 1, \cdots$, where $h_j^k(t)$ is the k-th order derivative of h_j and

$$\begin{cases} \phi_0(x) &= \phi(x) \\ \phi_k(x) &= -\left(\phi_{k-1}'''(x) - \phi_{k-1}''(x) + \phi_{k-1}'(x) + \sum_{j=0}^{k-1}(\phi_j(x)\phi_{k-j-1}(x))'\right) \end{cases} \tag{1.2}$$

for $k = 0, 1, \cdots$. When the well-posedness of eq.(1.1) is considered in the space $H^s(0,1)$ for some finite value $s \geq 0$, the following s-compatibility conditions thus arise naturally.

Definition 1.1 *(s-compatibility)* — Let $T > 0$ and $s \geq 0$ be given. We say the four-tuple $(\phi, \vec{h}) = (\phi, h_1, h_2, h_3) \in H^s(0,1) \times H^{\frac{s+1}{3}}(0,T) \times H^{\frac{s+1}{3}}(0,T) \times H^{\frac{s}{3}}(0,T)$ satisfies the s-compatibile condition if

$$\phi_k(0) = h_1^k(0), \quad \phi_k(1) = h_2^k(0) \tag{1.3}$$

holds for $k = 0, 1, \cdots, [\frac{s}{3}] - 1$ when $s - 3[\frac{s}{3}] \leq \frac{1}{2}$, or for $k = 0, 1, \cdots, [\frac{s}{3}]$ when $\frac{1}{2} \leq s - 3[\frac{s}{3}] \leq \frac{3}{2}$ and

$$\phi_k(0) = h_1^k(0), \quad \phi_k(1) = h_2^k(0), \quad \phi_k'(1) = h_3^k(0) \tag{1.4}$$

holds for $k = 0, 1, \cdots, [\frac{s}{3}]$ when $s - 3[\frac{s}{3}] \geq \frac{3}{2}$. We point out that eq.(1.3) does not include the case $[\frac{s}{3}] < 1$.

Since we have given the details of $s - compatibility$, we can give our main result of this article.

***Theorem* 1.2 (*Local well-posedness*)** — *Let $T > 0$ and $s \geq 0$ be given. Suppose that $(\phi, \vec{h}) \in H^s(0,1) \times H^{\frac{s+1}{3}}(0,T) \times H^{\frac{s+1}{3}}(0,T) \times H^{\frac{s}{3}}(0,T)$ is s-compatible. Then there exists a $T^* \in (0,T]$ depending only on the norm of $(\phi, \vec{h})$ in the space $H^s(0,1) \times H^{\frac{s+1}{3}}(0,T) \times H^{\frac{s+1}{3}}(0,T) \times H^{\frac{s}{3}}(0,T)$ such that eq. (1.1) admits a unique solution*

$$u \in C([0,T^*]; H^s(0,1)) \cap L^2([0,T^*]; H^{s+1}(0,1)) \tag{1.5}$$

***Theorem*1.3 (*Global well-posedness*)** — *Let $T>0$ and $s\geq 0$ be given. For any s-compatible*

$$(\phi,\vec{h})\in H^s(0,1)\times H^{\mu_1(s)}(0,T)\times H^{\mu_1(s)}(0,T)\times H^{\mu_2(s)}(0,T) \tag{1.6}$$

where

$$\mu_1(s)=\begin{cases}\epsilon+\frac{5s+9}{18} & if \quad 0\leq s<3\\ \frac{s+1}{3} & if \quad s\geq 3\end{cases} \tag{1.7}$$

$$\mu_2(s)=\begin{cases}\epsilon+\frac{5s+3}{18} & if \quad 0\leq s<3\\ \frac{s}{3} & if \quad s\geq 3\end{cases} \tag{1.8}$$

and ϵ is any positive constant, then the initial-boundary value problem of eq.(1.1) admits a unique solution

$$u\in C([0,T];H^s(0,1))\cap L^2([0,T];H^{s+1}(0,1)). \tag{1.9}$$

Remark 1.4 : The proof of local well-posedness and global well-posedness result of eq.(1.1) rely on the smoothing properties of the following linear problem:

$$\begin{cases} u_t+u_x-u_{xx}+u_{xxx}=f, \quad u(x,0)=\phi(x), \quad \text{for} \quad x\in[0,1],\\ u(0,t)=h_1(t), \quad u(1,t)=h_2(t), \quad u_x(1,t)=h_3(t), \quad \text{for} \quad t\geq 0.\end{cases} \tag{1.10}$$

We can decompose this linear problem associated to the eq.(1.1) into three parts:

(i) For $\phi\in L^2(0,1)$ with $f=0, h_j=0, j=1,2,3$, the solution *u* of eq.(1.10) belongs to the space $C(\mathbb{R}^+;L^2(0,1))\cap L^2(\mathbb{R}^+;H^1(0,1))$ and $u_x\in C([0,1];L^2_t(\mathbb{R}^+))$;

(ii) For $f\in L^1(\mathbb{R}^+;L^2(0,1))$ with $\phi=0, h_j=0, j=1,2,3$, the solution of eq.(1.10) belongs to the space $C(\mathbb{R}^+;L^2(0,1))\cap L^2(\mathbb{R}^+;H^1(0,1))$ and $u_x\in C([0,1];L^2_t(\mathbb{R}^+))$;

(iii) For $h_1,h_2\in H^{\frac{1}{3}}_{loc}(\mathbb{R}^+), h_3\in L^2_{loc}(\mathbb{R}^+)$ with $f=0$ and $\phi=0$, the solution *u* of eq.(1.10) belongs to the space $C(\mathbb{R}^+;L^2(0,1))\cap L^2_{loc}(\mathbb{R}^+;H^1(0,1))$ and $u_x\in C([0,1];L^2_t(\mathbb{R}^+))$;

In order to introduce our result conveniently, we define some spaces:

$$D_{s,T}=H^s(0,1)\times H^{\frac{s+1}{3}}(0,T)\times H^{\frac{s+1}{3}}(0,T)\times H^{\frac{s}{3}}(0,T).$$

The paper is organized as follows. In Section 2, we present various linear estimates associated to the IBVP of eq. (1.1). The local well-posedness result of eq. (1.1) is presented in Section 3. With the nonlinear interpolation theory [9], we will show the global well-posedness result of eq. (1.1).

2. Linear Estimates

In this section, we will discuss the linear problem of eq. (1.1). For the following linear problem, we will discuss the estimate of the solution u according to it:

$$\begin{cases} u_t + u_x - u_{xx} + u_{xxx} = 0, \quad u(x,0) = \phi(x) \quad \text{for} \quad x \in [0,1], \\ u(0,t) = 0, \quad u(1,t) = 0, \quad u_x(1,t) = 0, \quad \text{for} \quad t \geq 0. \end{cases} \tag{2.1}$$

Let A be the linear operator of the above eq.(2.1) in the space $L^2(0,1)$ and be defined as:

$$Af = -f''' + f'' - f'.$$

Consider A as an unbounded linear operator in the space $L^2(0,1)$ and define it's definition domain as:

$$\mathbb{D}(A) = \{f \in H^3(0,1), f(0) = f(1) = f'(1) = 0\}.$$

We can describe the eq.(2.1) as an abstract evolution equation in the space $L^2(0,1)$, viz:

$$\frac{du}{dt} = Au, \qquad u(0) = \phi,$$

and we can have the adjoint operator A^* of A in the space $L^2(0,1)$ and the definition domain of A^*

$$A^*g = g''' + g'' + g',$$
$$\mathbb{D}(A^*) = \{g \in H^3(0,1), g(0) = g(1) = g'(0) = 0\}.$$

It's very easy to check that both A and it's adjoint A^* are dissipative operators, which is to say:

$$\langle Af, f\rangle_{L^2(0,1)} \leq 0, \qquad \langle A^*g, g\rangle_{L^2(0,1)} \leq 0$$

for any $f \in \mathbb{D}(A)$ and $g \in \mathbb{D}(A^*)$.

So, the semigroup $W_0(t)$ generated by the infinitesimal generator A is C_0 in the space $L^2(0,1)$. We can use standard semigroup theory to write the mild solution of eq. (2.1) as follows:

$$u(t) = W_0(t)\phi,$$

which belongs to the space $C_b(R^+; L^2(0,1))$. If $\phi \in \mathbb{D}(A)$, then $u(t)$ belongs to the space $C(0,\infty; H^3(0,1)) \cap C^1(0,\infty; L^2(0,1))$ and $u(t) \in \mathbb{D}(A)$ for all $t \geq 0$ of course.

Proposition 2.1 — For any $\phi \in L^2(0,1)$, $\quad u(t) = W_0(t)\phi$ satisfies

$$\|u(\cdot,t)\|^2_{L^2(0,1)} + \int_0^t u_x^2(0,\tau)d\tau + 2\int_0^t\int_0^1 u_x^2(x,\tau)dxd\tau = \|\phi\|^2_{L^2(0,1)} \tag{2.2}$$

and

$$3\int_0^t\int_0^L u_x^2(x,\tau)dxd\tau + \int_0^1 xu^2dx \leq (1+\frac{1}{2}t)\|\phi\|^2_{L^2(0,1).} \tag{2.3}$$

Remark 2.2 : The equality (2.2) satisfies the trace property of eq. (2.1) and reveals the boundary smoothing effect. Combining the equality (2.2) and inequality (2.3), we can have the following estimate

$$\|u\|_{L^2(0,t;H^1(0,1))} \leq C(1+\frac{1}{2}t)^{1/2}\|\phi\|_{L^2(0,1)} \tag{2.4}$$

Inequality (2.4) is a Kato-type smoothing effect and this smoothing effect is global. We will show that it's enough to establish the well-posedness of eq.(1.1) in the space $H^s(0,1)$ for $s \geq 0$ only using this global Kato-smoothing effect.

PROOF : Since the proof if similar to that presented in [10], we omit it.

