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SOME CONTINUED FRACTIONS FOR π AND *G*

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We present here two classes of infinite series and the associated continued fractions involving π and Catalan's constant based on the work of Euler and Ramanujan. A few sundry continued fractions are also given.

Key words : Infinite series; continued fractions; Pi; Catalan's constant.

1. INTRODUCTION

A continued fraction is an expression of the general form

$$b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cdots}}}$$

with $a_1, a_2, \ldots$ and $b_0, b_1, b_2, \ldots$ real numbers. The following space-saving notation can be used:

$$b_0 + \frac{a_1}{b_1} \,{}_{+}\, \frac{a_2}{b_2} \,{}_{+}\, \frac{a_3}{b_3} \,{}_{+}\, \cdots$$

or the shorter notation $b_0 + \mathbf{K}_{n=1}^{\infty} \dfrac{a_n}{b_n}$, where the letter K comes from the German word *Kettenbrüche* (Ketten - chain, Brüche - fraction) for a continued fraction.

It is not known when, where and by whom continued fractions were first used, but the notion seems to be quite old. For an overview, see [3]. Euclid's algorithm for finding the greatest common divisor of two integers in effect converts a fraction into a terminating continued fraction. Continued fractions were used in India in the 6th century by Aryabhata to solve linear Diophantine equations and in the 12th century by Bhaskaracharya for solving the 'Pell' equation. Rafael Bombelli (1526-1572) and Pietro Antonio Cataldi (1548-1626), both of Bologna (Italy), gave continued fractions for $\sqrt{13}$ and $\sqrt{18}$ respectively. The first continued fraction for a number other than the quadratic irrationals is due to William Brouncker (1620-84) who converted the product $\frac{4}{\pi} = \prod_{n=1}^{\infty} \frac{(2n+1)^2}{2n(2n+2)}$ submitted to him by John Wallis (1616-1703) in 1655 (see [20, p. 182]):

$$\frac{4}{\pi} = 1 + \mathop{\mathbf{K}}_{n=1}^{\infty} \frac{(2n-1)^2}{2} = 1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \cdots}}}.$$

It was one of the sequence of continued fractions found by Brouncker [19, p. 307].

2. Euler's Continued Fractions for π

Leonhard Euler (1707–1783) laid the foundation of the theory of continued fractions in a dozen papers (six published in his life-time and six posthumously) and chapter 18 of his *Introductio in Analysin Infinitorum* (1748). Of these, five papers written between 1737 and 1780, with Eneström index numbers: E071, E123, E522, E593 and E745, are of interest to us here. Brouncker's fraction occurs in E071 [5, §4]. Euler explains the method of converting an infinite series into a continued fraction in E593 [8]. He gives among others this formula in [6, §7]:

$$\int_0^1 \frac{x^{n-1}dx}{1+x^m} = \cfrac{1}{n + \cfrac{n^2}{m + \cfrac{(m+n)^2}{m + \cfrac{(2m+n)^2}{m + \cfrac{(3m+n)^2}{m + \cdots}}}}}.$$

Now $\int \frac{xdx}{1+x^4} = \frac{1}{2} \arctan x^2 + C$. So taking $m = 4$, $n = 2$, we get Brouncker's fraction in disguise:

$$\frac{\pi}{8} = \cfrac{1}{2 + \cfrac{2^2}{4 + \cfrac{6^2}{4 + \cfrac{10^2}{4 + \cfrac{14^2}{4 + \cdots}}}}}.$$

We find in [8, §18] Euler's Theorem I:

If such an infinite series will have been proposed

$$s = \frac{1}{\alpha} - \frac{1}{\beta} + \frac{1}{\gamma} - \frac{1}{\delta} + \frac{1}{\epsilon} - \cdots,$$

a continued fraction of the form

$$\frac{1}{s} = \alpha + \cfrac{\alpha\alpha}{\beta - \alpha + \cfrac{\beta\beta}{\gamma - \beta + \cfrac{\gamma\gamma}{\delta - \gamma + \cdots}}}$$

can always be formed from it.

Taking the Leibnitz series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} = \frac{\pi}{4}$, Euler deduced Brouncker's fraction.

