Indian J. Pure Appl. Math., **48**(1): 133-146, March 2017

DOI: 10.1007/s13226-017-0215-x

MULTIPLE SOLUTIONS FOR NON-HOMOGENEOUS DEGENERATE SCHRÖDINGER EQUATIONS IN CONE SOBOLEV SPACES

Mohsen Alimohammady*, Ali Asghar Jafari* and Morteza Koozehgar Kalleji**

**Department of Mathematics, Faculty of Mathematical Sciences University of Mazandaran, Babolsar* 47416-1468, *Iran*

***Department of Mathematics, Faculty of Science, Arak University, Arak* 38156-8-8349, *Iran*

e-mails: amohsen@umz.ac.ir; aa.jafari@stu.umz.ac.ir; m.kalleji@yahoo.com

(*Received* 14 *January* 2016; *after final revision* 7 *June* 2016; *accepted* 27 *June* 2016)

The present paper deals with the study of semilinear and non-homogeneous Schrödinger equations on a manifold with conical singularity. We provide a suitable constant by Sobolev embedding constant and for $p \in (2, 2^*)$ with respect to non-homogeneous term $g(x) \in L_2^{\frac{n}{2}}(\mathbb{B})$, which helps to find multiple solutions of our problem. More precisely, we prove the existence of two solutions to the problem 1.1 with negative and positive energy in cone Sobolev space $\mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})$. Finally, we consider $p = 2$ and we prove the existence and uniqueness of Fuchsian-Poisson problem.

Key words : Semilinear elliptic equation; non-homogeneous Schrödinger equation, degenerate elliptic equations; con Sobolev space.

1. Introduction

In this paper, we show the existence of at least two weak solutions for semilinear and non-homogeneous Schrödinger equations on a manifold with conical degeneration as follows

$$\begin{cases} -\Delta_{\mathbb{B}} u + u = |u|^{p-2}u + g(x) & x \in int\mathbb{B}, \\ u = 0 & x \in \partial\mathbb{B} \end{cases} \tag{1.1}$$

where $2 < p < \frac{2n}{n-2} = 2^*$ is the critical cone Sobolev exponents and $g(x) = g(|x|) \in L_2^{\frac{n}{2}}(\mathbb{B})$, where $|x| = (x_1^2 + x_2^2 + +...x_n^2)^{\frac{1}{2}}$, which is considered by the authors in [4]. Here the domain $\mathbb{B}$ is $[0,1) \times X$ that X is an $(n-1)$-dimensional closed compact manifold, which is regarded as the local model near the conical points on manifolds with conical singularities, and $\partial\mathbb{B} = \{0\} \times X$. Moreover, the operator $\Delta_{\mathbb{B}}$ in 1.1 is defined by $(x_1\partial_{x_1})^2 + \partial_{x_2}^2 + ... + \partial_{x_n}^2$, which is an elliptic operator with totally characteristic degeneracy on the boundary $x_1 = 0$, we also call it Fuchsian type Laplace operator, and the corresponding gradient operator by $\nabla_{\mathbb{B}} := (x_1\partial_{x_1}, \partial_{x_2}, ..., \partial_{x_n})$. We recall more details about the manifolds with conical singularities in Section 2.

The problem 1.1 is related to the study of non-homogeneous and semilinear Schrödinger equation independent of time for a quantum particle in mechanics. If $g(x) \equiv 0$, the problem 1.1 reduces to problem (1.1) with potential function $V(x) \equiv 1$ in [1]. In the homogeneous case that is $g(x) \equiv 0$, and without potential term u in the problem 1.1 in the present manuscript one can reduce to the following Dirichlet problem,

$$\begin{cases} -\Delta_{\mathbb{B}} u = |u|^{p-2}u & x \in int\mathbb{B}, \\ u = 0 & x \in \partial\mathbb{B} \end{cases} \tag{1.2}$$

which the authors in [2] obtained a nontrivial weak solution for it.

This paper motivated by the paper [7] that the authors proved the existence of two weak solutions of the nonhomogeneous Schrödinger-Maxwell system in the presence Laplace operator Δ on $\mathbb{R}^n$ instead of Fuchsian-Laplace operator $\Delta_{\mathbb{B}}$ on conical manifold $\mathbb{B}$.

In this paper, we shall find multiple solutions for the problem 1.1 in cone Sobolev space $\mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})$ which will be given in the next section.

Corresponding to the problem 1.1, we define the energy functional $I : \mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B}) \longrightarrow \mathbb{R}$ by

$$\begin{aligned} I(u) &= \frac{1}{2}\int\limits_{\mathbb{B}} |\nabla_{\mathbb{B}} u|^2 \frac{dx_1}{x_1} dx' \\ &+ \frac{1}{2}\int\limits_{\mathbb{B}} u^2 \frac{dx_1}{x_1} dx' - \frac{1}{p}\int\limits_{\mathbb{B}} |u|^p \frac{dx_1}{x_1} dx' - \int\limits_{\mathbb{B}} g(x) u \frac{dx_1}{x_1} dx'. \end{aligned} \tag{1.3}$$

It is well known that $I \in C^1(\mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B}), \mathbb{R})$. Furthermore, by definition of the energy functional 1.3, one can get

$$\begin{aligned} \langle I'(u), v\rangle &= \int_{\mathbb{B}} \nabla_{\mathbb{B}} u \nabla_{\mathbb{B}} v \frac{dx_1}{x_1} dx' \\ &+ \int_{\mathbb{B}} uv \frac{dx_1}{x_1} dx' - \int_{\mathbb{B}} |u|^{p-2} uv \frac{dx_1}{x_1} dx' - \int_{\mathbb{B}} g(x) v \frac{dx_1}{x_1} dx'. \end{aligned} \tag{1.4}$$

Definition 1.1 — We say that u is a weak solution of problem 1.1 on $int\mathbb{B}$ if it satisfies in problem 1.1 in the sense of distribution that is

$$(\nabla_{\mathbb{B}} u, \nabla_{\mathbb{B}} v)_2 + (u, v)_2 = (|u|^{p-1}u, v)_2 + (g(x), v)_2 \qquad \forall v \in \mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B}).$$

For $p \in (2, 2^*)$, we set

$$C_p^* = \left(\frac{p-2}{2(p-1)}\right)\left[\frac{p}{2(p-1)C^P}\right]^{\frac{1}{p-2}}. \tag{1.5}$$

Now, we are in a position which we can express our main results in this paper.

