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ON THE STABILITY AND REGULARITY OF THE MULTIPLIER IDEALS OF MONOMIAL IDEALS

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Let $\mathfrak{a} \subseteq \mathbb{C}[x_1, \ldots, x_d]$ be a monomial ideal and $\mathcal{J}(\mathfrak{a})$ its multiplier ideal which is also a monomial ideal. It is proved that if $\mathfrak{a}$ is strongly stable or squarefree strongly stable then so is $\mathcal{J}(\mathfrak{a})$. Denote the maximal degree of minimal generators of $\mathfrak{a}$ by $d(\mathfrak{a})$. When $\mathfrak{a}$ is strongly stable or squarefree strongly stable, it is shown that the Castelnuovo-Mumford regularity of $\mathcal{J}(\mathfrak{a})$ is less than or equal to $d(\mathfrak{a})$. As a corollary, one gets a vanishing result on the ideal sheaf $\widetilde{\mathcal{J}(\mathfrak{a})}$ on $\mathbb{P}^{d-1}$ associated to $\mathcal{J}(\mathfrak{a})$ that $H^i(\mathbb{P}^{d-1}; \widetilde{\mathcal{J}(\mathfrak{a})}(s-i)) = 0$, for all $i > 0$ and $s \geq d(\mathfrak{a})$.

Key words : Stability; Castelnuovo-Mumford regularity; multiplier ideals.

1. INTRODUCTION

Multiplier ideal sheaves have become fundamental tools in higher dimensional algebraic geometry, which are found to be strongly related to adjoint ideals and test ideals in commutative algebra (cf. [2, 8-12]). Let $(X, \mathcal{O}_X)$ be a smooth quasiprojective complex variety and $\mathfrak{a} \subseteq \mathcal{O}_X$ an ideal sheaf on X. Then the multiplier ideal sheaf $\mathcal{J}(\mathfrak{a})$ of $\mathfrak{a}$ is also an ideal sheaf on X. When X is affine and $\mathfrak{a}$ is monomial, $\mathcal{J}(\mathfrak{a})$ can be described explicitly by a remarkable theorem of Howald [7]. In this case, $\mathcal{J}(\mathfrak{a})$ is also a monomial ideal. In [5], the authors discussed the problem when $\mathcal{J}(\mathfrak{a}) = \mathfrak{a}$. On the other hand, Castelnuovo-Mumford regularity (or regularity) is an important invariant for graded modules and coherent sheaves. It is worthwhile to estimate the regularity of multiplier ideals of monomial ideals and the ideal sheaves associated to these multiplier ideals.

There is a nice theory for the regularity of monomial ideals (cf. [6]). When I is a stable monomial ideal, Eliahou and Kervaire's [4] well-known result states that the regularity of I is equal to its generating degree. The main part of this paper focuses on the stability of the multiplier ideals of monomial ideals. We will show in Section 3 that if a monomial ideal $\mathfrak{a}$ is strongly stable or squarefree strongly stable, then so is its multiplier ideal $\mathcal{J}(\mathfrak{a})$. Section 4 is devoted to estimate the regularity of multiplier ideals. After comparing the generating degrees of $\mathfrak{a}$ and $\mathcal{J}(\mathfrak{a})$, we prove that the regularity of $\mathcal{J}(\mathfrak{a})$ is less than or equal to the generating degree of $\mathfrak{a}$ provided that $\mathfrak{a}$ is strongly stable or squarefree strongly stable. At the end, we get an unexpected vanishing result on the cohomology of the ideal sheaf associated to the multiplier ideal of a strongly stable or squarefree strongly stable monomial ideal.

2. PRELIMINARIES

Let K be a field and $K[x_1, \dots, x_d]$ a polynomial ring over K. Let I be an ideal of $K[x_1, \dots, x_d]$. When I is generated by monomials, we say that I is a monomial ideal, and its minimal generating set is denoted by $G(I)$. Further, if all the monomials in $G(I)$ are squarefree, I is said to be a squarefree monomial ideal.