We find this theorem as Theorem II in [8, §23]:

If the proposed series is of this form

$$s = \frac{1}{ab} - \frac{1}{bc} + \frac{1}{cd} - \frac{1}{de} + \frac{1}{ef} - \cdots,$$

from this, the following continued fraction springs forth

$$\frac{1}{as} = b + \cfrac{ab}{c - a + \cfrac{bc}{d - b + \cfrac{cd}{e - c + \cfrac{de}{f - d \cdots}}}}.$$

Noting that $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)(2n+1)} = \frac{\pi}{4} - \frac{1}{2}$, he deduced

$$\frac{4}{\pi - 2} = 3 + \cfrac{1 \cdot 3}{4 + \cfrac{3 \cdot 5}{4 + \cfrac{5 \cdot 7}{4 + \cfrac{7 \cdot 9}{4 + \cdots}}}}$$

or

$$\frac{\pi}{2} - 1 = \cfrac{2}{3 + \cfrac{1 \cdot 3}{4 + \cfrac{3 \cdot 5}{4 + \cfrac{5 \cdot 7}{4 + \cdots}}}}.$$

Note that the convergents of this continued fraction $c_n = \frac{p_n}{q_n}$ satisfy the following relations:

$$p_0 = q_0 = 1, \quad p_{n+1} = (2n+3)p_n + (-1)^n 2(2n-1)!!, \quad q_n = (2n+1)!!,$$

for $n = 0, 1, 2, \ldots$, where $n!! = n \cdot (n-2) \cdot (n-4) \cdot \ldots \cdot 1$ for n odd, with $(-1)!! = 1$.

This continued fraction is a special case of a more general one which can be deduced from a formula of Ramanujan giving a continued fraction for a product of quotients of gamma function values [17, p. 227]:

General Formula: Let

$$P_{-1} = \frac{1}{2}, \ P_{2m} = \prod_{k=1}^{m} \frac{2k(2k+2)}{(2k+1)^2}, \ P_{2m+1} = P_{2m} \cdot \frac{2m+2}{2m+3} \quad (m = 0, 1, 2, \ldots).$$

Then we have

$$\frac{\pi}{4P_n} - 1 = \cfrac{(-1)^{n+1} 2}{4(n+2) + (-1)^n + \cfrac{1 \cdot 3}{4(n+2) + \cfrac{3 \cdot 5}{4(n+2) + \cfrac{5 \cdot 7}{4(n+2) + \cdots}}}}$$

It may be compared with [17, p. 227, eq(30a)]. Note that these continued fractions give π as a combination of the partial products of Wallis's formula and a continued fraction. For instance, for

$n = 4$ we have:

$$\frac{\pi}{4} = \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot [1 - \cfrac{2}{25 + \cfrac{1 \cdot 3}{24 + \cfrac{3 \cdot 5}{24 + \cfrac{5 \cdot 7}{24 + \ddots}}}}].$$

Euler gave the following two continued fractions and a couple more in [6, §31, §33]:

$$\frac{\pi}{2} = 1 + \cfrac{1}{1 + \cfrac{1 \cdot 2}{1 + \cfrac{2 \cdot 3}{1 + \cfrac{3 \cdot 4}{1 + \cdots}}}},$$

$$\frac{\pi}{2} = 2 - \cfrac{1}{2 + \cfrac{1^2}{2 + \cfrac{2^2}{2 + \cfrac{3^2}{2 + \cdots}}}}.$$

He obtained this fraction with partial denominators $4n$ from his general formula in [7, §18] and [8, §36]:

$$\frac{6\sqrt{3}}{\pi} = 3 + \cfrac{3 \cdot 1^2}{8 + \cfrac{3 \cdot 3^2}{12 + \cfrac{3 \cdot 5^2}{16 + \cfrac{3 \cdot 7^2}{20 + \cdots}}}}.$$

These continued fractions with partial denominators $3n - 2$ and $5n - 3$ are due to Glaisher [10, 11].

$$\frac{2}{\pi} = 1 - \cfrac{1 \cdot 1}{4 - \cfrac{2 \cdot 3}{7 - \cfrac{3 \cdot 5}{10 - \cfrac{4 \cdot 7}{13 - \cdots}}}},$$

$$\frac{3\sqrt{3}}{\pi} = 2 - \cfrac{2(1\cdot 1)}{7 - \cfrac{2(2\cdot 3)}{12 - \cfrac{2(3\cdot 5)}{17 - \cdots}}}.$$