***Theorem* 1.2** — *If $p \in (2, 2^*)$, and $g(x) \in C^1(\mathbb{B}) \bigcap L_2^{\frac{n}{2}}(\mathbb{B})$ is a nonnegative function satisfying the following conditions:*

(G1): $g(x) = g(|x|) \neq 0$,

(G2): $(\nabla_{\mathbb{B}} g(x), g(x))_2 \in L_2^{\frac{n}{2}}(\mathbb{B})$,

(G3): $\|g\|_{L_2^{\frac{n}{2}}(\mathbb{B})} < C_p^*$,

where C_p^* given by 1.5. Then the problem 1.1 has at least two nontrivial weak solutions u_0 and u_1 such that $I(u_0) < 0 < I(u_1)$.

In the last section we consider the case $p = 2$, so we deal with Poisson equation on the manifold $\mathbb{B}$. In this case, we present a unique weak solution of the problem 1.1.

***Theorem* 1.3** — *If $p = 2$ the problem 1.1 has exactly one solution.*

In Section 2, we will introduce the manifolds with conical singularities, the stretched manifold associated to the conic manifold, cone Sobolev spaces and the corresponding properties of them. In Section 3, we use the Ekeland's variational principle to obtain a weak solution with negative energy. In Section 4, we show that the problem 1.1 admits a positive energy solution. To end this, we apply mountain pass Theorem 4.1 and consider the appropriate conditions on the non-homogeneous term of the problem 1.1. In the last section, we study the problem 1.1 in case $p = 2$. To compare with the case $2 < p < 2^*$ we prove that the problem 1.1 has exactly one solution in case $p = 2$.

2. Cone Sobolev Spaces

In this section, we recall some definitions and notations from Sobolev spaces on manifolds with conical singularities. We refer enthusiastic reader to [2, 3, 5, 9, 10] and the references therein.

Let B be a manifold with conical singularities $x_1, ..., x_N$. First, for simplicity let us consider the case $N = 1$ and set $x = x_1$. If X is C^∞ closed compact manifold, the cone $X^\Delta := \frac{\bar{\mathbb{R}}_+ \times X}{\{0\} \times X}$ is an example of such an B. In this case the conical singularity is represented by $\{0\} \times X$ in the quotient space. In general, B is locally near x modelled on such a cone. More precisely, $B - \{x\}$ is smooth, and we have a singular chart

$$\chi : G \to X^{\Delta}$$

for some neighborhood G of x in M and a smooth manifold $X = X(x)$ where $\chi(x)$ is equal to the vertex of X^{Δ}, $\varphi = \chi|_{G-\{x\}} : G - \{x\} \to X^{\wedge} := \mathbb{R}_+ \times X$ is a diffeomorphism [2]. More precisely, a finite dimensional manifold B with conical singularities, is a topological space with a finite subset $B_0 = \{x_1, ..., x_N\} \subset B$ of conical singularities, which has the following two properties:

(a) $B - B_0$ is a C^{∞} manifold.

(b) any $x \in B_0$ has an open neighborhood G in B, such that there is a homeomorphism $\chi : G \to X^{\Delta}$ for some closed compact C^{∞} manifold $X = X(x)$ and φ restricts a diffeomorphism $\varphi' : G - \{x\} \to X^{\wedge}$.

For such a manifold, let $n \geq 2$ and $X \subset S^{n-1}$ be a bounded open set in the unit sphere of $\mathbb{R}^n_x$. The set $B := \left\{ x \in \mathbb{R}^n - \{0\} \quad ; \quad \frac{x}{|x|} \in X \right\} \cup \{0\}$ is an infinite cone with the base X and the critical point $\{0\}$. Using the polar coordinates, one can get a description of $B - \{0\}$ in the form $X^{\wedge} = \mathbb{R}_+ \times X$, which is called the open stretched cone with the base X, and $\{0\} \times X$ is the boundary of $X^{\wedge}$.

Now, we assume that the manifold B is paracompact and of dimension n. By this assumption we can define the stretched manifold associated with B. Let $\mathbb{B}$ be a C^{∞} manifold with compact C^{∞} boundary $\partial\mathbb{B} \cong \bigcup_{x \in B_0} X(x)$ for which there exists a diffeomorphism $B - B_0 \cong \mathbb{B} - \partial\mathbb{B} := int\mathbb{B}$, the restriction of which to $G_1 - B_0 \cong U_1 - \partial\mathbb{B}$ for an open neighborhood $G_1 \subset B$ near the points of B_0 and a collar neighborhood $U_1 \subset \mathbb{B}$ with $U_1 \cong \bigcup_{x \in B_0} \{[0, 1) \times X(x)\}$. The typical differential operators on a manifold with conical singularities, called Fuchs type, are operators that are in a neighborhood of $x_1 = 0$ of the following form

$$A = x_1^{-m} \sum_{k=0}^{m} a_k(x_1)(-x_1 \partial_{x_1})^k$$

with $(x_1, x) \in X^{\wedge}$ and $a_k(x_1) \in C^{\infty}(\bar{\mathbb{R}}_+, \mathit{Diff}^{m-k}(X))$ [9]. The differential $x_1 \partial x_1$ in Fuchs type operators provokes us to apply the Mellin transform $M : C_0^{\infty}(\mathbb{R}_+) \to \mathcal{A}(\mathbb{C})$, for $u(x_1) \in C_0^{\infty}(\mathbb{R}_+)$, $z \in \mathbb{C}$, defined as

$$Mu(z) := \int_0^{+\infty} x_1^z u(x_1) \frac{dx_1}{x_1}, \tag{2.1}$$

where $\mathcal{A}(\mathbb{C})$ denotes the space of entier functions.