Let $I \subseteq K[x_1, \dots, x_d]$ be a monomial ideal. Every monomial $x_1^{a_1} \cdots x_d^{a_d} \in I$ corresponds to its exponent vector $(a_1, \dots, a_d) \in \mathbb{N}^d$ where $\mathbb{N}$ contains 0. The convex hull in $\mathbb{R}^d$ of the set of all the exponent vectors of monomials of I is called the Newton polygon of I, denoted by $P(I)$. Then the set of monomials in the integral closure $\overline{I}$ of I is just the set of all the monomials $x_1^{a_1} \cdots x_d^{a_d}$ with $(a_1, \dots, a_d) \in P(I)$ (cf. [13, Proposition 1.4.6]).

Let X be a smooth quasiprojective complex algebraic variety and $\mathfrak{a} \subseteq \mathcal{O}_X$ an ideal sheaf on X. Let $f : Y \to X$ be a log resolution of $\mathfrak{a}$ with $f^{-1}(\mathfrak{a}) = \mathcal{O}_Y(-E)$. The multiplier ideal of $\mathfrak{a}$ is defined to be

$$\mathcal{J}(\mathfrak{a}) = f_*\mathcal{O}_Y(K_{Y/X} - E),$$

where $K_{Y/X} = K_Y - f^*K_X$ is the relative canonical divisor (cf. [9]). Then $\mathcal{J}(\mathfrak{a})$ is an ideal sheaf on X.

In the case $X = \mathbb{A}^d$, Howald [7] gave an explicit description of $\mathcal{J}(\mathfrak{a})$.

Howald Theorem — *Let $\mathfrak{a} \subseteq \mathbb{C}[x_1, \dots, x_d]$ be a monomial ideal. Then $\mathcal{J}(\mathfrak{a})$ is also a monomial ideal in $\mathbb{C}[x_1, \dots, x_d]$ and*

$$\mathcal{J}(\mathfrak{a}) = (x_1^{a_1} \cdots x_d^{a_d} : (a_1, \dots, a_d) + (1, \dots, 1) \in \operatorname{Int}(P(\mathfrak{a})) \cap \mathbb{N}^d),$$

where $\operatorname{Int}(P(\mathfrak{a}))$ denotes the interior of $P(\mathfrak{a})$.

3. Stability of Multiplier Ideals

For any monomial $u \in K[x_1, \ldots, x_d]$, set $m(u) = \max\{i : x_i \mid u\}$. Let $I \subseteq K[x_1, \ldots, x_d]$ be a monomial ideal. I is called to be stable if for any monomial $u \in I$ and any $i < m(u)$ one has $\frac{x_i u}{x_{m(u)}} \in I$, and strongly stable if for any monomial $u \in I$ and any $i < t$ with $x_t \mid u$ one has $\frac{x_i u}{x_t} \in I$. Similarly, for any squarefree monomial ideal, we say that I is squarefree stable if for any squarefree monomial $u \in I$ and any $i < m(u)$ with $x_i \nmid u$ one has $\frac{x_i u}{x_{m(u)}} \in I$, and squarefree strongly stable if for any squarefree monomial $u \in I$ and any $i < t$ with $x_t \mid u$ and $x_i \nmid u$ one has $\frac{x_i u}{x_t} \in I$. Let $\overline{I}$ be the integral closure of I, which is also monomial. The following two lemmas are well-known (cf. [6, Theorem 1.4.2] and [13, Proposition 1.4.6]).

Lemma 3.1 — A monomial $u \in \overline{I}$ if and only if there exist $r > 0$ and monomials $u_1, \ldots, u_r \in I$ such that $u^r = u_1 \cdots u_r$.