They were converted from the following series:

$$\frac{\pi}{2} = 1 + \frac{1}{3} + \frac{1\cdot 2}{3\cdot 5} + \frac{1\cdot 2\cdot 3}{3\cdot 5\cdot 7} + \frac{1\cdot 2\cdot 3\cdot 4}{3\cdot 5\cdot 7\cdot 9} + \cdots ,$$

$$\frac{2\pi}{3\sqrt{3}} = 1 + \frac{1}{2\cdot 3} + \frac{1\cdot 2}{3\cdot 4\cdot 5} + \frac{1\cdot 2\cdot 3}{4\cdot 5\cdot 6\cdot 7} + \frac{1\cdot 2\cdot 3\cdot 4}{5\cdot 6\cdot 7\cdot 8\cdot 9} + \cdots .$$

We will now obtain some new continued fractions for π.

3. Series with Linear Factors and Continued Fractions

If we define:

$$y_k = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)(2n+1)(2n+3)\cdots(2n+2k-3)(2n+2k-1)}$$

then the following recurrence relation with $k \in \mathbb{N}$ is easy to establish:

$$y_k = \frac{1}{k}\cdot y_{k-1} - \frac{1}{2k\cdot 1\cdot 3\cdot 5\cdots(2k-3)\cdot(2k-1)}.$$

Indeed, we have that:

$$\begin{aligned}
2ky_k &= 2k\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)(2n+1)\cdots(2n+2k-3)(2n+2k-1)} \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)(2n+1)\cdots(2n+2k-5)(2n+2k-3)} \\
&\qquad - \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n+1)\cdots(2n+2k-3)(2n+2k-1)}.
\end{aligned}$$

By shifting the index of summation, the last sum on the right can be rewritten as:

$$\begin{aligned}
&- \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{(2n-1)(2n+1)\cdots(2n+2k-5)(2n+2k-3)} \\
&= \frac{1}{1\cdot 3\cdot 5\cdots(2k-1)} - \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)(2n+1)\cdots(2n+2k-5)(2n+2k-3)}
\end{aligned}$$

immediately leading to the expected result. See also [15].

Using this recurrence with $y_0 = \dfrac{\pi}{4}$ (Leibnitz's series), we get a class of series with an increasing number of factors in the denominator:

$$\frac{\pi}{8} - \frac{1}{3} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)(2n+1)(2n+3)} \ , \tag{1}$$

$$\frac{\pi}{24} - \frac{11}{90} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)(2n+1)(2n+3)(2n+5)} \ ,$$

$$\frac{\pi}{96} - \frac{2}{63} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)(2n+1)(2n+3)(2n+5)(2n+7)}.$$

Following Euler's method, we obtain

$$\frac{8}{3\pi - 8} = 5 + \cfrac{1 \cdot 5}{6 + \cfrac{3 \cdot 7}{6 + \cfrac{5 \cdot 9}{6 + \cfrac{7 \cdot 11}{6 + \cdots}}}} \ ,$$

$$\frac{24}{15\pi - 44} = 7 + \cfrac{1 \cdot 7}{8 + \cfrac{3 \cdot 9}{8 + \cfrac{5 \cdot 11}{8 + \cfrac{7 \cdot 13}{8 + \cdots}}}} \ ,$$

$$\frac{96}{105\pi - 320} = 9 + \cfrac{1 \cdot 9}{10 + \cfrac{3 \cdot 11}{10 + \cfrac{5 \cdot 13}{10 + \cfrac{7 \cdot 15}{10 + \cdots}}}}.$$

It may be noted that the numerator of the left hand side is $4(k!)$ while the multiple of π is $(2k-1)!!$, where $(2k-1)$ is the constant in the largest factor $2n+2k-1$ in the denominator of the series. These continued fractions are all special cases of:

***Theorem* 1** — *Let* $s = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\prod_{j=1}^{k}(2n+2j-3)}$, *then we have*

$$\frac{1}{(2k-1)!!\,s} = 2k+1+\cfrac{1\cdot(2k+1)}{2k+2+\cfrac{3\cdot(2k+3)}{2k+2+\cfrac{5\cdot(2k+5)}{2k+2+\cfrac{7\cdot(2k+7)}{2k+2+\cdots}}}}.$$