One can find further details on Fuchs type operators and all implications and definitions of the cone Sobolev spaces in [2, 3, 9, 10]. We use the so-called weighted Meline transform

$$M_\gamma u := Mu_{|\Gamma_{\frac{1}{2}-\gamma}} = \int_0^{+\infty} x_1^{\frac{1}{2}-\gamma+i\tau} u(x_1)\frac{dx_1}{x_1},$$

where $\Gamma_\beta := \{z \in \mathbb{C} \quad ; \quad Rez = \beta\}$. The inverse weighted Meline transform is defined as

$$(M_\gamma^{-1} g)(x_1) = \frac{1}{2i\pi}\int_{\Gamma_{\frac{1}{2}}} x_1^{-z} g(z)dz.$$

In order to define cone Sobolev spaces on the stretched manifolds, at the first we introduce the weighted Sobolev spaces on $\mathbb{R}^n$ and then by using of partition unity, we introduce suitable weighted cone Sobolev space on the stretched manifold $\mathbb{B}$.

Definition 2.1 — For $(x_1, x') \in \mathbb{R}_+ \times \mathbb{R}^{n-1} = \mathbb{R}_+^n$ we say that $u(x_1, x') \in L_p(\mathbb{R}_+^n, \frac{dx_1}{x_1}dx')$ if

$$\|u\|_{L_p} = \left(\int_{\mathbb{R}_+}\int_{\mathbb{R}^{n-1}} x_1^n |u(x_1, x')|^p \frac{dx_1}{x_1}dx'\right)^{\frac{1}{p}} < \infty.$$

The weighted L_p-spaces with weight data $\gamma \in \mathbb{R}$ is denoted by $L_p^\gamma(\mathbb{R}_+^n, \frac{dx_1}{x_1}dx')$. In fact, if $u(x_1, x') \in L_p^\gamma(\mathbb{R}_+^n, \frac{dx_1}{x_1}dx')$, then $x_1^{-\gamma}u(x_1, x') \in L_p(\mathbb{R}_+^n, \frac{dx_1}{x_1}dx')$, and

$$\|u\|_{L^p}^\gamma = \left(\int_{\mathbb{R}_+}\int_{\mathbb{R}^{n-1}} x_1^n |x_1^{-\gamma}u(x_1, x')|^p \frac{dx_1}{x_1}dx'\right)^{\frac{1}{p}} < \infty.$$

Now, we can define the weighted $p-$Sobolev spaces for $1 \le p < \infty$.

Definition 2.2 — For $m \in \mathbb{N}$, $\gamma \in \mathbb{R}$ and $1 \le p < \infty$, the spaces

$$\mathcal{H}_p^{m,\gamma}(\mathbb{R}_+^n) := \left\{u \in \mathcal{D}'(\mathbb{R}_+^n) \quad ; \quad x_1^{\frac{n}{p}-\gamma}(x_1\partial_{x_1})^\alpha \partial_{x'}^\beta u \in L_p(\mathbb{R}_+^n, \frac{dx_1}{x_1}dx')\right\}, \tag{2.2}$$

for any $\alpha \in \mathbb{N}$, $\beta \in \mathbb{N}^{n-1}$ and $|\alpha| + |\beta| \le m$. In other words, if $u(x_1, x) \in \mathcal{H}_p^{m,\gamma}(\mathbb{R}_+^n)$, then $(x_1\partial_{x_1})^\alpha \partial_{x'}^\beta u \in L_p^\gamma(\mathbb{R}_+^n, \frac{dx_1}{x_1}dx')$.

Hence, $\mathcal{H}_p^{m,\gamma}(\mathbb{R}_+^n)$ is a Banach space with norm

$$\|u\|_{\mathcal{H}_p^{m,\gamma}(\mathbb{R}_+^n)} = \sum_{|\alpha|+|\beta|\le m}\left(\int\int_{\mathbb{R}_+^n} x_1^n |x_1^{-\gamma}(x_1\partial_{x_1})^\alpha \partial_{x'}^\beta u(x_1, x')|^p \frac{dx_1}{x_1}dx'\right)^{\frac{1}{p}}.$$

Let X be a closed compact C^∞ manifold, and $\mathcal{U} = \{U_1, ..., U_N\}$ an open covering of X by coordinate neighborhoods. If we fix a subordinate partition of unity $\{\varphi_1, ..., \varphi_N\}$ and charts

$\chi_j : U_j \to \mathbb{R}^{n-1}$, $j = 1, ..., N$. Then we say that $u \in \mathcal{H}_p^{m,\gamma}(X^\wedge)$ if and only if $u \in \mathcal{D}'(X^\wedge)$ with the norm

$$\|u\|_{\mathcal{H}_p^{m,\gamma}(X^\wedge)} = \left\{ \sum_{j=1}^{N} \|(1 \times \chi_j^*)^{-1} \varphi_j u\|^p_{\mathcal{H}_p^{m,\gamma}(\mathbb{R}_+^n)} \right\}^{\frac{1}{p}} < \infty,$$

where $1 \times \chi_j^* : C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^{n-1}) \to C_0^\infty(\mathbb{R}_+ \times U_j)$ is the pull-back function with respect to $1 \times \chi_j : \mathbb{R}_+ \times U_j \to \mathbb{R}_+ \times \mathbb{R}^{n-1}$. Denote the $\mathcal{H}_{p,0}^{m,\gamma}(X^\wedge)$ as subspace of $\mathcal{H}_p^{m,\gamma}(X^\wedge)$ which is defined as the closure of $C_0^\infty(X^\wedge)$ with respect to the norm $\|.\|_{\mathcal{H}_p^{m,\gamma}(X^\wedge)}$. Now, we have the following definition