Lemma 3.2 — Let $G(I) = \{u_1, \ldots, u_s\}$ where $u_j = x_1^{n_{j1}} \cdots x_d^{n_{jd}}$, $j = 1, \ldots, s$. Then a monomial $u = x_1^{n_1} \cdots x_d^{n_d} \in \overline{I}$ if and only if there exist nonnegative rational numbers $c_1, \ldots, c_s$ such that $\sum_{j=1}^{s} c_j = 1$ and component-wise,

$$(n_1, \ldots, n_d) \geq \sum_{j=1}^{s} c_j (n_{j1}, \ldots, n_{jd}).$$

In order to discuss the stability of multiplier ideals, we need the stability of integral closures.

Proposition 3.3 — Let I be a monomial ideal. If I is (strongly) stable, then $\overline{I}$ is also (strongly) stable.

Proof : Let us show the statement for stability. The argument for the strongly stability case is similar.

Let $u \in \overline{I}$ be a monomial. Then, by Lemma 3.1, there exist $r > 0$ and monomials $u_1, \ldots, u_r \in I$ such that $u^r = u_1 \cdots u_r$. Notice that $m(u_i) \leq m(u)$, $i = 1, \ldots, r$, and if $x_{m(u)} \mid u_i$, then $m(u_i) = m(u)$. Further, for any $j < m(u)$, if $x_{m(u)}^s \mid u_i$, then $\frac{x_j^s u_i}{x_{m(u)}^s} \in I$ by the stability of I. From $u^r = u_1 \cdots u_r$, we see that there exist $s_1, \ldots, s_r \geq 0$ such that $r = s_1 + \cdots + s_r$ and $x_{m(u)}^{s_i} \mid u_i$, $i = 1, \ldots, r$, where we use the convention that $x_{m(u)}^0 = 1$. Then, for any $j < m(u)$, one has that $\frac{x_j^{s_i} u_i}{x_{m(u)}^{s_i}} \in I$, $i = 1, \ldots, r$. It follows that

$$\left(\frac{x_j u}{x_{m(u)}}\right)^r = \frac{x_j^{s_1} u_1}{x_{m(u)}^{s_1}} \cdots \frac{x_j^{s_r} u_r}{x_{m(u)}^{s_r}}.$$

Then, by Lemma 3.1 again, we get that $\frac{x_j u}{x_{m(u)}} \in \overline{I}$. Hence $\overline{I}$ is stable. □

Proposition 3.4 — Let I be a squarefree monomial ideal. Then $\overline{I}$ is also squarefree, and if I is squarefree (strongly) stable then $\overline{I}$ is also squarefree (strongly) stable.

PROOF : Firstly, we show that $\overline{I}$ is squarefree. Suppose that $G(I) = \{u_1, \ldots, u_s\}$ and $u_j = x_1^{n_{j1}} \cdots x_d^{n_{jd}}$, $j = 1, \ldots, s$, where $n_{ji} \leq 1$. Let $u = x_1^{n_1} \cdots x_d^{n_d} \in G(\overline{I})$, let us show that u is squarefree.

Since $u \in \overline{I}$, it follows from Lemma 3.2 that there exist rational numbers $c_j \geq 0$, $j = 1, \ldots, s$, such that $\sum_{j=1}^{s} c_j = 1$ and

$$(n_1, \ldots, n_d) \geq \sum_{j=1}^{s} c_j (n_{j1}, \ldots, n_{jd}).$$

We claim that $n_i < 1 + \sum_{j=1}^{s} c_j n_{ji}$ holds for any i. Otherwise, there is some i such that $(n_1, \ldots, n_{i-1}, n_i - 1, n_{i+1}, \ldots, n_d) \geq \sum_{j=1}^{s} c_j (n_{j1}, \ldots, n_{jd})$, which implies that $\frac{u}{x_i} \in \overline{I}$. This contradicts $u \in G(\overline{I})$. Therefore, for any i,

$$n_i < 1 + \sum_{j=1}^{s} c_j n_{ji} \leq 1 + \sum_{j=1}^{s} c_j = 2,$$

thus $n_i \leq 1$. Hence u is squarefree and $\overline{I}$ is a squarefree monomial ideal.