Now let's consider the inhomogeneous linear problem

$$\begin{cases} u_t + u_x - u_{xx} + u_{xxx} = f, \quad u(x,0) = 0 \quad \text{for} \quad x \in [0,1], \\ u(0,t) = 0, \quad u(1,t) = 0, \quad u_x(1,t) = 0 \quad \text{for} \quad t \geq 0 \end{cases} \tag{2.5}$$

Since we have defined the linear operator A above, we may write (2.6) as an abstract nonhomogeneous evolution equation with initial-value equals zero, viz.

$$\frac{du}{dt} = Au + f, \qquad u(0) = 0. \tag{2.6}$$

We can write the mild solution of the abstract equation (2.7) as follow:

$$u(t) = \int_0^t W_0(t-\tau)f(\tau)d\tau \tag{2.7}$$

for any $f \in L^1_{loc}(\mathbb{R}^+; L^2(0,1))$ and this mild solution belongs to the space $C(\mathbb{R}^+; L^2(0,1))$. It is a weak solution of (2.6) in the sense of distribution. If we want to get the strong solution of (2.7), we should let $f \in \mathbb{D}(A)$ for $t > 0$ and check $Af \in L^1_{loc}(\mathbb{R}^+; L^2(0,1))$. □

Proposition 2.3 — There exists a constant C such that for any $f \in L^1_{loc}(\mathbb{R}^+; L^2(0,1))$, the solution of eq.(2.5) satisfies:

$$\|u(\cdot,t)\|_{L^2(0,1)} + \|u_x(0,\cdot)\|_{L^2(0,t)} + \|u_x\|_{L^2(0,t;L^2(0,1))} \leq C\|f\|_{L^1(0,t;L^2(0,1))} \tag{2.8}$$

and

$$\int_0^1 xu^2(x,t)dx + 4\int_0^t\int_0^1 u_x^2(x,\tau)dxd\tau \le (1+t)\|f\|^2_{L^1(0,t;L^2(0,1))}. \tag{2.9}$$

PROOF : We may assume that u is a strong solution to eq. (2.5). Using the method of Carleman estimate, multiply the equation in eq. (2.5) by u and integrate over (0,1) with respect to x. Integration by parts leads to

$$\frac{d}{dt}\int_0^1 u^2dx + u_x^2(0,t) + 2\int_0^1 u_x^2dx \le 2\|f(\cdot,t)\|_{L^2(0,1)}\|u(\cdot,t)\|_{L^2(0,1)}. \tag{2.10}$$

Integrate the two sides of (2.10) with respect to t, we have

$$\begin{aligned}\|u(\cdot,t)\|^2_{L^2(0,1)} + \|u_x(0,\cdot)\|^2_{L^2(0,t)} \quad &+ \quad 2\int_0^t\int_0^1 u_x^2(x,\tau)dxd\tau \\ &\le \quad 2\|f\|_{L^1(0,t;L^2(0,1))}\|u\|_{L^\infty(0,t;L^2(0,1))}\end{aligned} \tag{2.11}$$

Since $u(t) = \int_0^t W_0(t-\tau)f(\tau)d\tau$ and $W_0(t)$ is a contractive semigroup, we have

$$\begin{aligned}\|u\|_{L^\infty(0,t;L^2(0,1))} &\le \|u(\cdot,t')\|_{L^2(0,1)} \qquad \text{for} \quad \text{some} \quad t' \in [0,t] \\ &\le C\int_0^{t'} \|f(\cdot,\tau)\|_{L^2(0,1)}d\tau \\ &\le C\|f\|_{L^1(0,t;L^2(0,1))}.\end{aligned}$$

Then we have the following inequality:

$$\|u(\cdot,t)\|^2_{L^2(0,1)} + \|u_x(0,\cdot)\|^2_{L^2(0,t)} + 2\|u_x\|^2_{L^2(0,t;L^2(0,1))} \le C\|f\|^2_{L^1(0,t;L^2(0,1))},$$

and after simple computation, we have :

$$\|u(\cdot,t)\|_{L^2(0,1)} + \|u_x(0,\cdot)\|_{L^2(0,t)} + 2\|u_x\|_{L^2(0,t;L^2(0,1))} \le C\|f\|_{L^1(0,t;L^2(0,1))},$$

and we have proved (2.8). Next, let's prove (2.9).

Multiply both sides of the equation of eq. (2.5) with $2xu$, and integrate over $(0,1)$ with respect to x and over $(0,t)$ with respect to t, we have

$$\begin{aligned}\int_0^1 xu^2(x,t)dx + 4\int_0^t\int_0^1 u_x^2(x,\tau)dxd\tau &= 2\int_0^t\int_0^1 xf(x,\tau)u(x,\tau)dxd\tau + \int_0^t\int_0^1 u^2dxd\tau \\ &\le \int_0^t \|x^{1/2}f(\cdot,\tau)\|_{L^2(0,1)}\|x^{1/2}u(\cdot,\tau)\|_{L^2(0,1)}d\tau \\ &\quad + \int_0^t\int_0^1 u^2dxd\tau \\ &\le \|x^{1/2}f(\cdot,\tau)\|_{L^1(0,t;L^2(0,1))}\|x^{1/2}u(\cdot,\tau)\|_{L^\infty(0,t;L^2(0,1))} \\ &\quad + \int_0^t\int_0^1 u^2dxd\tau \\ &\le \frac{1}{2}\|x^{1/2}f(\cdot,\tau)\|^2_{L^1(0,t;L^2(0,1))} + \frac{1}{2}\|x^{1/2}u(\cdot,\tau)\|^2_{L^\infty(0,t;L^2(0,1))} \\ &\quad + \int_0^t\int_0^1 u^2dxd\tau.\end{aligned}$$

For $u(t)=\int_0^t W_0(t-\tau)f(\tau)d\tau$, and $W_0(t)$ is a contractive semigroup, we can have the following estimates:

$$\begin{aligned}&\|x^{1/2}f(\cdot,\tau)\|^2_{L^1(0,t;L^2(0,1))} \le \|f\|^2_{L^1(0,t;L^2(0,1))}, \\ &\|x^{1/2}u(\cdot,\tau)\|^2_{L^\infty(0,t;L^2(0,1))} \le \|f\|^2_{L^1(0,t;L^2(0,1))}, \\ &\int_0^t\int_0^1 u^2dxd\tau \le t\int_0^1 \|\sup_{0\le\tau\le t} u(x,\tau)\|^2dx \le t\|f\|^2_{L^1(0,t;L^2(0,1))}.\end{aligned}$$

So we have

$$\int_0^1 xu^2(x,t)dx + 4\int_0^t\int_0^1 u_x^2(x,\tau)dxd\tau \le (1+t)\|f\|^2_{L^1(0,t;L^2(0,1))},$$

this is (2.9), and we have finished the proof of Proposition 2.3. □

Next, let's consider the non-homogeneous boundary-value linear problem

$$\begin{cases} u_t+u_x-u_{xx}+u_{xxx}=0, \quad u(x,0)=0, \quad \text{for} \quad x\in[0,1], \\ u(0,t)=h_1(t), \quad u(1,t)=h_2(t), \quad u_x(1,t)=h_3(t), \quad \text{for} \quad t\ge 0. \end{cases} \tag{2.12}$$

We will use Laplace transform to seek an explicit solution formula in terms of the boundary-values problem of eq.(2.12) as Bona did in [10]. After Laplace transform, eq.(2.12) is converted to the following form:

$$\begin{cases} s\hat{u}(x,s)+\hat{u}_x(x,s)-\hat{u}_{xx}+\hat{u}_{xxx}=0, \\ \hat{u}(0,s)=\hat{h}_1(s), \hat{u}(1,s)=\hat{h}_2(s), \hat{u}_x(1,s)=\hat{h}_3(s), \end{cases} \tag{2.13}$$

where

$$\hat{u}(x,s) = \int_0^{+\infty} e^{-st}u(x,t)dt$$

and

$$\hat{h}_j(s) = \int_0^{+\infty} e^{-st}h_j(t)dt, \quad j = 1,2,3.$$

We can write the solution $\hat{u}(x,s)$ of eq. (2.13) in the form:

$$\hat{u}(x,s) = \sum_{j=1}^{3} c_j(s)e^{\lambda_j(s)x},$$

where $\lambda_j(s), \quad j = 1,2,3$, are the three solutions of the characteristic equation

$$s + \lambda - \lambda^2 + \lambda^3 = 0, \tag{2.14}$$

and $c_j(s), \quad j = 1,2,3$, solve the following linear system

$$\begin{cases} c_1(s) + c_2(s) + c_3(s) = \hat{h}_1(s), \\ c_1(s)e^{\lambda_1(s)} + c_2(s)e^{\lambda_2(s)} + c_3(s)e^{\lambda_3(s)} = \hat{h}_2(s), \\ c_1(s)\lambda_1(s)e^{\lambda_1(s)} + c_2(s)\lambda_2(s)e^{\lambda_2(s)} + c_3(s)\lambda_3(s)e^{\lambda_3(s)} = \hat{h}_3(s), \end{cases} \tag{2.15}$$

Let $\lambda_1 = a + bi$ be a solution of eq.(2.14), the parameter a and b are to be determined by the following discussion:

According to eq.(2.14) we have:

$$\begin{aligned} s &= -[a(a^2 - b^2 - a + 1) - b(2ab - b)] - [b(a^2 - b^2 - a + 1) + a(2ab - b)]i \\ &= A + Bi, \end{aligned}$$

where

$$A = -[a(a^2 - b^2 - a + 1) - b(2ab - b)]$$

and

$$B = -[b(a^2 - b^2 - a + 1) + a(2ab - b)].$$

Let $A = 0$, then we have $b = \pm\sqrt{\frac{a(a^2-a+1)}{3a-1}}$. We choose $b = \sqrt{\frac{a(a^2-a+1)}{3a-1}}$, then we have

$$\begin{aligned} B &= -\sqrt{\frac{a(a^2 - a + 1)}{3a - 1}}\,\frac{4a(2a^2 - 2a + 1) - 1}{3a - 1}, \\ s = s(a) &= -\frac{(8a^3 - 8a^2 + 4a - 1)\sqrt{a(3a - 1)(a^2 - a + 1)}}{(3a - 1)^2}i, \end{aligned}$$

and

$$s \to -\frac{8}{9}a^3 i, \qquad \text{as} \quad a \to +\infty.$$

Solving the equation $B(a) = 0$ with variable a, we have its maximum root is $a = \frac{1}{2}$. So, for $s \in [0, +\infty)$, we have $a \in [\frac{1}{2}, +\infty)$ accordingly. With easy discussion, we know that the other two roots of eq.(2.14) have the following forms

$$\lambda_2(a) = -\frac{2\sqrt[3]{3}}{2} ai + \overrightarrow{\alpha},$$
$$\lambda_3(a) = \overrightarrow{\beta},$$

where

$$\overrightarrow{\alpha} \quad \text{and} \quad \overrightarrow{\beta} \in \mathbb{C}, \quad \mathrm{Re}(\overrightarrow{\alpha}) > 0, \quad \frac{|\overrightarrow{\alpha}|}{a} \to 0, \quad as \quad a \to +\infty, \quad \text{and} \quad \mathrm{Re}(\overrightarrow{\beta}) < 0.$$

Now, let's consider the linear system (2.15), and use Cramer's rule to get $c_j(s), \quad j = 1, 2, 3$. We have following results:

$$c_j = \frac{\Delta_j(s)}{\Delta(s)}, \quad j = 1, 2, 3,$$

where, $\Delta(s)$ is the determinant of the coefficient matrix of eq.(2.15) and $\Delta_j(s)$ the determinants of the matrices that are obtained by replacing the $ith-column$ of $\Delta(s)$ by the column vector $(\hat{h}_1(s), \hat{h}_2(s), \hat{h}_3(s)), j = 1, 2, 3.$

Taking the inverse Laplace transform of $\hat{u}$ yields

$$u(x,t) = \frac{1}{2\pi i}\int_{\omega - i\infty}^{\omega + i\infty} e^{st}\hat{u}(x,s)ds = \sum_{j=1}^{3}\frac{1}{2\pi i}\int_{\omega - i\infty}^{\omega + i\infty} e^{st}\frac{\Delta_j(s)}{\Delta(s)}e^{\lambda_j(s)x}ds$$

for any $\omega > 0$. We can write the solution of eq.(2.13) in the form

$$u(x,t) = u_1(x,t) + u_2(x,t) + u_3(x,t),$$

where $u_k(x,t)$ is the solution of eq.(2.12) with $h_j \equiv 0$ when $j \neq k, \quad k, j = 1, 2, 3$; and so u_k has the following representation

$$u_k(x,t) = \sum_{j=1}^{3}\frac{1}{2\pi i}\int_{\omega - i\infty}^{\omega + i\infty} e^{st}\frac{\Delta_{j,k}(s)}{\Delta(s)}e^{\lambda_j(s)x}\hat{h}_k(s)ds \equiv W_k(t)h_k, \tag{2.16}$$

for $k = 1, 2, 3$. $\Delta_{j,k}(s)$ is obtained from $\Delta_j(s)$ when $\hat{h}_k(s) = 1$ and $\hat{h}_i(s) \equiv 0$ for $i \neq k, i, k = 1, 2, 3$. It's obvious that the right-hand side of eq. (2.16) is continuous with respect to

ω for any $\omega \geq 0$ and the left-hand side of eq. (2.16) does not depend on ω. So we can take $\omega = 0$ and write eq. (2.16) in the following form

$$\begin{aligned} u_k(x,t) &= \sum_{j=1}^{3} \frac{1}{2\pi i} \int_0^{+i\infty} e^{st} \frac{\Delta_{j,k}(s)}{\Delta(s)} e^{\lambda_j(s)x} \hat{h}_k(s) ds + \sum_{j=1}^{3} \frac{1}{2\pi i} \int_{-i\infty}^{0} e^{st} \frac{\Delta_{j,k}(s)}{\Delta(s)} e^{\lambda_j(s)x} \hat{h}_k(s) ds \\ &\equiv I_k(x,t) + II_k(x,t), \end{aligned}$$

for $k = 1, 2, 3$.

Since we have given $s = s(\zeta)$ (i.e. let $a = \zeta$) and $\zeta \geq \frac{1}{2}$, we can write $I_k(x,t)$ and $II_k(x,t)$ in the following form

$$I_k(x,t) = \sum_{j=1}^{3} \frac{1}{2\pi i} \int_{\frac{1}{2}}^{+\infty} e^{s(\zeta)t} e^{\lambda_j^+(\zeta)x} \frac{\Delta_{j,k}^+(\zeta)}{\Delta^+(\zeta)} \frac{\partial s(\zeta)}{\partial \zeta} \hat{h}_k^+(\zeta) d\zeta$$

and

$$II_k(x,t) = \sum_{j=1}^{3} \frac{1}{2\pi i} \int_{\frac{1}{2}}^{+\infty} e^{-s(\zeta)t} e^{\lambda_j^-(\zeta)x} \frac{\Delta_{j,k}^-(\zeta)}{\Delta^-(\zeta)} \frac{\partial s(\zeta)}{\partial \zeta} \hat{h}_k^-(\zeta) d\zeta$$

where $\hat{h}_k^+(\zeta) = \hat{h}_k(s(\zeta))$, $\Delta^+(\zeta)$ and $\Delta_{j,k}^+(\zeta)$ are obtained from $\Delta(s)$ and $\Delta_{j,k}(s)$ by replacing s with $s(\zeta)$ and $\lambda_j(s)$ with $\lambda_j^+(\zeta)$, for $j = 1, 2, 3$, respectively. We also point out that $\Delta^-(\zeta) = \overline{\Delta^+}(\zeta)$ and $\Delta_{j,k}^-(\zeta) = \overline{\Delta_{j,k}^+}(\zeta)$ for $j = 1, 2, 3$, and $\hat{h}_k^-(\zeta) = \overline{\hat{h_k^+}}(\zeta)$.

In the following discussion, we will give a lemma which is some different to the result given by Bona in [10].

Lemma 2.4 — Let Kf be the function defined by

$$Kf(x) = \int_0^{+\infty} e^{\gamma(\nu)x} f(\nu) d\nu, \quad \text{for} \quad \text{any} \quad f \in L^2(0, +\infty),$$

where $\gamma(\nu)$ is a continuous complex-valued function defined on $(0, +\infty)$ and satisfies the following conditions:

(i) There exist $\delta > 0$ and $b > 0$ such that

$$\operatorname*{Inf}_{0<\nu<\delta} \frac{\|\mathrm{Re}\gamma(\nu)\|}{\nu} \geq b \quad , \quad \text{and} \quad \mathrm{Re}\gamma(\nu) > 0 \quad ;$$

(ii) There exist a complex number $\alpha + i\beta$ such that

$$\lim_{\nu \to +\infty} \frac{\gamma(\nu)}{\nu} = \alpha + i\beta.$$

Then there exists a constant C such that for all $f \in L^2(0,+\infty)$,

$$\|Kf\|_{L^2(0,1)} \leq C(\|e^{\mathrm{Re}\gamma(\cdot)} f(\cdot)\|_{L^2(0,+\infty)} + \|f\|_{L^2(0,+\infty)}).$$

PROOF : Using Hölder inequality, we have

$$\begin{aligned}
\|Kf\|^2_{L^2(0,1)} &\leq \int_0^1 \left(\int_0^{+\infty} e^{\mathrm{Re}(\gamma(\nu))x} \mid f(\nu) \mid d\nu \right)^2 dx \\
&= \int_0^1 \int_0^{+\infty} e^{\mathrm{Re}(\gamma(\nu))x} \mid f(\nu) \mid d\nu \int_0^{+\infty} e^{\mathrm{Re}(\gamma(\xi))x} \mid f(\xi) \mid d\xi dx \\
&= \int_0^{+\infty} \int_0^{+\infty} \left(\int_0^1 e^{\mathrm{Re}(\gamma(\nu)+\gamma(\xi))x} dx \right) \mid f(\nu) \mid\mid f(\xi) \mid d\nu d\xi \\
&\leq \int_0^{+\infty} \int_0^{+\infty} \frac{(e^{\mathrm{Re}(\gamma(\nu)+\gamma(\xi))} + 1)}{\mathrm{Re}(\gamma(\nu)+\gamma(\xi))} \mid f(\nu) \mid\mid f(\xi) \mid d\nu d\xi \\
&\leq \left\| \int_0^{+\infty} \frac{e^{\mathrm{Re}\gamma(\nu)} \mid f(\nu) \mid}{\mid \mathrm{Re}(\gamma(\nu)+\gamma(\xi)) \mid} d\nu \right\|_{L^2(0,+\infty)} \|e^{\mathrm{Re}\gamma(\cdot)} f(\cdot)\|_{L^2(0,+\infty)} \\
&\quad + \left\| \int_0^{+\infty} \frac{\mid f(\nu) \mid}{\mid \mathrm{Re}(\gamma(\nu)+\gamma(\xi)) \mid} d\nu \right\|_{L^2(0,+\infty)} \|f(\cdot)\|_{L^2(0,+\infty)}.
\end{aligned}$$

Observe that

$$\|e^{\mathrm{Re}\gamma(\mu\xi)} f(\mu\xi)\|_{L^2(0,\infty)} \leq \frac{1}{\sqrt{\mu}} \|e^{\mathrm{Re}\gamma(\cdot)} f(\cdot)\|_{L^2(0,+\infty)}, \qquad \text{for} \quad \text{any} \quad \mu \in (0,+\infty)$$

and under the condition (i), we have

$$\frac{\xi}{\mid \mathrm{Re}(\gamma(\mu\xi)+\gamma(\xi)) \mid} \leq \frac{C}{\xi+1}, \qquad \text{for} \quad \text{any} \quad \xi \in (0,+\infty).$$