Definition 2.3 — Let $\mathbb{B} = [0, 1) \times X$ be the stretched manifold of the manifold B with conical singularity. Then the cone Sobolev space $\mathcal{H}_p^{m,\gamma}(\mathbb{B})$, for $m \in \mathbb{N}, \gamma \in \mathbb{R}$ and $p \in (1, \infty)$, is defined by

$$\mathcal{H}_p^{m,\gamma}(\mathbb{B}) = \{u \in W_{loc}^{m,p}(int\mathbb{B}) \quad ; \quad \omega u \in \mathcal{H}_p^{m,\gamma}(X^\wedge)\},$$

for any cut-off function ω supported by a collar neighborhood of $[0, 1) \times \partial\mathbb{B}$. Moreover, the subspace $\mathcal{H}_{p,0}^{m,\gamma}(\mathbb{B})$ of $\mathcal{H}_p^{m,\gamma}(\mathbb{B})$ is defined by

$$\mathcal{H}_{p,0}^{m,\gamma}(\mathbb{B}) := (\omega)\mathcal{H}_{p,0}^{m,\gamma}(X^\wedge) + (1 - \omega)W_0^{m,p}(int\mathbb{B}),$$

where $W_0^{m,p}(int\mathbb{B})$ denotes the closure of $C_0^\infty(int\mathbb{B})$ in Sobolev space $W^{m,p}(\tilde{X})$ when $\tilde{X}$ is closed compact C^∞manifold of dimension n containing $\mathbb{B}$ as a submanifold with boundary.

Definition 2.4 — Let $\mathbb{B} = [0, 1) \times X$. We say that $u(x) \in L_p^\gamma(\mathbb{B})$ with $1 < p < \infty, \gamma \in \mathbb{R}$, if

$$\|u\|_{L_p^\gamma(\mathbb{B})} = \int_{\mathbb{B}} x_1^n |x_1^{-\gamma} u(x)|^p (\frac{dx_1}{x_1}) dx' < \infty.$$

For $\gamma = \frac{n}{p}$ and $\gamma = \frac{n}{q}$ such that $\frac{1}{p} + \frac{1}{q} = 1$, we have the following Hölders inequality

$$\int_{\mathbb{B}} |u(x)v(x)| \frac{dx_1}{x_1} dx' \leq \left(\int_{\mathbb{B}} |u(x)|^p \frac{dx_1}{x_1} dx' \right)^{\frac{1}{p}} \left(\int_{\mathbb{B}} |v(x)|^q \frac{dx_1}{x_1} dx' \right)^{\frac{1}{q}}. \tag{2.3}$$

In the sequel, for convenience we denote

$$(u, v)_2 = \int_{\mathbb{B}} u(x)v(x) \ \frac{dx_1}{x_1} dx' \quad \text{and} \quad \|u\|_{L_p^{\frac{n}{p}}(\mathbb{B})} = \int_{\mathbb{B}} |u(x)|^p \ \frac{dx_1}{x_1} dx'.$$

Proposition 2.5 — (Poincaré inequality) [2]. Let $\mathbb{B} = [0, 1) \times X$ be a bounded subspace in $\mathbb{R}_+^n$ with $X \subset \mathbb{R}^{n-1}$, $\gamma \in \mathbb{R}$ and $p \in (1, \infty)$. If $u \in \mathcal{H}_p^{1,\gamma}(\mathbb{B})$ then

$$\|u(x)\|_{L_p^\gamma(\mathbb{B})} \leq c\|\nabla_{\mathbb{B}} u(x)\|_{L_p^\gamma(\mathbb{B})} \tag{2.4}$$

where $\nabla_{\mathbb{B}} u = (x_1\partial_{x_1}, \partial_{x_2}, ..., \partial_{x_n})$ and the constant c depending only on $\mathbb{B}$.

Proposition 2.6 [3] — For $2 < p < 2^*$ the embedding $\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B}) \hookrightarrow \mathcal{H}^{0,\frac{n}{p}}_{p,0}(\mathbb{B})$ is compact.

3. A Weak Solution with Negative Energy

In this section, we use the Ekeland's variational principle to show the existence a weak solution u_0 which the energy functional is negative in this point. To express our main result, we need the following lemma.

Lemma 3.1 — Let $p \in (2, 2^*)$ and $\|g\|_{L_2^{\frac{n}{2}}(\mathbb{B})} < C_p^*$ with C_p^* given by 1.5. Then for the energy functional I defined by 1.3, there exist $\alpha > 0$ and $\rho > 0$ such that

$$I(u) \geq \rho > 0$$

for any $\|u\|_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} = \alpha$.

Proof : By cone Sobolev embedding Theorem, we have

$$\begin{aligned} I(u) &\geq \frac{1}{2}\|u\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} - \frac{C^p}{p}\|u\|^p_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} - \|g\|_{L_2^{\frac{n}{2}}(\mathbb{B})}\|u\|_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} \\ &= \|u\|_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}\left(\frac{1}{2}\|u\|_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} - \frac{C^p}{p}\|u\|^{p-1}_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} - \|g\|_{L_2^{\frac{n}{2}}(\mathbb{B})}\right). \end{aligned} \tag{3.1}$$

Set $h(t) = \frac{1}{2}t - \frac{C^P}{p}t^{p-1}$ for $t \geq 0$. By calculations, we see that

$$\max_{t\geq 0} h(t) = h(\alpha) = C_p^*,$$

where $\alpha = (\frac{p}{2C^p(p-1)})^{\frac{1}{p-2}}$. Therefore, it follows from 3.1 that if $\|g\|_{L_2^{\frac{n}{2}}(\mathbb{B})} < C_p^*$, there exists $\rho = \alpha(h(\alpha) - \|g\|_{L_2^{\frac{n}{2}}(\mathbb{B})}) < C_p^*$ such that

$$I(u) \geq \rho > 0,$$

for any $u \in \mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})$ with $\|u\|_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} = \alpha$. □