Now we prove that $\overline{I}$ is squarefree stable provided that I is squarefree stable, and omit the argument for the statement on the squarefree strongly stability. Let $u \in \overline{I}$ be a squarefree monomial. Then by Lemma 3.1 that there exists $r > 0$ such that $u^r = v u_1 \cdots u_r$, where v is a monomial and $u_i \in I$, $i = 1, \ldots, r$, are squarefree monomials. For any $j < m(u)$ with $x_j \nmid u$, one has that $x_j \nmid u_i$, $i = 1, \ldots, r$. Without loss of generality, suppose that $x_{m(u)} \mid u_i$, $i = 1, \ldots, s$, and $x_{m(u)} \nmid u_i$, $i = s+1, \ldots, r$. Then

$$\begin{aligned} \left(\frac{x_j u}{x_{m(u)}}\right)^r &= \frac{x_j^{r-s} v}{x_{m(u)}^{r-s}} \cdot \frac{x_j u_1}{x_{m(u)}} \cdots \frac{x_j u_s}{x_{m(u)}} \cdot u_{s+1} \cdots u_r \\ &\in I^r. \end{aligned}$$

Hence by Lemma 3.1 again we have that $\frac{x_j u}{x_{m(u)}} \in \overline{I}$. Therefore $\overline{I}$ is squarefree stable. □

Let $\mathfrak{a} \subseteq \mathbb{C}[x_1, \ldots, x_d]$ be a monomial ideal and $P(\mathfrak{a})$ the Newton polygon of $\mathfrak{a}$. Recall that

$$\overline{\mathfrak{a}} = (x_1^{a_1} \cdots x_d^{a_d} : (a_1, \ldots, a_d) \in P(\mathfrak{a}))$$

and

$$\mathcal{J}(\mathfrak{a}) = (x_1^{a_1} \cdots x_d^{a_d} : (a_1, \ldots, a_d) + (1, \ldots, 1) \in \mathrm{Int}(P(\mathfrak{a})) \cap \mathbb{N}^d).$$

Let us discuss the stability of $\mathcal{J}(\mathfrak{a})$.

***Theorem* 3.5 —** *If $\mathfrak{a} \subseteq \mathbb{C}[x_1, \ldots, x_d]$ is a strongly stable monomial ideal, then $\mathcal{J}(\mathfrak{a})$ is also strongly stable.*

PROOF : Let $u \in \mathcal{J}(\mathfrak{a})$ be a monomial. Suppose that $x_t \mid u$ and $j < t$. Let us show that $\frac{x_j u}{x_t} \in \mathcal{J}(\mathfrak{a})$, then $\mathcal{J}(\mathfrak{a})$ is strongly stable.

Since $\overline{\mathfrak{a}}$ is strongly stable by Proposition 3.3 and $x_1 \cdots x_d u \in \overline{\mathfrak{a}}$, it follows that $\overline{\mathfrak{a}}$ contains

$$x_1 \cdots x_d \left(\frac{x_j u}{x_t} \right) = \frac{x_j (x_1 \cdots x_d u)}{x_t}$$

and

$$\frac{x_j (x_1 \cdots x_d (\frac{x_j u}{x_t}))}{x_t}.$$

Set $x_1 \cdots x_d u = x_1^{\alpha_1} \cdots x_d^{\alpha_d}$. Then $P(\mathfrak{a})$ contains points

$$(\alpha_1, \ldots, \alpha_{j-1}, \alpha_j + 1, \alpha_{j+1}, \ldots, \alpha_{t-1}, \alpha_t - 1, \alpha_{t+1}, \ldots, \alpha_d)$$

and

$$(\alpha_1, \ldots, \alpha_{j-1}, \alpha_j + 2, \alpha_{j+1}, \ldots, \alpha_{t-1}, \alpha_t - 2, \alpha_{t+1}, \ldots, \alpha_d).$$