Using the generalized Minkowski's inequality, we have

$$\begin{aligned}
\left\| \int_0^{+\infty} \frac{e^{\mathrm{Re}\gamma(\nu)} \mid f(\nu) \mid}{\mathrm{Re}(\gamma(\nu)+\gamma(\xi))} d\nu \right\|_{L^2(0,+\infty)} &= \left\| \int_0^{+\infty} \frac{e^{\mathrm{Re}\gamma(\mu\xi)} \mid f(\mu\xi) \mid \xi d\mu}{\mathrm{Re}(\gamma(\mu\xi)+\gamma(\xi))} \right\|_{L^2(0,+\infty)} \\
&\leq \int_0^{+\infty} \left\| \frac{e^{\mathrm{Re}\gamma(\mu\xi)} \mid f(\mu\xi) \mid \xi d\mu}{\mathrm{Re}(\gamma(\mu\xi)+\gamma(\xi))} \right\|_{L^2(0,+\infty)} d\mu \\
&\leq C \int_0^{+\infty} \frac{1}{\sqrt{\mu}(1+\mu)} d\mu \|e^{\mathrm{Re}\gamma(\cdot)} f(\cdot)\|_{L^2(0,+\infty)} \\
&\leq C\|e^{\mathrm{Re}\gamma(\cdot)} f(\cdot)\|_{L^2(0,+\infty)}
\end{aligned}$$

for some constant $C > 0$. Using the same argument, we can also have the following inequality

$$\left\| \int_0^{+\infty} \frac{\mid f(\nu) \mid}{\mathrm{Re}(\gamma(\nu)+\gamma(\xi))} d\nu \right\|_{L^2(0,+\infty)} \leq C\|f(\cdot)\|_{L^2(0,+\infty)}.$$

So, we complete the proof. □

Using Lemma 2.4, we can directly have the following Lemma 2.5.

Lemma 2.5 — For any $a > 0$ be given. We define function Gf as follows

$$Gf(x) = \int_0^a e^{i\gamma(\mu)x} f(\mu)d\mu \qquad \text{for} \quad \text{any} \quad f \in L^2(0,a),$$

where $r(\cdot)$ is a continuous and real-valued function defined on the interval $[0, a]$ which is also C^1 on $(0, a)$. $r(\mu)$ also satisfies the condition that: there exist a constant C_1 such that

$$\frac{1}{\mid \gamma'(\mu) \mid} \leq C_1, \quad \text{for} \quad \text{any} \quad 0 < \mu < a.$$

Then there exist a constant C_2 such that the following inequality holds

$$\|Gf\|_{L^2(0,a)} \leq C_2 \|f\|_{L^2(0,a)}$$

In the next part, we will discuss the smoothing properties of the linear system eq. (1.10) promoted by only one boundary value, that means

$$f \equiv 0; \quad \phi \equiv 0; \quad h_i \not\equiv 0 \quad \text{and} \quad h_j \equiv 0, \quad i = 1, 2, \text{ or } 3 \ \text{ with } \quad j \in \{1, 2, 3\} \backslash \{i\}.$$

Proposition 2.6 — There exists a constant C such that

$$\|u_1\|_{L^2(0,+\infty;H^1(0,1))} + \sup_{0 \leq t < +\infty} \|u_1(\cdot, t)\|_{L^2(0,1)} \leq C \|h_1\|_{H^{\frac{1}{3}}(0,+\infty)} \tag{2.17}$$

and $\partial_x u_1 \in C_b([0, 1]; L^2(0, +\infty))$ with

$$\sup_{x \in [0,1]} \|\partial_x u_1(x, \cdot)\|_{L^2(0,+\infty)} \leq C \|h_1\|_{H^{\frac{1}{3}}(0,+\infty)}, \quad \text{for} \quad \text{all} \quad h_1 \in H^{\frac{1}{3}}(0, +\infty). \tag{2.18}$$

PROOF : In order to show the details of the proof, we give the explicit estimate of $\lambda_j^+(\zeta), \quad \Delta^+(\zeta)$, and $\frac{\Delta_{j,1}^+(\zeta)}{\Delta^+(\zeta)}, \quad j = 1, 2, 3$. We write $\Delta^+(\zeta)$ in another form

$$\Delta^+(\zeta) = A_1 + A_2 + A_3,$$

where

$$A_1 = \Delta_{1,1}^+(\zeta) = (\lambda_3^+(\zeta) - \lambda_2^+(\zeta))e^{\lambda_2^+(\zeta)+\lambda_3^+(\zeta)},$$
$$A_2 = \Delta_{2,1}^+(\zeta) = (\lambda_1^+(\zeta) - \lambda_3^+(\zeta))e^{\lambda_1^+(\zeta)+\lambda_3^+(\zeta)},$$
$$A_3 = \Delta_{3,1}^+(\zeta) = (\lambda_2^+(\zeta) - \lambda_1^+(\zeta))e^{\lambda_1^+(\zeta)+\lambda_2^+(\zeta)},$$

and we have the following estimates

$$\frac{\Delta_{1,1}^{+}(\zeta)}{\Delta^{+}(\zeta)} \to 0, \qquad \frac{\Delta_{2,1}^{+}(\zeta)}{\Delta^{+}(\zeta)} \to 0, \qquad \frac{\Delta_{3,1}^{+}(\zeta)}{\Delta^{+}(\zeta)} \sim 1.$$

as $\zeta \to +\infty$.

Under the help of Lemma 2.4 and Lemma 2.5, we have the result that there exists a constant C such that

$$\|I_1(\cdot,t)\|_{L^2(0,1)}^2 \le C\sum_{j=1}^{3}\int_{\frac{1}{2}}^{+\infty}\left|\frac{\Delta_{j,1}^{+}(\zeta)}{\Delta^{+}(\zeta)}\right|^2\left(e^{\mathrm{Re}\lambda_j^{+}(\zeta)}+1\right)^2\left|\hat{h_1}^{+}(\zeta)\frac{\partial s(\zeta)}{\partial\zeta}\right|^2 d\zeta,$$

for $\frac{\partial s(\zeta)}{\partial \zeta} \sim \zeta^2$ as $\zeta \to +\infty$, we have

$$\|I_1(\cdot,t)\|_{L^2(0,1)}^2 \le C\int_{\frac{1}{2}}^{+\infty}|\zeta|^4|\hat{h_1}^{+}(\zeta)|^2 d\zeta.$$

Let $\xi i = s(\zeta)i$, we have $\zeta \sim \xi^{\frac{1}{3}}, \quad i.e. \quad \zeta^2 \sim \xi^{\frac{2}{3}}$, and $d\xi = s'(\zeta)d\zeta \sim \zeta^2 d\zeta$ as $\zeta \to +\infty$. So, we have the following estimate:

$$\begin{aligned}\|I_1(\cdot,t)\|_{L^2(0,1)}^2 &\le C\int_0^{\infty}|\zeta|^2|\hat{h_1}^{+}(\zeta)|^2 d\xi\\ &\le C\int_0^{\infty}(1+\xi^2)^{\frac{1}{3}}|\hat{h_1}(\xi)|^2 d\xi\\ &\le C\|h_1\|_{H^{\frac{1}{3}}(0,+\infty)}^2.\end{aligned}$$

We can use the same strategy as what we have done above to have the estimate of $II_1(\cdot,t)$ as follows

$$\|II_1(\cdot,t)\|_{L^2(0,1)}^2 \le C\|h_1\|_{H^{\frac{1}{3}}(0,+\infty)}^2.$$

Now, let's discuss what regularity $I_1(x,t)$ has.

$$\begin{aligned}\partial_x I_1(x,t) &= \sum_{j=1}^{3}\frac{1}{2\pi}\int_{\frac{1}{2}}^{+\infty}e^{is(\zeta)t}\lambda_j^{+}(\zeta)e^{\lambda_j^{+}(\zeta)x}\frac{\Delta_{j,1}^{+}(\zeta)}{\Delta^{+}(\zeta)}s'(\zeta)\hat{h_1}^{+}(\zeta)d\zeta\\ &= \sum_{j=1}^{3}\frac{1}{2\pi}\int_0^{+\infty}e^{i\omega t}\lambda_j^{+}(\theta(\omega))e^{\lambda_j^{+}(\theta(\omega))x}\frac{\Delta_{j,1}^{+}(\theta(\omega))}{\Delta^{+}(\theta(\omega))}\hat{h_1}(i\omega)d\omega,\end{aligned}$$

where $\theta(\omega)$ is the solution of the equation $\omega = s(\zeta)$. Let $\mathbb{F}[G_j^+(\cdot)] = \lambda_j^+(\theta(\omega))e^{\lambda_j^+(\theta(\omega))x}\frac{\Delta_{j,1}^+(\theta(\omega))}{\Delta^+(\theta(\omega))}\hat{h_1}(i\omega)$ be the Fourier transform of some function $G(\cdot)$, we have

$$\partial_x I_1(x,t) = \sum_{j=1}^{3}\frac{1}{2\pi}\int_0^{+\infty} e^{i\omega t}\mathbb{F}[G_j^+(\cdot)]d\omega.$$

Using Plancherel theorem on t, we have

$$\begin{aligned}\|\partial_x I_1(x,\cdot)\|^2_{L^2(0,+\infty)} &\le \sum_{j=1}^{3}\int_0^{+\infty}\left|\lambda_j^+(\theta(\omega))e^{\lambda_j^+(\theta(\omega))x}\frac{\Delta_{j,1}^+(\theta(\omega))}{\Delta^+(\theta(\omega))}\right|^2|\hat{h_1}(i\omega)|^2 d\omega\\ &\le C\sum_{j=1}^{3}\int_0^{+\infty}|\lambda_j^+(\theta(\omega))|^2\left(\sup_{0\le x\le 1}|e^{\lambda_j^+(\theta(\omega))x}|^2\right)\left|\frac{\Delta_{j,1}^+(\theta(\omega))}{\Delta^+(\theta(\omega))}\right|^2|\hat{h_1}(i\omega)|^2 d\omega.\end{aligned}$$