***Theorem* 3.2** — *Suppose* $p \in (2, 2^*)$, $0 \leq g(x) = g(|x|) \in L_2^{\frac{n}{2}}(\mathbb{B}) - \{0\}$ *and* $\|g\|_{L_2^{\frac{n}{2}}(\mathbb{B})} < C_p^*$. *Then there exists* $u_0 \in \mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})$, *such that*

$$I(u_0) = \inf\left\{I(u) : u \in \mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B}) \text{ and } \|u\|_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} = \alpha\right\} < 0, \tag{3.2}$$

where α *is given by Lemma 3.1. Moreover,* u_0 *is a solution of problem 1.1.*

PROOF : Since $0 \leq g(x) = g(|x|) \in L_2^{\frac{n}{2}}(\mathbb{B}) - \{0\}$ and $g(x) \neq 0$, we can choose a function $v \in \mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})$ such that $\int_{\mathbb{B}} g(x)v(x)\,\frac{dx_1}{x_1}dx' > 0$. Then for $t > 0$ small enough, we have

$$I(tv) = \frac{t^2}{2}\int_{\mathbb{B}} |\nabla_{\mathbb{B}} v|^2 \frac{dx_1}{x_1}dx' + \frac{t^2}{2}\int_{\mathbb{B}} v^2 \frac{dx_1}{x_1}dx' \tag{3.3}$$

$$- \frac{t^p}{p}\int_{\mathbb{B}} |v|^p \frac{dx_1}{x_1}dx' - t\int_{\mathbb{B}} g(x)v\frac{dx_1}{x_1}dx' < 0. \tag{3.4}$$

This shows that

$$c_0 = \inf\Big\{ I(u) \ : \ u \in \bar{B_\alpha}\Big\} < 0,$$

where $\bar{B_\alpha} = \{u \in \mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B}) : \|u\|_{\mathcal{H}_{2,0}^{1,\frac{n}{2}}} \leq \alpha\}$. By Ekeland's variational principle, there exists $\{u_n\} \subset \bar{B_\alpha}$ such that:

(i) $c_0 \leq I(u_n) \leq c_0 + \frac{1}{n}$,

(ii) $I(w) \geq I(u_n) - \frac{1}{n}\|w - u_n\|_{\mathcal{H}_{2,0}^{1,\frac{n}{2}}}$ for all $w \in \bar{B_\alpha}$.

From a standard procedure, see for example [11], we can prove that $\{u_n\}$ is a bounded (PS)-sequence of I. Therefore by compactness of embedding $\mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B}) \hookrightarrow \mathcal{H}_{p,0}^{0,\frac{n}{p}}(\mathbb{B}) = L_p^{\frac{n}{p}}(\mathbb{B})$ for $2 < p < 2^*$, there exists $u_0 \in \mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})$ such that $u_n \to u_0$ strongly in $\mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})$ when $n \to \infty$. Hence $I(u_0) = c_0 < 0$ and $I'(u_0) = 0$.

4. A WEAK SOLUTION WITH POSITIVE ENERGY

This section deals with the existence of weak solution to problem 1.1 with positive energy. To approach this, we will apply the result from the functional analysis. Furthermore, we use the $PS-$sequence property to get our aim.

***Theorem* 4.1** [8] — *Let $(X, \|.\|)$ be a Banach space, $J \subset \mathbb{R}^+$ an interval and $\{I_\mu\}_{\mu\in J}$ a family of C^1-functionals on X of the form*

$$I_\mu(u) = A(u) - \mu B(u), \quad \forall \mu \in J,$$

where $B(u) \geq 0$, $\forall u \in X$ and $B(u) \to +\infty$ or $A(u) \to +\infty$ as $\|u\| \to \infty$.

Assume that there are two points $v_1, v_2 \in X$ such that

$$c(\mu) = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I_\mu(\gamma(t)) > \max\{I_\mu(v_1), I_\mu(v_2)\}, \quad for \ \ \mu \in J,$$

where

$$\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = v_1, \gamma(1) = v_2\}.$$

Then, for almost every $\mu \in J$, there is a sequence $\{v_n\} \subset X$ such that

(i) $\{v_n\}$ is bounded,

(ii) $I_\mu(v_n) \to c(\mu)$,

(iii) $I'_\mu(v_n) \to 0$ in the dual of X.

In order to applying Theorem 4.1 to get a solution to our problem 1.1, we introduce the following approximation problem

$$\begin{cases} -\Delta_{\mathbb{B}} u + u = \mu|u|^{p-2}u + g(x) & x \in int\mathbb{B}, \\ u = 0 & x \in \partial\mathbb{B}, \end{cases} \tag{4.1}$$

where $\mu \in [\frac{1}{2}, 1]$, $p \in (2, 2^*)$ and $g(x) = g(|x|) \in L_2^{\frac{n}{2}}(\mathbb{B}) - \{0\}$.

Let $X = \mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})$ and $J = [\frac{1}{2}, 1]$, and define $I_\mu : X \longrightarrow \mathbb{R}$ by

$$I_\mu(u) = A(u) - \mu B(u),$$

with $A(u) = \frac{1}{2}\int_{\mathbb{B}} |\nabla_{\mathbb{B}} u|^2 \frac{dx_1}{x_1} dx' + \frac{1}{2}\int_{\mathbb{B}} u^2 \frac{dx_1}{x_1} dx' - \int_{\mathbb{B}} g(x) u \frac{dx_1}{x_1} dx'$ and $B(u) = \frac{1}{p}\int_{\mathbb{B}} |u|^p \frac{dx_1}{x_1} dx'$.

Then $\{I_\mu\}_{\mu\in J}$ is a family of C^1-functionals on X, $B(u) \geq 0, \ \ \forall u \in X$ and $A(u) \geq \frac{1}{2}\|u\|^2_{\mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})} - \|g\|_{L_2^{\frac{n}{2}}(\mathbb{B})}\|u\|_{\mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})} \to +\infty$ as $\|u\|_{\mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})} \to \infty$.