In order to show that $\frac{x_j u}{x_t} \in \mathcal{J}(\mathfrak{a})$, it is enough to show that the point $(\alpha_1, \ldots, \alpha_{j-1}, \alpha_j + 1, \alpha_{j+1}, \ldots, \alpha_{t-1}, \alpha_t - 1, \alpha_{t+1}, \ldots, \alpha_d)$ is not on the boundary of $P(\mathfrak{a})$. Note that it is not on any coordinate hyperplane as its components are all positive. Suppose, on the contrary, that $(\alpha_1, \ldots, \alpha_{j-1}, \alpha_j + 1, \alpha_{j+1}, \ldots, \alpha_{t-1}, \alpha_t - 1, \alpha_{t+1}, \ldots, \alpha_d)$ is on some hyperplane H which bounds $P(\mathfrak{a})$. Assume the equation of H is

$$p_1 x_1 + \cdots + p_d x_d = p, \qquad p_i \geq 0, p > 0.$$

Since $(\alpha_1, \ldots, \alpha_d)$ is an interior point, it follows that

$$\begin{aligned} & p_1 \alpha_1 + \cdots + p_d \alpha_d > p \\ & p_1 \alpha_1 + \cdots + p_j(\alpha_j + 1) + \cdots + p_t(\alpha_t - 1) + \cdots + p_d \alpha_d = p \\ & p_1 \alpha_1 + \cdots + p_j(\alpha_j + 2) + \cdots + p_t(\alpha_t - 2) + \cdots + p_d \alpha_d \geq p. \end{aligned}$$

It turns out that $p_t > p_j$ from the first and second formulae, and $p_t \leq p_j$ from the second and third formulae, a contradiction. The result follows. □

Remark 3.6 : When $\mathfrak{a} \subseteq \mathbb{C}[x_1, \ldots, x_d]$ is a stable monomial ideal, $\overline{\mathfrak{a}}$ is also stable by Proposition 3.3. It is reasonable that Theorem 3.5 should be true for stable monomial ideals, but the authors can not show this. The following is an example that $\mathfrak{a}$ is a stable, but not strongly stable, monomial ideal and $\mathcal{J}(\mathfrak{a})$ is not strongly stable.

Set $\mathfrak{a} \subseteq \mathbb{C}[x_1, x_2, x_3]$ and $\mathfrak{a} = (x_1x_2^4, x_1^2x_2^3, x_1^3x_2^2, x_1^4x_2, x_1^5, x_1x_2^3x_3, x_1^2x_2^2x_3, x_1x_2^2x_3^2)$. Then $\mathfrak{a}$ is stable, but not strongly stable. It is not difficult to see that $P(\mathfrak{a})$ is the convex hull bounded by the following four hyperplanes:

$$\begin{aligned} H_1 &: \quad x_1 = 1 \\ H_2 &: \quad x_2 = 0 \\ H_3 &: \quad x_1 + 2x_2 = 5 \\ H_4 &: \quad x_1 + x_2 + x_3 = 5. \end{aligned}$$

Then the point $(2, 2, 2)$ is an interior point, while the point $(3, 1, 2)$ is on the boundary. It turns out that $x_1x_2x_3 \in \mathcal{J}(\mathfrak{a})$, while $x_1^2x_3 \notin \mathcal{J}(\mathfrak{a})$. This means that $\mathcal{J}(\mathfrak{a})$ is not a strongly stable monomial ideal.

***Theorem* 3.7** — *Let $\mathfrak{a} \subseteq \mathbb{C}[x_1, \ldots, x_d]$ be a squarefree monomial ideal. Then $\mathcal{J}(\mathfrak{a})$ is also squarefree, and if $\mathfrak{a}$ is squarefree strongly stable then $\mathcal{J}(\mathfrak{a})$ is also squarefree strongly stable.*