Using the estimate of $\lambda_j^+(\zeta), \quad \frac{\Delta_{j,1}^+}{\Delta^+}, \quad j = 1,2,3$, we have the following inequality

$$\begin{aligned}\|\partial_x I_1(x,\cdot)\|^2_{L^2(0,+\infty)} &\le C\sum_{j=1}^{3}\int_0^{+\infty}(1+\omega^2)^{\frac{1}{3}}|\hat{h_1}(i\omega)|^2 d\omega\\ &= C\sum_{j=1}^{3}\int_0^{+\infty}(1+\omega^2)^{\frac{1}{3}}\left|\int_0^{+\infty} e^{-i\omega\tau}h_1(\tau)d\tau\right|^2 d\omega\\ &= C\|h_1\|^2_{H^{\frac{1}{3}}(0,+\infty).}\end{aligned}$$

Using the same strategy, we can also get the following estimate

$$\|I_1(x,\cdot)\|^2_{L^2(0,+\infty)} \le C\|h_1\|_{H^{\frac{1}{3}}(0,+\infty)},$$

and

$$\begin{aligned}\|\partial_x II_1(x,\cdot)\|^2_{L^2(0,+\infty)} &\le C\|h_1\|_{H^{\frac{1}{3}}(0,+\infty)},\\ \|II_1(x,\cdot)\|^2_{L^2(0,+\infty)} &\le C\|h_1\|_{H^{\frac{1}{3}}(0,+\infty)}.\end{aligned}$$

So, we have

$$\begin{aligned}\|I_1\|_{L^2(0,+\infty;H^1(0,1))} &\le C\|h_1\|_{H^{\frac{1}{3}}(0,+\infty)},\\ \|II_1\|_{L^2(0,+\infty;H^1(0,1))} &\le C\|h_1\|_{H^{\frac{1}{3}}(0,+\infty)}.\end{aligned}$$

We can have the follow equality directly

$$\begin{aligned}&\partial_x I_1(x,t)-\partial_x I_1(x_0,t)\\&\quad=\sum_{j=1}^{3}\frac{1}{2\pi}\int_0^{+\infty}e^{i\omega t}\lambda_j^+(\theta(\omega))(e^{\lambda_j^+(\theta(\omega))x}-e^{\lambda_j^+(\theta(\omega))x_0})\frac{\Delta_{j,1}^+(\theta(\omega))}{\Delta^+(\theta(\omega))}\hat{h_1}(i\omega)d\omega.\end{aligned}$$

Observe that

$$\begin{aligned}&\|\partial_x I_1(x,t)-\partial_x I_1(x_0,t)\|^2_{L^2(0,+\infty)}\\&\quad\le\sum_{j=1}^{3}\frac{1}{2\pi}\int_0^{+\infty}\left|\lambda_j^+(\theta(\omega))(e^{\lambda_j^+(\theta(\omega))x}-e^{\lambda_j^+(\theta(\omega))x_0})\right|^2|\frac{\Delta_{j,1}^+(\theta(\omega))}{\Delta^+(\theta(\omega))}|^2d\omega\quad\le C||h_1||^2_{H^{\frac{1}{3}}}(0,+\infty).\end{aligned}$$

Using Fatous's lemma, we have

$$\begin{aligned}&\lim_{x\to 0}\|\partial_x I_1(x,t)-\partial_x I_1(x_0,t)\|^2_{L^2(0,+\infty)}\\&\qquad\le\sum_{j=1}^{3}\frac{1}{2\pi}\int_0^{+\infty}\left|\lambda_j^+(\theta(\omega))\lim_{x\to 0}(e^{\lambda_j^+(\theta(\omega))x}-e^{\lambda_j^+(\theta(\omega))x_0})\right|^2|\frac{\Delta_{j,1}^+(\theta(\omega))}{\Delta^+(\theta(\omega))}|^2d\omega\\&\qquad=0\end{aligned}$$

so we have proved $\partial_x I_1(x,\cdot)\in C_b([0,1];L^2(0,+\infty))$ with

$$\sup_{x\in[0,1]}\|\partial_x I_1(x,\cdot)\|_{L^2(0,+\infty)}\le C\|h_1\|_{H^{\frac{1}{3}}(0,+\infty)}$$

for all $h_1\in H^{\frac{1}{3}}(0,+\infty)$. Using the same strategy, we can also get the following estimate

$$\partial_x II_1(x,\cdot)\in C_b([0,1];L^2(0,+\infty)),$$

with

$$\sup_{x\in[0,1]}\|\partial_x II_1(x,\cdot)\|_{L^2(0,+\infty)}\le C\|h_1\|_{H^{\frac{1}{3}}(0,+\infty)}.$$

So, we have proved (2.18) and the proof of Proposition(2.6) is finished. □

Proposition 2.7 — There exists a constant C such that

$$\|u_2\|_{L^2(0,+\infty;H^1(0,1))}+\sup_{0\le t<+\infty}\|u_2(\cdot,t)\|_{L^2(0,1)}\le C\|h_2\|_{H^{\frac{1}{3}}(0,+\infty)}\tag{2.19}$$

and $\partial_x u_2 \in C_b([0,1]; L_t^2(0,+\infty))$ with

$$\sup_{x\in[0,1]} \|\partial_x u_2(x,\cdot)\|_{L^2(0,+\infty)} \le C\|h_2\|_{H^{\frac{1}{3}}(0,+\infty)}, \quad \text{for} \quad \text{all} \quad h_2 \in H^{\frac{1}{3}}(0,+\infty). \tag{2.20}$$

PROOF : Since we already have

$$\begin{aligned}\Delta^+(\zeta) = \quad & (\lambda_3^+(\zeta) - \lambda_2^+(\zeta))e^{\lambda_2^+(\zeta)+\lambda_3^+(\zeta)} + (\lambda_1^+(\zeta) - \lambda_3^+(\zeta))e^{\lambda_1^+(\zeta)+\lambda_3^+(\zeta)} \\ & +(\lambda_2^+(\zeta) - \lambda_1^+(\zeta))e^{\lambda_1^+(\zeta)+\lambda_2^+(\zeta)}.\end{aligned}$$

and we can also have

$$\begin{aligned}\Delta_{1,2}^+(\zeta) &= \lambda_2^+(\zeta)e^{\lambda_2^+(\zeta)} - \lambda_3^+(\zeta)e^{\lambda_3^+(\zeta)}, \\ \Delta_{2,2}^+(\zeta) &= \lambda_3^+(\zeta)e^{\lambda_3^+(\zeta)} - \lambda_1^+(\zeta)e^{\lambda_1^+(\zeta)}, \\ \Delta_{3,2}^+(\zeta) &= \lambda_1^+(\zeta)e^{\lambda_1^+(\zeta)} - \lambda_2^+(\zeta)e^{\lambda_2^+(\zeta)}.\end{aligned}$$

Using the results of $\lambda_j^+(\zeta), \quad j = 1,2,3$, we can have the following estimates

$$\begin{aligned}\frac{\Delta_{1,2}^+(\zeta)}{\Delta^+(\zeta)} &\sim \frac{\lambda_2^+(\zeta)e^{\lambda_2^+(\zeta)}}{(\lambda_2^+(\zeta) - \lambda_1^+(\zeta))e^{\lambda_1^+(\zeta)+\lambda_2^+(\zeta)}} \sim \frac{1}{e^{\lambda_1^+(\zeta)}}, \\ \frac{\Delta_{2,2}^+(\zeta)}{\Delta^+(\zeta)} &\sim \frac{-\lambda_1^+(\zeta)e^{\lambda_1^+(\zeta)}}{(\lambda_2^+(\zeta) - \lambda_1^+(\zeta))e^{\lambda_1^+(\zeta)+\lambda_2^+(\zeta)}} \sim \frac{1}{e^{\lambda_2^+(\zeta)}}, \\ \frac{\Delta_{3,2}^+(\zeta)}{\Delta^+(\zeta)} &\sim \frac{\lambda_1^+(\zeta)e^{\lambda_1^+(\zeta)}}{(\lambda_2^+(\zeta) - \lambda_1^+(\zeta))e^{\lambda_1^+(\zeta)+\lambda_2^+(\zeta)}} \sim \frac{1}{e^{\lambda_2^+(\zeta)}},\end{aligned}$$

as $\zeta \to +\infty$. Under the help of Lemma 2.4, Lemma 2.5 and estimates of $\frac{\Delta_{j,2}^+(\zeta)}{\Delta^+(\zeta)}, \quad j = 1,2,3$ above, we have the result that there exists a constant C such that

$$\begin{aligned}\|I_2(\cdot,t)\|_{L^2(0,1)}^2 &\le \sum_{j=1}^{3} \int_{\frac{1}{2}}^{+\infty} \left|\frac{\Delta_{j,2}^+(\zeta)}{\Delta^+(\zeta)}\right|^2 \left(e^{\mathrm{Re}\lambda_j^+(\zeta)} + 1\right)^2 \left|\hat{h_2}^+(\zeta)\frac{\partial s(\zeta)}{\partial \zeta}\right|^2 d\zeta \\ &\le C\int_0^{+\infty} (\zeta^2+1)^2 |\hat{h_2}^+(\zeta)|^2 d\xi \\ &\le C\|h_2\|_{H^{\frac{1}{3}}(0,+\infty)}^2.\end{aligned}$$

The second inequality was obtained by the following estimates

$$\left|\frac{\Delta^+_{1,2}(\zeta)}{\Delta^+(\zeta)}\right||e^{\mathrm{Re}\lambda_1^+(\zeta)}| \sim \frac{\lambda_2^+(\zeta)e^{\lambda_1^+(\zeta)}}{(\lambda_2^+(\zeta)-\lambda_1^+(\zeta))e^{\lambda_1^+(\zeta)}} \sim 1,$$
$$\left|\frac{\Delta^+_{2,2}(\zeta)}{\Delta^+(\zeta)}\right||e^{\mathrm{Re}\lambda_2^+(\zeta)}| \sim \frac{-\lambda_1^+(\zeta)e^{\lambda_2^+(\zeta)}}{(\lambda_2^+(\zeta)-\lambda_1^+(\zeta))e^{\lambda_2^+(\zeta)}} \sim 1,$$
$$\left|\frac{\Delta^+_{3,2}(\zeta)}{\Delta^+(\zeta)}\right||e^{\mathrm{Re}\lambda_3^+(\zeta)}| \sim \frac{\lambda_1^+(\zeta)}{(\lambda_2^+(\zeta)-\lambda_1^+(\zeta))e^{\lambda_2^+(\zeta)}} \to 0,$$

as $\zeta \to +\infty$. We can use the same strategy as what we have done above to have the estimate of $II_1(\cdot,t)$ as follow

$$\|II_2(\cdot,t)\|^2_{L^2(0,1)} \le C\|h_2\|^2_{H^{\frac{1}{3}}(0,+\infty)}.$$

Now, let's discuss what regularity $I_2(x,t)$ has.