Lemma 4.2 — Suppose $p \in (2, 2^*)$, $0 \leq g(x) = g(|x|) \in L_2^{\frac{n}{2}}(\mathbb{B}) - \{0\}$ and $\|g\|_{L_2^{\frac{n}{2}}(\mathbb{B})} < C_p^*$, then

(i) there exist $a, b > 0$ and $e \in \mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})$ with $\|e\|_{\mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})} > b$ such that $I_\mu(u) \geq a > 0$ with $\|u\|_{\mathcal{H}_{2,0}^{1,\frac{n}{2}}} = b$ and $I_\mu(e) < 0$ for all $\mu \in [\frac{1}{2}, 1]$,

(ii) for any $\mu \in [\frac{1}{2}, 1]$, we have

$$c_\mu = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I_\mu(\gamma(t)) > \max\{I_\mu(0), I_\mu(e)\},$$

where

$$\Gamma = \left\{\gamma \in C([0,1], \mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})) : \gamma(0) = 0, \gamma(1) = e\right\}.$$

PROOF : (i) Since $I_\mu(u) \geq I_1(u)$ for all $u \in \mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})$ and $\mu \in [\frac{1}{2}, 1]$, by Lemma 3.1 there exist $a, b > 0$, which are independent of $\mu \in [\frac{1}{2}, 1]$, such that $I_1(u) \geq a > 0$ with $\|u\|_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}} = b$. Take $0 \leq w \in \mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})$, $\forall t > 0$ setting $w_t(x) = t^s w(tx)$ where $s \in (0, \infty)$. Since $2 + \frac{2}{s} < p < 2^*$, there exists $t_0 > 0$ large enough such that for all $\mu \in [\frac{1}{2}, 1]$,

$$\begin{aligned} I_\mu(w_{t_0}) &\leq \frac{t_0^{2s+2}}{2} \int\limits_{\mathbb{B}} |\nabla_{\mathbb{B}} w|^2 \frac{dx_1}{x_1} dx' + \frac{t_0^{2s}}{2} \int\limits_{\mathbb{B}} w^2 \frac{dx_1}{x_1} dx' \\ &- \frac{t_0^{ps}}{p} \int\limits_{\mathbb{B}} |w|^p \frac{dx_1}{x_1} dx' < 0, \end{aligned} \tag{4.2}$$

which is independent of $\mu \in [\frac{1}{2}, 1]$. Take $e = w_{t_0}$, hence (i) is valid.

(ii) By the definition of c_μ, we have for all $\mu \in [\frac{1}{2}, 1]$,

$$c_\mu \geq c_1 \geq a > 0.$$

Since $I_\mu(0) = 0$ and $I_\mu(e) < 0$ for all $\mu \in [\frac{1}{2}, 1]$, we see that (ii) is valid. □

By Theorem 4.1 and Lemma 4.2, there exists $\mu_j \in [\frac{1}{2}, 1]$ such that

(i) $\mu_j \to 1$ as $j \to +\infty$,

(ii) I_{μ_j} has a bounded $PS-$sequence u_n^j at the level c_{μ_j}.

Since the embedding $\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B}) \hookrightarrow \mathcal{H}^{0,\frac{n}{p}}_{p,0}(\mathbb{B}) = L_p^{\frac{n}{p}}(\mathbb{B})$ for $2 < p < 2^*$, is compact, for each $j \in \mathbb{N}$, there exists $u_j \in \mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})$ such that $u_n^j \to u_j$ strongly in $\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})$ as $n \to \infty$. u_j is a solution of problem 4.1 with $\mu = \mu_j$. Moreover, we have

$$0 < a \leq I_{\mu_j}(u_j) = c_{\mu_j} \leq c_{\frac{1}{2}} \text{ and } I'_{\mu_j}(u_j) = 0, \text{ for all } j \in \mathbb{N}. \tag{4.3}$$

Lemma 4.3 — Under the condition of Theorem 1.2 $\{u_j\}$ is a bounded (PS)-sequence of functional I.

PROOF : Let $I(u_j) \to c$, $\|I'(u_j)\|_{\mathcal{H}^{-1,-\frac{n}{2}}_{2,0}(\mathbb{B})} \to 0$, where $I'(.)$ is the Fréchet differentiation, and $\mathcal{H}^{-1,-\frac{n}{2}}_{2,0}(\mathbb{B})$ the dual space of $\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})$, and

$$\|I'(u)\|_{\mathcal{H}^{-1,-\frac{n}{2}}_{2,0}(\mathbb{B})} = \sup_{\varphi \in C_0^\infty(\mathbb{B})} \frac{|\langle I'(u), \varphi \rangle|}{\|\varphi\|_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}}.$$

First we show that $\{u_j\}$ is bounded in $\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})$. In fact, if we assume that $\{u_j\}$ is non-bounded in $\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})$, by Poincaré inequality, the norm of $\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})$ is equivalent to the norm $\|\nabla_{\mathbb{B}} u_j\|_{L_2^{\frac{n}{2}}(\mathbb{B})}$, thus we have

$$\|\nabla_{\mathbb{B}} u_j\|_{L_2^{\frac{n}{2}}(\mathbb{B})} \to +\infty \text{ as } j \to +\infty.$$

Let $v_j = \frac{u_j}{\|\nabla_{\mathbb{B}} u_j\|_{L_2^{\frac{n}{2}}(\mathbb{B})}}$ then, $\|\nabla_{\mathbb{B}} v_j\|_{L_2^{\frac{n}{2}}(\mathbb{B})} = 1$, which means v_j being bounded in $\mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})$, i.e., there exist $v \in \mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})$, and a subsequence v_{j_k}, such that

$$v_{j_k} \rightharpoonup v \text{ in } \mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B}).$$