PROOF : Firstly, let us show that $\mathcal{J}(\mathfrak{a})$ is squarefree. Suppose that $G(\mathfrak{a}) = \{u_1, \ldots, u_s\}$ and $u_j = x_1^{n_{j1}} \cdots x_d^{n_{jd}}$, $j = 1, \ldots, s$. Let $u = x_1^{n_1} \cdots x_d^{n_d} \in G(\mathcal{J}(\mathfrak{a}))$. Then $(n_1 + 1, \ldots, n_d + 1) \in \mathrm{Int}(P(\mathfrak{a}))$. It follows from Lemma 3.2 that there exist rational numbers $c_j \geq 0$, $j = 1, \ldots, s$, such that $\sum_{j=1}^{s} c_j = 1$ and

$$(n_1 + 1, \ldots, n_d + 1) \geq \sum_{j=1}^{s} c_j(n_{j1}, \ldots, n_{jd}).$$

By the assumption that $(n_1 + 1, \ldots, n_d + 1)$ is an interior point of $P(\mathfrak{a})$, we may assume that $(n_1 + 1, \ldots, n_d + 1) \in \mathrm{Int}(Q)$ where

$$Q = \{(e_1, \ldots, e_d) \in \mathbb{R}^d_{\geq 0} : (e_1, \ldots, e_d) \geq \sum_{j=1}^{s} c_j(n_{j1}, \ldots, n_{jd})\},$$

for some $c_j \geq 0$, $j = 1, \ldots, s$, with $\sum_{j=1}^{s} c_j = 1$. Thus $n_i + 1 > \sum_{j=1}^{s} c_jn_{ji}$, $i = 1, \ldots, d$.

Claim : one has that $n_i \leq \sum_{j=1}^{s} c_jn_{ji}$ for any i. Otherwise, there exists some i such that $n_i > \sum_{j=1}^{s} c_jn_{ji}$. Then

$$(n_1 + 1, \ldots, n_{i-1} + 1, n_i, n_{i+1} + 1, \ldots, n_d + 1) > \sum_{j=1}^{s} c_j(n_{j1}, \ldots, n_{jd}).$$

It follows that $(n_1+1,\ldots,n_{i-1}+1,n_i,n_{i+1}+1,\ldots,n_d+1)\in \mathrm{Int}(Q)\subseteq \mathrm{Int}(P(\mathfrak{a}))$, so that $\frac{u}{x_i}\in \mathcal{J}(\mathfrak{a})$. This contradicts $u\in G(\mathcal{J}(\mathfrak{a}))$.

From this claim, we have that, for any i, $n_i\leq \sum_{j=1}^{s} c_j=1$. Hence u is squarefree, and $\mathcal{J}(\mathfrak{a})$ is a squarefree monomial ideal.

Now, we prove that $\mathcal{J}(\mathfrak{a})$ is squarefree strongly stable. Let $u\in\mathcal{J}(\mathfrak{a})$ be a squarefree monomial. Suppose that $j<t$ such that $x_t\mid u$ but $x_j\nmid u$. From $u\in\mathcal{J}(\mathfrak{a})$, we have

$$x_1\cdots x_d u=v'v,$$

where $v\in G(\overline{\mathfrak{a}})$ which is squarefree, then $x_t\mid v'$. In fact, for any $v\in G(\overline{\mathfrak{a}})$, as $v\mid x_1\cdots x_d$, there exists some monomial v' such that $x_1\cdots x_d u=v'v$. Take any such an equality, we get

$$x_1\cdots x_d(\frac{x_j u}{x_t})=\left(\frac{x_j v'}{x_t}\right)v\in\overline{\mathfrak{a}}.$$

Let $x_1\cdots x_d u=x_1^{\alpha_1}\cdots x_d^{\alpha_d}$. In order to show that $\frac{x_j u}{x_t}\in\mathcal{J}(\mathfrak{a})$, we have to show that the point $(\alpha_1,\ldots,\alpha_{j-1},\alpha_j+1,\alpha_{j+1},\ldots,\alpha_{t-1},\alpha_t-1,\alpha_{t+1},\ldots,\alpha_d)$ is not on the boundary of $P(\mathfrak{a})$. This is true when $\overline{\mathfrak{a}}$ contains the following element

$$\frac{x_j(x_1\cdots x_d(\frac{x_j u}{x_t}))}{x_t},$$

by a similar argument to the proof of Theorem 3.5.