$$\begin{aligned}\partial_x I_2(.,t) &= \sum_{j=1}^{3}\frac{1}{2\pi}\int_{\frac{1}{2}}^{+\infty} e^{is(\zeta)t}\lambda_j^+(\zeta)e^{\lambda_j^+(\zeta)x}\frac{\Delta^+_{j,2}(\zeta)}{\Delta^+(\zeta)}s'(\zeta)\hat{h_2}^+(\zeta)d\zeta\\ &= \sum_{j=1}^{3}\frac{1}{2\pi}\int_{0}^{+\infty} e^{i\omega t}\lambda_j^+(\theta(\omega))e^{\lambda_j^+(\theta(\omega))x}\frac{\Delta^+_{j,2}(\theta(\omega))}{\Delta^+(\theta(\omega))}\hat{h_2}(i\omega)d\omega,\end{aligned}$$

Using the estimate of $\lambda_j^+(\zeta)$, $\frac{\Delta^+_{j,2}(\zeta)}{\Delta^+(\zeta)}$, $j=1,2,3$ and the same method we used in the proof of Proposition 2.6, we have the following inequalities

$$\begin{aligned}\|\partial_x I_2(x,\cdot)\|^2_{L^2(0,+\infty)} &\le C\sum_{j=1}^{3}\int_0^{+\infty}(1+\omega^2)^{\frac{1}{3}}|\hat{h_2}(i\omega)|^2 d\omega\\ &\le C\|h_2\|^2_{H^{\frac{1}{3}}(0,+\infty)},\\ \|I_2(x,\cdot)\|^2_{L^2(0,+\infty)} &\le C\|h_2\|^2_{H^{\frac{1}{3}}(0,+\infty)},\\ \|I_2\|_{L^2(0,+\infty;H^1(0,1))} &\le C\|h_2\|_{H^{\frac{1}{3}}(0,+\infty)},\end{aligned}$$

and

$$\begin{aligned}\|\partial_x II_2(x,\cdot)\|^2_{L^2(0,+\infty)} &\le C\|h_2\|^2_{H^{\frac{1}{3}}(0,+\infty)},\\ \|II_2(x,\cdot)\|^2_{L^2(0,+\infty)} &\le C\|h_2\|^2_{H^{\frac{1}{3}}(0,+\infty)},\\ \|II_2\|_{L^2(0,+\infty;H^1(0,1))} &\le C\|h_2\|_{H^{\frac{1}{3}}(0,+\infty)}.\end{aligned}$$

So, we have proved (2.19). We can also use the method we used in the proof of Proposition 2.6 to have the result that both $I_2(x,\cdot)$ and $II_2(x,\cdot)$ are continuous from $[0,1]$ to the space $L^2(0,+\infty)$ and we have finished the proof of Proposition 2.7. □

Proposition 2.8 — There exists a constant C such that

$$\|u_3\|_{L^2(0,+\infty;H^1(0,1))} + \sup_{0\le t<+\infty} \|u_3(\cdot,t)\|_{L^2(0,1)} \le C\|h_3\|_{L^2(0,+\infty)} \tag{2.21}$$

and $\partial_x u_3 \in C_b([0,1];L^2(0,+\infty))$ with

$$\sup_{x\in[0,1]} \|\partial_x u_3(x,\cdot)\|_{L^2(0,+\infty)} \le C\|h_3\|_{L^2(0,+\infty)}, \quad \text{for} \quad all \quad h_3 \in L^2(0,+\infty). \tag{2.22}$$

PROOF : With the same strategy, we also write $u_3(x,t)$ in the form $u_3(x,t) = I_3(x,t)+II_3(x,t)$, where

$$I_3(x,t) = \sum_{j=1}^{3} \frac{1}{2\pi i} \int_{\frac{1}{2}}^{+\infty} e^{s(\zeta)t} e^{\lambda_j^+(\zeta)x} \frac{\Delta_{j,3}^+(\zeta)}{\Delta^+(\zeta)} e^{\lambda_j(s(\zeta))x} \frac{\partial s(\zeta)}{\partial \zeta} \hat{h}_3^+(\zeta) d\zeta$$

and

$$II_3(x,t) = \sum_{j=1}^{3} \frac{1}{2\pi i} \int_{\frac{1}{2}}^{+\infty} e^{-s(\zeta)t} e^{\lambda_j^-(\zeta)x} \frac{\Delta_{j,3}^-(\zeta)}{\Delta^-(\zeta)} e^{\lambda_j(s(\zeta))x} \frac{\partial s(\zeta)}{\partial \zeta} \hat{h}_3^-(\zeta) d\zeta.$$

With the estimate of $\lambda_j^+(\zeta), \quad \Delta^+(\zeta), \quad j = 1,2,3$ and

$$\begin{aligned} \Delta_{1,3}^+(\zeta) &= e^{\lambda_3^+(\zeta)} - e^{\lambda_2^+(\zeta)}, \\ \Delta_{2,3}^+(\zeta) &= e^{\lambda_1^+(\zeta)} - e^{\lambda_3^+(\zeta)}, \\ \Delta_{3,3}^+(\zeta) &= e^{\lambda_2^+(\zeta)} - e^{\lambda_1^+(\zeta)}, \end{aligned}$$

we have the following estimates

$$\begin{aligned} \frac{\Delta_{1,3}^+(\zeta)}{\Delta^+(\zeta)} &\sim \frac{1}{\zeta e^{\lambda_1^+(\zeta)}}, \\ \frac{\Delta_{2,3}^+(\zeta)}{\Delta^+(\zeta)} &\sim \frac{1}{\zeta e^{\lambda_2^+(\zeta)}}, \\ \frac{\Delta_{3,3}^+(\zeta)}{\Delta^+(\zeta)} &\sim \frac{1}{\zeta e^{\lambda_2^+(\zeta)}}, \end{aligned}$$

as $\zeta \to +\infty$. Using the same strategy we have developed in the proofs of Proposition 2.6 and Proposition 2.7, we can prove the Proposition 2.8. □

We can use semigroup theory to write the solution of eq.(2.12) in the following form

$$u(t) = \sum_{j=1}^{3} W_j(t)h_j, \tag{2.23}$$

where W_j is defined in (2.16). Let's define a space

$$\mathcal{G}_{s,T} = H^{\frac{s+1}{3}}(0,T) \times H^{\frac{s+1}{3}}(0,T) \times H^{\frac{s}{3}}(0,T),$$

for $s \geq 0$ and $T > 0$, and we define the norm of $\overrightarrow{h}(t) = (h_1, h_2, h_3)$ in the space $\mathcal{G}_{s,T}$ as follows

$$\|\overrightarrow{h}\|_{\mathcal{G}_{s,T}} \equiv (\|h_1\|^2_{H^{\frac{s+1}{3}}(0,T)} + \|h_2\|^2_{H^{\frac{s+1}{3}}(0,T)} + \|h_3\|^2_{H^{\frac{s}{3}}(0,T)})^{\frac{1}{2}}.$$

Using the estimates we have got in Proposition 2.6, Proposition 2.7 and Proposition 2.8, we can have the following theorem directly.

***Theorem* 2.9** — *There exists a unique solution $u(x,t)$ of eq. (2.13) with*

$$u(x,t) \in C_b(0,+\infty; L^2(0,1)) \cap L^2(0,+\infty; H^1(0,1)) \quad \textit{and} \quad u_x \in C_b([0,1]; L^2(0,+\infty)).$$

Moreover there exists a constant C, such that

$$\|u\|_{L^2(0,+\infty;H^1(0,1))} + \sup_{0\leq t<+\infty} \|u(\cdot,t)\|_{L^2(0,1)} \leq C\|\overrightarrow{h}\|_{\mathcal{G}_{0,+\infty}}$$

and

$$\sup_{x\in[0,1]} \|u_x(x,\cdot)\|_{L^2(0,+\infty)} \leq C\|\overrightarrow{h}\|_{\mathcal{G}_{0,+\infty}}.$$

Now, let's discuss the Kato-smoothing property of the eq.(2.1). we extend ϕ which is defined on the interval $[0,1]$ to the whole line $\mathbb{R}$ by zero and let ϕ^* denote it. Then, we have

$$\tilde{v}(x,t) = \tilde{W}(t)\phi^*,$$

where the semigroup $\tilde{W}(t)$ is generated by the linear operator $\tilde{A}$, which is defined by

$$\tilde{A}f = -f' + f'' - f''', \quad \text{for} \quad f \in \mathbb{D}(\tilde{A}) = H^3(\mathbb{R}).$$

We also assume that v_b is the following problem's solution

$$\begin{cases} v_t + v_x - v_{xx} + v_{xxx} = 0, \quad v(x,0) = 0, \quad \text{for} \quad x \in [0,1], \\ v(0,t) = \tilde{v}(0,t) = \xi_1(t), \\ v(1,t) = \tilde{v}(1,t) = \xi_2(t), \\ v_x(1,t) = \tilde{v}_x(1,t) = \xi_3(t) \quad \text{for} \quad t \geq 0 \end{cases} \tag{2.24}$$

As what we did in (2.16), we have $v_b = W_b(t)\overrightarrow{\xi}$. So according to the discussion above, we have the following result.