Here, for simplicity, we still denote v_{j_k}, by v_j, then from $\|I'(u_j)\|_{\mathcal{H}_{2,0}^{-1,-\frac{n}{2}}(\mathbb{B})} \to 0$ we know that for any $\varphi \in C_0^\infty(\mathbb{B})$,

$$\begin{aligned}\int_{\mathbb{B}} \nabla_{\mathbb{B}} u_j \nabla_{\mathbb{B}} \varphi \frac{dx_1}{x_1} dx' &+ \int_{\mathbb{B}} u_j \varphi \frac{dx_1}{x_1} dx' - \int_{\mathbb{B}} |u_j|^{p-2} u_j \varphi \frac{dx_1}{x_1} dx' \\ &= \int_{\mathbb{B}} g(x)\varphi \frac{dx_1}{x_1} dx' + o(1)\|\varphi\|_{\mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})}, \end{aligned} \tag{4.4}$$

and then

$$\begin{aligned}\int_{\mathbb{B}} \nabla_{\mathbb{B}} v_j \nabla_{\mathbb{B}} \varphi \frac{dx_1}{x_1} dx' &+ \int_{\mathbb{B}} v_j \varphi \frac{dx_1}{x_1} dx' - \int_{\mathbb{B}} |u_j|^{p-2} v_j \varphi \frac{dx_1}{x_1} dx' \\ &= \frac{\int_{\mathbb{B}} g(x)\varphi \frac{dx_1}{x_1} dx' + o(1)\|\varphi\|_{\mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})}}{\|\nabla_{\mathbb{B}} u_j\|_{L_2^{\frac{n}{2}}(\mathbb{B})}}, \end{aligned} \tag{4.5}$$

i.e.,

$$\int_{\mathbb{B}} \nabla_{\mathbb{B}} v_j \nabla_{\mathbb{B}} \varphi \frac{dx_1}{x_1} dx' + \int_{\mathbb{B}} v_j \varphi \frac{dx_1}{x_1} dx' - \int_{\mathbb{B}} |u_j|^{p-2} v_j \varphi \frac{dx_1}{x_1} dx' = o(1). \tag{4.6}$$

From $I(u_j) \to c$ we know

$$\frac{1}{2}\int_{\mathbb{B}} |\nabla_{\mathbb{B}} u_j|^2 \frac{dx_1}{x_1} dx' + \frac{1}{2}\int_{\mathbb{B}} u_j^2 \frac{dx_1}{x_1} dx' - \frac{1}{p}\int_{\mathbb{B}} |u_j|^p \frac{dx_1}{x_1} dx' = \int_{\mathbb{B}} g(x) u_j \frac{dx_1}{x_1} dx' + o(1) + c,$$

that means

$$\frac{1}{2}\int_{\mathbb{B}} |\nabla_{\mathbb{B}} v_j|^2 \frac{dx_1}{x_1} dx' + \frac{1}{2}\int_{\mathbb{B}} v_j^2 \frac{dx_1}{x_1} dx' - \frac{1}{p}\int_{\mathbb{B}} |u_j|^{p-2}|v_j|^2 \frac{dx_1}{x_1} dx' = o(1). \tag{4.7}$$

Take $\varphi = v_j$ in (4.6), then we have

$$\int_{\mathbb{B}} |\nabla_{\mathbb{B}} v_j|^2 \frac{dx_1}{x_1} dx' + \int_{\mathbb{B}} v_j^2 \frac{dx_1}{x_1} dx' - \int_{\mathbb{B}} |u_j|^{p-2}|v_j|^2 \frac{dx_1}{x_1} dx' = o(1). \tag{4.8}$$

From (4.7) and (4.8) obtain that $p=2$, which contradicts with the assumption $p>2$. Therefore $\{u_j\}$ is bounded in $\mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})$. Thus there exist $u_1 \in \mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})$ and a subsequence, which still denotes by u_j, such that $u_j \rightharpoonup u_1$ in $\mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})$. We want to prove that $u_j \to u_1$ in $\mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})$. According to definition of the norm space $\mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})$ we have

$$\|u_j - u_1\|_{\mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})} \approx \|\nabla_{\mathbb{B}} u_j - \nabla_{\mathbb{B}} u_1\|_{L_2^{\frac{n}{2}}(\mathbb{B})}.$$

Then, by 1.4 we can obtain

$$\begin{aligned} \|u_j - u_1\|^2_{\mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})} &\approx \|\nabla_{\mathbb{B}} u_j - \nabla_{\mathbb{B}} u_1\|^2_{L_2^{\frac{n}{2}}(B)} \leq \langle I'(u_j) - I'(u_1), u_j - u_1\rangle \\ &+ \int_{\mathbb{B}} (|u_j|^{p-2}u_j - |u_1|^{p-2}u_1)(u_j - u_1)\frac{dx_1}{x_1}dx'. \end{aligned} \tag{4.9}$$

Since $u_j \rightharpoonup u_1$ in $\mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})$, $\|I'(u_j)\|_{\mathcal{H}_{2,0}^{-1,-\frac{n}{2}}(\mathbb{B})} \to 0$, and $C_0^\infty(\mathbb{B})$ is dense in $\mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})$, we can deduce that $\langle I'(u_j) - I'(u_1), u_j - u_1\rangle \to 0$ as $j \to +\infty$. So from Proposition 2.6, we have $u_j \to u_1$ in $L_p^{\frac{n}{p}}(\mathbb{B})$, that is, $|u_j|^{p-2}u_j \to |u_1|^{p-2}u_1$ in $L_p^{\frac{n(p-1)}{p}}(\mathbb{B})$. Therefore, one can imply by Hölder inequality

$$\begin{aligned} &\int_{\mathbb{B}} (|u_j|^{p-2}u_j - |u_1|^{p-2}u_1)(u_j - u_1)\frac{dx_1}{x_1}dx' \\ &\leq \int_{\mathbb{B}} ||u_j|^{p-2}u_j - |u_1|^{p-2}u_1||u_j - u_1|\frac{dx_1}{x_1}dx' \\ &\leq \||u_j|^{p-2}u_j - |u_1|^{p-2}u_1\|_{L_p^{\frac{n(p-1)}{p}}(\mathbb{B})} \|u_j - u_1\|_{L_p^{\frac{n}{p}}(\mathbb{B})}. \end{aligned}$$

Hence, $u_j \to u_1$ in $L_p^{\frac{n}{p}}(\mathbb{B})$, implies that

$$\|u_j - u_1\|_{\mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})} \approx \|\nabla_{\mathbb{B}} u_j - \nabla_{\mathbb{B}} u_1\|_{L_2^{\frac{n}{2}}(\mathbb{B})} \to 0 \;\; as \;\; j \to +\infty. \square$$

PROOF OF THEOREM 1.2 : From Lemma 4.3, we proved that $\{u_j\}$ is a $(PS)-$sequence of the functional I_λ. Moreover, it implies from compactness of the embedding mapping (see Proposition 2.6)

$$\mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B}) \hookrightarrow L_p^{\frac{n}{p}}(\mathbb{B}) \qquad 2 < p < 2^*,$$

that there exists a solution u_1 for which $I(u_1) > 0$. Finally, one can obtain the statement of Theorem 1.2 in combining of Theorem 4.1.