Suppose that there is some $v\in G(\overline{\mathfrak{a}})$ such that $x_t\nmid v$. Then $x_t^2\mid v'$ for any monomial v' such that $x_1\cdots x_d u=v'v$. In this case

$$\frac{x_j(x_1\cdots x_d(\frac{x_j u}{x_t}))}{x_t}=\left(\frac{x_j^2 v'}{x_t^2}\right)v\in\overline{\mathfrak{a}}.$$

Hence $\frac{x_j u}{x_t}\in\mathcal{J}(\mathfrak{a})$. In the following, we assume that $x_t\mid v$ for any $v\in G(\overline{\mathfrak{a}})$.

If there exists some $v\in G(\overline{\mathfrak{a}})$ such that $x_j\nmid v$, then $\frac{x_j v}{x_t}\in\overline{\mathfrak{a}}$ by the strongly stability of $\overline{\mathfrak{a}}$, which is proved by Proposition 3.4. It follows that we also have

$$\frac{x_j(x_1\cdots x_d(\frac{x_j u}{x_t}))}{x_t}=\left(\frac{x_j v'}{x_t}\right)\left(\frac{x_j v}{x_t}\right)\in\overline{\mathfrak{a}}.$$

The remainder case is that for any $v\in G(\overline{\mathfrak{a}})$, $x_j\mid v$ and $x_t\mid v$. In this case, the Newton polygon $P(\mathfrak{a})$ is symmetric between x_j-axis and x_t-axis. On the other hand, $x_1\cdots x_d(\frac{x_j u}{x_t})$ is obtained from $x_1\cdots x_d u$ by exchanging between x_j and x_t. Then the point $(\alpha_1,\ldots,\alpha_{j-1},\alpha_j+1,\alpha_{j+1},\ldots,\alpha_{t-1},\alpha_t-1,\alpha_{t+1},\ldots,\alpha_d)$ is not on the boundary of $P(\mathfrak{a})$ because that neither is the point $(\alpha_1,\ldots,\alpha_d)$ by the assumption on u. The proof is complete. □

4. Regularity of Multiplier Ideals

Let $\mathfrak{a}$ be a monomial ideal, its generating degree $d(\mathfrak{a})$ is defined as:

$$d(\mathfrak{a}) = \max\{\deg(u) : u \in G(\mathfrak{a})\}.$$

Proposition 4.1 — Let $\mathfrak{a} \subseteq \mathbb{C}[x_1, \ldots, x_d]$ be a monomial ideal. Then

$$d(\mathcal{J}(\mathfrak{a})) \leq d(\mathfrak{a}).$$

Proof : Suppose that $G(\mathfrak{a}) = \{u_1, \ldots, u_s\}$ and $u_j = x_1^{n_{j1}} \cdots x_d^{n_{jd}}$, $j = 1, \ldots, s$. Let $u = x_1^{n_1} \cdots x_d^{n_d} \in G(\mathcal{J}(\mathfrak{a}))$. Then $(n_1 + 1, \ldots, n_d + 1) \in \operatorname{Int}(P(\mathfrak{a}))$ and, as in the proof of Theorem 3.7, we have that $(n_1 + 1, \ldots, n_d + 1) \in \operatorname{Int}(Q)$ where

$$Q = \{(e_1, \ldots, e_d) \in \mathbb{R}^d_{\geq 0} : (e_1, \ldots, e_d) \geq \sum_{j=1}^{s} c_j (n_{j1}, \ldots, n_{jd})\},$$

for some $c_j \geq 0$, $j = 1, \ldots, s$, with $\sum_{j=1}^{s} c_j = 1$.