Proposition 2.10 — We can write the solution of eq. (2.1) in another form as

$$u(x,t) = W_0(t)\phi = \tilde{W}(t)\phi^* - W_b(t)\overrightarrow{\xi},$$

for any $\phi \in L^2(0,1)$. Where ϕ^* and $\overrightarrow{\xi} = (\xi_1, \xi_2, \xi_3)$ are defined as above.

In order to have more precise estimate of the solution $u(t)$ of eq. (2.1), the following Lemma will be used.

Lemma 2.11 — There exists a constant C, such that for any $\phi^* \in L^2(\mathbb{R})$, the solution $\tilde{v}(x,t) = \tilde{W}(t)\phi^*$ satisfies the following estimate

$$\sup_{x\in\mathbb{R}} \|\tilde{v}(x,\cdot)\|_{L^2(0,+\infty)} \le C\|\phi^*\|_{H^{-1}(\mathbb{R})}, \tag{2.25}$$

with

$$\tilde{v}_x \in C_b(\mathbb{R}; L^2_t(0,+\infty)) \quad \text{and} \quad \sup_{x\in\mathbb{R}} \|\tilde{v}_x(x,\cdot)\|_{L^2(0,+\infty)} \le C\|\phi^*\|_{L^2(\mathbb{R})}, \tag{2.26}$$

PROOF : We first solute the following linear equation

$$\tilde{v}_t + \tilde{v}_x - \tilde{v}_{xx} + \tilde{v}_{xxx} = 0, \quad \tilde{v}(x,0) = \phi^*(x), \tag{2.27}$$

where $x \in (-\infty, +\infty)$, $\quad t \ge 0$. Using Fourier transform on x, we can have the solution of the eq.(2.27) as follows

$$\tilde{v}(x,t) = K(x,t) * \phi^*(x), \quad \text{where} \quad \mathbb{F}[K(x,t)] = \hat{K}(\xi,t) = e^{(i\xi^3 - \xi^2 - i\xi)t},$$

and

$$\begin{aligned}\mathbb{F}[\tilde{v}(x,t)] &= \hat{\tilde{v}}(\xi,t)\\ &= \mathbb{F}(\phi^*)\cdot\mathbb{F}[K(x,t)]\\ &= \hat{\phi}^*(\xi)\cdot e^{(i\xi^3-\xi^2-i\xi)t}.\end{aligned}$$

Then, we have the following estimate

$$\begin{aligned}\int_0^{+\infty} |\tilde{v}(x,t)|^2 dt &= \int_0^{+\infty} \left| \int_{-\infty}^{+\infty} e^{ix\xi}\hat{\tilde{v}}(\xi,t)d\xi \right|^2 dt\\ &= \int_0^{+\infty} \left| \int_{-\infty}^{+\infty} e^{it(\xi^3-\xi)}e^{-\xi^2 t}e^{ix\xi}\hat{\phi}^*(\xi)d\xi \right|^2 dt\end{aligned}$$

we use the variable substitution with $\xi = \psi(\eta)$, where ξ is the solution of the equation $\xi^3 - \xi = \eta$ and the generalized Minkowski inequality, then, we have the following estimate

$$\begin{aligned}
\int_0^{+\infty} \left| \int_{-\infty}^{+\infty} e^{it(\xi^3-\xi)} e^{-\xi^2 t} e^{ix\xi} \hat{\phi}^*(\xi) d\xi \right|^2 dt &= \int_0^{+\infty} \left| \int_{-\infty}^{+\infty} e^{it\eta} e^{-\psi^2(\eta)t} e^{ix\psi(\eta)} \hat{\phi}^*(\psi(\eta))\psi'(\eta) d\eta \right|^2 dt \\
&\le \int_{-\infty}^{+\infty} \left| \int_0^{+\infty} e^{it\eta} e^{-\psi^2(\eta)t} e^{ix\psi(\eta)} \hat{\phi}^*(\psi(\eta))\psi'(\eta) dt \right|^2 d\eta \\
&\le \int_{-\infty}^{+\infty} \left| \int_0^{+\infty} e^{it\eta} e^{ix\psi(\eta)} \hat{\phi}^*(\psi(\eta))\psi'(\eta) dt \right|^2 d\eta \\
&= \int_{-\infty}^{+\infty} \left| \mathbb{F}^{-1} \left(e^{ix\psi(\eta)} \hat{\phi}^*(\psi(\eta))\psi'(\eta) \right) \right|^2 d\eta \\
&= \int_{-\infty}^{+\infty} |e^{ix\psi(\eta)} \hat{\phi}^*(\psi(\eta))\psi'(\eta)|^2 d\eta \\
&= \int_{-\infty}^{+\infty} |\hat{\phi}^*(\psi(\eta))\psi'(\eta)|^2 d\eta \\
&= \int_{-\infty}^{+\infty} |\hat{\phi}^*(\xi)|^2 \left| \frac{\partial \xi}{\partial \eta} \right| d\xi \\
&\le C \int_{-\infty}^{+\infty} |\hat{\phi}^*(\xi)|^2 (1+\xi^2)^{-1} d\xi \\
&= C \|\phi^*\|^2_{H^{-1}(\mathbb{R})}.
\end{aligned}$$

Using the same strategy we can prove that

$$\begin{aligned}
\int_0^{+\infty} |\partial_x \tilde{v}(x,t)|^2 dt &= \int_0^{+\infty} \left| \int_{-\infty}^{+\infty} e^{ix\xi} i\xi \hat{\tilde{v}}(\xi,t) d\xi \right|^2 dt \\
&\le \int_{-\infty}^{+\infty} |\hat{\phi^*}(\xi)|^2 \xi^2 \left| \frac{\partial \xi}{\partial \eta} \right| d\xi \\
&\le C \int_{-\infty}^{+\infty} |\hat{\phi^*}(\xi)|^2 d\xi \\
&= C \|\phi^*\|^2_{L^2(\mathbb{R})}.
\end{aligned}$$

So, we have proved (2.25). Using Fatou's lemma as what we applied in the proof of Proposition 2.6, we can have the continuity of $\partial_x \tilde{v}(x,t)$ and we have also proved (2.26). The proof of Lemma 2.11 was finished. □

Using Theorem 2.9, Proposition 2.10 and Lemma 2.11, we can have the following two propositions directly.

Proposition 2.12 — For any $\phi(x) \in L^2(0,1)$, the solution $u(x,t)$ of eq. (2.1) satisfies the following properties

$$u_x \in \quad C_{b,x}([0,1]; L_t^2(0,+\infty)),$$

and we have the following estimate: there exists a constant C such that

$$\sup_{x\in[0,1]} \|u_x(x,\cdot)\|_{L^2(0,+\infty)} \le C\|\phi\|_{L^2(0,1)}.$$

Proposition 2.13 — For any $T > 0$ be given, then there exists a constant C such that the solution $u(x,t)$ of eq. (2.5) satisfies the following estimate

$$\sup_{x\in[0,1]} \|u_x(x,\cdot)\|_{L_t^2(0,T)} \le C \int_0^T \|f(\cdot,\tau)\|_{L^2(0,1)} d\tau.$$

3. Nonlinear Problem

In section 2, we have discussed many kinds of estimates of the linear problem and with these estimates we can begin our discussion about the full nonlinear IBVP problem as follow

$$\begin{cases} u_t + u_x - u_{xx} + u_{xxx} + uu_x = 0, \quad u(x,0) = \phi(x), \quad \text{for} \quad x \in [0,1], \\ u(0,t) = h_1(t), \quad u(1,t) = h_2(t), \quad u_x(1,t) = h_3(t), \quad \text{for} \quad t \ge 0 \end{cases} \tag{3.1}$$

Our strategy to get the well-posedness of the nonlinear problem eq.(3.1) contains two parts. First, we will discuss the local well-posedness of the nonlinear problem eq.(3.1) and then, with the help of the local well-posedness results, we will show the global well-posedness of the eq.(3.1). Before our discussion, we give the definition of a new space and it's norm which will be needed in the following discussion. For any $T > 0$ and $s \ge 0$ be given, let

$$K_{s,T} = \{u \in C([0,T]; H^s(0,1)) \cap L^2([0,T]; H^{s+1}(0,1)), \quad \text{and} \quad u_x \in C([0,1]; L^2(0,T))\},$$

and we define it's norm on the space $K_{s,T}$ as

$$\|u\|_{K_{s,T}} = \|u\|_{C([0,T];H^s(0,1))} + \|u\|_{L^2([0,T];H^{s+1}(0,1))} + \|u_x\|_{C([0,1];L^2(0,T))}$$

for any $u \in K_{s,T}$ accordingly. Following the similar arguments as that in [10], we have the result that the space $K_{s,T}$ has the following properties of Lemma 3.1 and Proposition 3.2 which shows the local well-posedness result of eq.(3.1) in the space $D_{0,T}$.

Lemma 3.1 — For any $s \geq 0$ and $T > 0$ be given, there exists a constant C such that

$$\int_0^T \|\partial_x(u(\cdot,t)v(\cdot,t))\|_{H^s(0,1)}dt \leq C(T^{\frac{1}{2}} + T^{\frac{1}{3}})\|u\|_{K_{s,T}}\|v\|_{K_{s,T}}, \quad \text{for} \quad \text{any} \quad u, v \in K_{s,T}.$$

Proposition 3.2 — For any $T > 0$ and $(\phi, \overrightarrow{h}) \in D_{0,T}$ be given with $\overrightarrow{h} = (h_1, h_2, h_3)$, there exists a $T^* \in (0, T]$ which only depends on $\|(\phi, \overrightarrow{h})\|_{D_{0,T}}$, such that eq.(3.1) admits a unique solution $u(x,t) \in K_{0,T^*}$. Moreover, for any $T' < T^*$, there is a neighbourhood U of $(\phi, \overrightarrow{h})$ such that the eq.(1.1) admits a unique solution in the space $K_{0,T'}$ for any $(\psi, \overrightarrow{h_1}) \in U$ and the corresponding solution map from U to $K_{0,T'}$ is Lipschitz continuous.