5. A UNIQUE SOLUTION FOR FUCHSIAN-POISSON EQUATION

In this section, we aim to prove that problem 1.1 admits exactly one solution for $p = 2$. In this case, problem 1.1 reduces to the Fuchsian-Poisson equation as follows:

$$\begin{cases} -\Delta_{\mathbb{B}} u = g(x) & x \in int\mathbb{B}, \\ u = 0 & x \in \partial\mathbb{B} \end{cases} \tag{5.1}$$

In order to prove the existence and uniqueness of weak solution problem 5.1 we apply some fundamental results about functional calculus in [6].

PROOF OF THEOREM 1.3 : Consider the functional : $\mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B}) \longrightarrow \mathbb{R}$ defined as

$$I(u) = \frac{1}{2}\int_{\mathbb{B}} |\nabla_{\mathbb{B}} u|^2 \frac{dx_1}{x_1} dx' - \int_{\mathbb{B}} g(x) u \frac{dx_1}{x_1} dx'.$$

We know that I is differentiable on $\mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})$ such that

$$\langle I'(u), v\rangle = \int_{\mathbb{B}} \nabla_{\mathbb{B}} u \nabla_{\mathbb{B}} v \frac{dx_1}{x_1} dx' - \int_{\mathbb{B}} g(x) v \frac{dx_1}{x_1} dx'.$$

For arbitrary $u, v \in \mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})$ that $u \neq v$, we can obtian

$$\begin{aligned} ((I'(u) - I'(v))(u - v) &= \int_{\mathbb{B}} (\nabla_{\mathbb{B}} u - \nabla_{\mathbb{B}} v)(\nabla_{\mathbb{B}} u - \nabla_{\mathbb{B}} v) \frac{dx_1}{x_1} dx' \\ &= \int_{\mathbb{B}} |\nabla_{\mathbb{B}} (u - v)|^2 \frac{dx_1}{x_1} dx' = \|u - v\|_{\mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})} > 0. \end{aligned} \tag{5.2}$$

Therefore, I is strictly convex by Proposition (1.5.7) in [6]. By Poincaré and Hölder inequality, we get

$$\begin{aligned} I(u) &\geq \frac{1}{2}\int_{\mathbb{B}} |\nabla_{\mathbb{B}} u|^2 \frac{dx_1}{x_1} dx' - \|g\|_{L_2^{\frac{n}{2}}(\mathbb{B})} \|u\|_{L_2^{\frac{n}{2}}(\mathbb{B})} \\ &\geq \frac{1}{2}\|u\|^2_{\mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})} - C_{emb} C_p^* \|u\|_{\mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})} \end{aligned} \tag{5.3}$$

where C_{emb} is the embedding constant and C_p^* is given by 1.5. According to 5.3, one can get

$$I(u) \to \infty \qquad as \quad \|u\|_{\mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})} \to \infty,$$

that is, the functional I is coercive. By Theorems (1.5.6) and (1.5.8) in [6], I admits a unique global minimum. Hence, the proof is complete.

ACKNOWLEDGEMENT

This research work has been supported by a research grant from the University of Mazandaran. The authors are grateful to the referees for their valuable suggestions which helped improving the original manuscript.

REFERENCES

1. M. Alimohammady and M. K. Kalleji, Existence result for a class of semilinear totally characteristic hypoelliptic equations with conical degeneration, *J. Funct. Anal.*, **265** (2013), 2331-2356.
2. H. Chen, X. Liu and Y. Wei, Cone Sobolev inequality and Dirichlet problem for nonlinear elliptic equations on manifold with conical singularities, *Calc. Var.*, **43** (2012), 463-484.
3. H. Chen, X. Liu and Y. Wei, Existence theory for a class of semilinear totally characteristic elliptic equations with critical cone Sobolev exponents, *Ann. Glob. Anal. Geom.*, **37** (2011), 27-43.
4. H. Chen, Y. Wei and B. Zhou, Existence of solutions for degenerate elliptic equations with singular potential on conical singular manifolds, *Math. Nachr.*, **25** (No.11-12) (2012), 1370-1384.
5. J. Dávila and I. Perla, Nonlinear elliptic problems with a singular weight on the boundary, *Calc. Var. Partial Differential Equations*, **41**(3-4) (2011), 567-586.
6. M. Badiale and E. Serra, Semilinear elliptic equation for beginners, DOI: 10.1007/978-0-85729-227-8, *Springer London*, **20** (2011).
7. Y. Jiang, Z. Wang and H. S. Zhou, Multiple solution for a nonhomogeneous Schrödinger-Maöxwell system in $\mathbb{R}^3$, *Nonlinear Anal.*, **83** (2013), 50-57.
8. L. Jeanjean, On the existence of bounded Palais-Smale sequences and application to a Landsman-Lazer-type problem on $\mathbb{R}^N$, *Proc. Roy. Soc. Edinburgh Sect. A*, **129**(4), (2004), 787-809.
9. B. W. Schulze, Boundary value problems and singular pseudo-differential operators, *Wiley, Chichester*, (1998).
10. E. Schrohe and J. Seiler, Ellipticity and invertibility in the cone algebra on L_p-Sobolev spaces, *Integr. Equ. Oper. Theory*, **41** (2001), 93-114.
11. X. P. Zhu and H. S. Zhou, Existence of multiple positive solutions of inhomogeneous semilinear elliptic problems in unbounded domains, *Proc. Roy. Soc. Edinburgh Sect. A*, **115**(2), (1990), 301-318.