By the claim in the proof of Theorem 3.7, we have that $n_i \leq \sum_{j=1}^{s} c_j n_{ji}$ for any i. It follows that

$$\begin{aligned}
\deg(u) &= \sum_{i=1}^{d} n_i \leq \sum_{i=1}^{d} \sum_{j=1}^{s} c_j n_{ji} \\
&= \sum_{j=1}^{s} c_j (\sum_{i=1}^{d} n_{ji}) = \sum_{j=1}^{s} c_j \deg(m_j) \\
&\leq \sum_{j=1}^{s} c_j d(\mathfrak{a}) = d(\mathfrak{a}) \sum_{j=1}^{s} c_j = d(\mathfrak{a}).
\end{aligned}$$

This proves that $d(\mathcal{J}(\mathfrak{a})) \leq d(\mathfrak{a})$. □

This upper bound for $d(\mathcal{J}(\mathfrak{a}))$ is helpful for computing $\mathcal{J}(\mathfrak{a})$.

Let $S = K[x_1, \ldots, x_d]$ be a polynomial ring over a field K and M a finitely generated graded S-module. Let

$$\cdots \to F_j \to \cdots \to F_0 \to M \to 0$$

be a minimal free resolution of M, where $F_j = \oplus_i S(-a_{ji})$. One says that M is m-regular if $a_{ji} - j \leq m$ for all i, j and defines the Castelnuovo-Mumford regularity (or regularity) of M by

$$\operatorname{reg}(M) = \min\{m : M \text{ is } m\text{-regular }\}.$$

For the properties of the regularity, we refer to [3]. The paper [14] contains some new results about the regularity of operations of ideals. For the regularity of monomial ideals, the following result is well-known.

Lemma 4.2 ([4, Theorem 2.1], [1, Corollary 2.6]) — Let $I \subseteq S$ be a monomial ideal. If I is stable or squarefree stable, then

$$\mathrm{reg}(I) = \max\{\deg(u) : u \in G(I)\}.$$

Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^n$. We say that $\mathcal{F}$ is m-regular if $H^i(\mathbb{P}^n; \mathcal{F}(m-i)) = 0$ for all $i > 0$. Its Castelnuovo-Mumford regularity (or regularity) is defined as

$$\mathrm{reg}(\mathcal{F}) = \min\{m : \ \mathcal{F} \text{ is } m\text{-regular } \}.$$

The following result is also well-known (cf. [3, Proposition 4.16]).

Lemma 4.3 — Let $\widetilde{M}$ be the coherent sheaf on $\mathbb{P}^{d-1}$ associated to M. Then

$$\mathrm{reg}(\widetilde{M}) \leq \mathrm{reg}(M).$$

Then we get our main theorem.

***Theorem* 4.4** — *Let $\mathfrak{a} \subseteq \mathbb{C}[x_1, \ldots, x_d]$ be a monomial ideal. Suppose that $\mathfrak{a}$ is strongly stable or squarefree strongly stable. Then*

$$\mathrm{reg}(\mathcal{J}(\mathfrak{a})) \leq d(\mathfrak{a}).$$

PROOF : By Theorems 3.5 and 3.7, $\mathcal{J}(\mathfrak{a})$ is also strongly stable or squarefree strongly stable. On the other hand, $d(\mathcal{J}(\mathfrak{a})) \leq d(\mathfrak{a})$ by Proposition 4.1. Then the result follows from Lemma 4.2. □

As a corollary, we have the following vanishing result.

Corollary 4.5 — Let $\mathfrak{a} \subseteq \mathbb{C}[x_1, \ldots, x_d]$ be a monomial ideal. Suppose that $\mathfrak{a}$ is strongly stable or squarefree strongly stable. Then, for all $i > 0$ and $s \geq d(\mathfrak{a})$,

$$H^i(\mathbb{P}^{d-1}; \widetilde{\mathcal{J}(\mathfrak{a})}(s-i)) = 0.$$

PROOF : This is because that $\mathrm{reg}(\widetilde{\mathcal{J}(\mathfrak{a})}) \leq \mathrm{reg}(\mathcal{J}(\mathfrak{a})) \leq d(\mathfrak{a})$. □

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