For the following linear system with initial value, boundary conditions and nonhomogeneous term,

$$\begin{cases} u_t + u_x - u_{xx} + u_{xxx} = f, \quad u(x,0) = \phi(x), \quad \text{for} \quad x \in [0,1], \\ u(0,t) = h_1(t), \quad u(1,t) = h_2(t), \quad u_x(1,t) = h_3(t), \quad \text{for} \quad t \geq 0 \end{cases} \tag{3.2}$$

we have the following estimates

Proposition 3.3 — Let $T > 0$, $0 \leq s \leq 3$, $f \in W^{\frac{s}{3},1}([0,T]; H^{\frac{s}{3}}(0,1))$ and $(\phi, \overrightarrow{h}) \in D_{s,T}$ be given and they satisfy the $s-compatibility$ conditions. Then there exists a unique solution of eq. (3.2) and a constant C such that

$$\|u(x,t)\|_{K_{s,T}} \leq C(\|(\phi, \overrightarrow{h})\|_{D_{s,T}} + \|f\|_{W_t^{\frac{s}{3},1}(0,T;H^{\frac{s}{3}}(0,1))}), \tag{3.3}$$

where the constant C only depends on $(\phi, \overrightarrow{h})$ and f. If $s = 3$, then $u_t \in K_{0,T}$ with

$$\|u_t(x,t)\|_{K_{0,T}} \leq C(\|(\phi, \overrightarrow{h})\|_{D_{3,T}} + \|f\|_{W_t^{1,1}(0,T;H^1(0,1))}).$$

Remark 3.4 : The condition $f \in W^{\frac{s}{3},1}([0,T]; H^{\frac{s}{3}}(0,1))$ here is different from the condition $f \in W^{\frac{s}{3},1}([0,T]; L^2(0,1))$ which is presented in [10]. The authors find that the condition in Proposition 3.3 of f here is necessary and we will find its importance in the proof as follows.

PROOF : we will only discuss the case $s = 0$ and $s = 3$, since the other case $0 < s < 3$ can be got by the interpolation theory. Using the linear estimates we have got in Section 2, for any $(\phi, \overrightarrow{h}) \in D_{0,T}$ and $f \in L^1(0,T; L^2(0,1))$, we can have the following result directly

$$\|u\|_{K_{0,T}} \leq C(\|(\phi, \overrightarrow{h})\|_{D_{0,T}} + \|f\|_{L^1(0,T;L^2(0,1))}), \tag{3.4}$$

where C is a constant which only depends on $(\phi, \overrightarrow{h})$ and f. So, we have proved the case $s = 0$. For the case $s = 3$, we let u be the solution of eq.(3.2), $v = u_t$ and v be the solution of the following equation accordingly.

$$\begin{cases} v_t + v_x - v_{xx} + v_{xxx} = f_t, \quad v(x,0) = f(x,0) - \phi'''(x) + \phi''(x) - \phi'(x), \quad \text{for} \quad x \in [0,1], \\ v(0,t) = h_1'(t), \quad v(1,t) = h_2'(t), \quad v_x(1,t) = h_3'(t), \quad \text{for} \quad t \geq 0 \end{cases} \tag{3.5}$$

According to the estimate (3.4), we have the estimate of the solution $v(x,t)$ of eq. (3.5) directly as follows

$$\begin{aligned} \|v\|_{K_{0,T}} &\leq C(\|(f(x,0) - \phi'''(x) + \phi''(x) - \phi'(x), \vec{h'})\|_{D_{0,T}} + \|f_t\|_{L^1(0,T;L^2(0,1))}) \\ &\leq C(\|f(\cdot,0)\|_{H^1(0,1)} + \|(\phi, \vec{h})\|_{D_{3,T}} + \|f\|_{W^{1,1}([0,T];L^2(0,1))}) \\ &\leq C(\|(\phi, \vec{h})\|_{D_{3,T}} + \|f\|_{W^{1,1}([0,T];H^1(0,1))}). \end{aligned}$$

With the definition of $\|u\|_{K_{3,T}}$, we have the following estimates

$$\begin{aligned} \|u\|_{K_{3,T}} &= \|u\|_{C([0,T];H^3(0,1))} + \|u\|_{L^2([0,T];H^4(0,1))} + \|u_x\|_{C([0,1];L^2(0,T))} \\ &\leq \|u\|_{C([0,T];L^2(0,1))} + \|u_x\|_{C([0,T];L^2(0,1))} + \|u_{xx}\|_{C([0,T];L^2(0,1))} + \|u_{xxx}\|_{C([0,T];L^2(0,1))} \\ &\quad + \|u\|_{L^2([0,T];H^1(0,1))} + \|u_x\|_{L^2([0,T];H^1(0,1))} + \|u_{xx}\|_{L^2([0,T];H^1(0,1))} \\ &\quad + \|u_{xxx}\|_{L^2([0,T];H^1(0,1))} + \|u_x\|_{C([0,1];L^2(0,T))}. \end{aligned}$$

For

$$\begin{aligned} \|u_x\|_{L^2([0,T];H^1(0,1))} &\leq C(\|(\phi, \vec{h})\|_{D_{3,T}} + \|f\|_{W^{1,1}([0,T];H^1(0,1))}) \\ \|v_x\|_{L^2([0,T];H^1(0,1))} &= \|u_{tx}\|_{L^2([0,T];H^1(0,1))} \\ &\leq C(\|(\phi, \vec{h})\|_{D_{3,T}} + \|f\|_{W^{1,1}([0,T];H^1(0,1))}), \end{aligned}$$

we have

$$\begin{aligned} \|u_{xx}\|_{C([0,T];L^2(0,1))} &\leq \|u_x\|_{C([0,T];H^1(0,1))} \\ &\leq C\|u_x\|_{H^1([0,T],H^1(0,1))} \\ &\leq C(\|(\phi, \vec{h})\|_{D_{3,T}} + \|f\|_{W^{1,1}([0,T];H^1(0,1))}). \end{aligned}$$

Because u is the solution of eq. (3.2), we have

$$\begin{aligned} u_{xxx} &= -u_t - u_x + u_{xx} - f \\ &= -v - u_x + u_{xx} - f, \end{aligned}$$

and

$$\begin{aligned}\|u_{xxx}\|_{C([0,T];L^2(0,1))} &= \|-v-u_x+u_{xx}-f\|_{C([0,T];L^2(0,1))}\\ &\le \|v\|_{C([0,T];L^2(0,1))}+\|u_x\|_{C([0,T];L^2(0,1))}+\|u_{xx}\|_{C([0,T];L^2(0,1))}\\ &+\|f\|_{C([0,T];L^2(0,1))}\\ &\le C(\|(\phi,\overrightarrow{h})\|_{D_{3,T}}+\|f\|_{W^{1,1}([0,T];H^1(0,1))}).\end{aligned}$$

So we have proved

$$\|u\|_{C([0,T];H^3(0,1))} \le C(\|(\phi,\overrightarrow{h})\|_{D_{3,T}}+\|f\|_{W^{1,1}([0,T];H^1(0,1))}).$$

For the term $\|u\|_{L^2([0,T];H^4(0,1))}$, with the same strategy used as above, we only need the following estimate

$$\begin{aligned}\|f\|_{L^2([0,T];H^1(0,1))} &\le C\|f\|_{C([0,T];H^1(0,1))}\\ &\le C\|f\|_{W^{1,1}([0,T];H^1(0,1))}),\end{aligned}$$

and we can have

$$\|u\|_{L^2([0,T];H^4(0,1))} \le C(\|(\phi,\overrightarrow{h})\|_{D_{3,T}}+\|f\|_{W^{1,1}([0,T];H^1(0,1))}).$$

Now, we have proved the case $s=3$. The proof was finished. □

Since we have proved the local well-posedness of the eq.(3.2) in the case $0\le s\le 3$, we can use the method of induction to give the result in the case $0\le s\le +\infty$. With the similar discussion to that presented in [10], we can have the following result directly.

***Theorem* 3.5** — *For any $T>0$, $s\ge 0$ be given, and any $(\phi,\overrightarrow{h})\in D_{s,T}$ satisfying the s-compatibility conditions. Then there exists a $T^*\in[0,T]$ which depends only on $\|(\phi,\overrightarrow{h})\|_{D_{s,T}}$, such that there exists a unique solution $u\in K_{s,T^*}$ of the eq.(3.1) with $\partial_t^j u\in K_{s-3j,T^*}$ for $j=0,1,2,...,[\frac{s}{3}-1],[\frac{s}{3}]$.*

In Theorem 3.5, we know that $T^*\le T$ and we also want to know whether it is possible that $T^*=T$ or not for any $(\phi,\overrightarrow{h})\in D_{s,T}$. We say that the IBVP(3.2) is globally well-posed if $T^*=T$ for any $(\phi,\overrightarrow{h})\in D_{s,T}$. We first give an very useful estimate of the solution u of eq.(3.2) in the following lemma and the details of the proof is similar to that presented in [10].

Lemma 3.6 — For any $T>0$, $s\ge 0$ be given, and any $(\phi,\overrightarrow{h})\in D_{s,T}$ satisfying the s-compatibility conditions, then there exists a continuous and nondecreasing function $\gamma_s: R^+\to R^+$ such that

$$\sup_{0\le t\le T}\|u(\cdot,t)\|_{H^s(0,1)} \le \gamma_s(\|(\phi,\overrightarrow{h})\|_{D_{s,T}}) \tag{3.6}$$

for any smooth solution u of eq. (3.1).

Using Lemma 3.6, the real interpolation theory in [9] and with the similar discussion in [10], we can have the global well-posedness of eq.(3.1) as follow:

***Theorem* 3.7** — *For any $T > 0$, $s \geq 0$ be given, and any $(\phi, \vec{h}) \in D_{s,T}$ satisfying the s-compatibility conditions. Then there exists a unique solution $u \in K_{s,T}$ of the eq.(3.1) with $\partial_t^j u \in K_{s-3j,T}$ for $j = 0, 1, 2, ..., [\frac{s}{3} - 1], [\frac{s}{3}]$.*

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