To illustrate how the continued fractions are derived from the corresponding series (Euler's theorem I), we give an example (see also [14]). We will convert the following series given in [12, p. 269, Ex. 109(c)]:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n(2n+1)(2n+2)} = \frac{\pi-3}{4} \tag{2}$$

into

$$\frac{1}{\pi-3} = 6+\cfrac{1^2}{6+\cfrac{3^2}{6+\cfrac{5^2}{6+\cdots}}},$$

the "new" continued fraction derived in 1999 by Lange [13] though it already occurs in a 1988 paper of Castellanos [4] who rightly ascribed it to Euler. This continued fraction is related to Brouncker's.

Let us define $z_n = (-1)^n \left(\frac{\pi-3}{4} - \sum_{k=1}^{n} \frac{(-1)^{k-1}}{2k(2k+1)(2k+2)} \right)$. From this it follows that:

$$z_n + z_{n-1} = \frac{1}{2n(2n+1)(2n+2)}, \text{ and } z_{n+1} + z_n = \frac{1}{(2n+2)(2n+3)(2n+4)}.$$

Dividing both equations, we find:

$$\frac{z_{n+1}+z_n}{z_n+z_{n-1}} = \frac{n(n+1)(2n+1)}{(n+1)(n+2)(2n+3)} = \frac{n(2n+1)}{(n+2)(2n+3)}.$$

We now get rid of the denominators:

$$(2n+3)(n+2)z_{n+1} + [(2n+3)(n+2) - (2n+1)n]z_n = (2n+1)nz_{n-1}$$

or

$$(2n+3)(n+2)z_{n+1} + 6(n+1)z_n = (2n+1)nz_{n-1}.$$

If we use the transformation $w_n = (n+1)z_n$, we have:

$$(2n+3)w_{n+1} + 6w_n = (2n+1)w_{n-1} \Rightarrow (2n+3)\frac{w_{n+1}}{w_n} + 6 = \frac{2n+1}{\frac{w_n}{w_{n-1}}}$$

giving the relation that generates the continued fraction on taking $n = 1, 2, 3, \cdots$

$$\frac{w_n}{w_{n-1}} = \frac{(2n+1)}{6 + (2n+3)\frac{w_{n+1}}{w_n}}.$$

The value of the continued fraction obtained in this way is given by:

$$\frac{w_1}{w_0} = \frac{2z_1}{z_0} = 2 \cdot \frac{-(\frac{\pi-3}{4} - \frac{1}{2\cdot 3\cdot 4})}{\frac{\pi-3}{4}}.$$

Some manipulations lead to the desired form. Other continued fractions for π can be obtained in a similar way. We derived this interesting series by combining (1) and (2):

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)2n(2n+1)(2n+2)(2n+3)} = \frac{10-3\pi}{72} \tag{3}$$

which gives the second convergent for π :

$$\pi = \frac{22}{7} - 24\sum_{n=2}^{\infty} \frac{(-1)^n}{(2n+1)(2n+2)(2n+3)(2n+4)(2n+5)}.$$

We converted the preceding series into the following continued fraction:

$$\frac{6}{10-3\pi} = 10 + \cfrac{1\cdot 5}{10 + \cfrac{3\cdot 7}{10 + \cfrac{5\cdot 9}{10 + \cfrac{7\cdot 11}{10 + \cfrac{9\cdot 13}{10 + \cdots}}}}}$$

which may be compared with the continued fraction derived earlier (with $a_k = 10,\ \forall\ k \in \mathbb{N}$) and one given by Euler in [9, §12] and also by Osler [16]:

$$\frac{16}{\pi} = 5 + \cfrac{1^2}{10 + \cfrac{3^2}{10 + \cfrac{5^2}{10 + \cfrac{7^2}{10 + \cfrac{9^2}{10 + \cdots}}}}}$$

Osler [16] (also in [17, p. 226, eq(28a)]) gives the following two classes of continued fractions related to Brouncker's. They were already known to John Wallis [20]. Let

$$P_0 = 1, \text{ and } P_n = \prod_{k=1}^{n} \frac{(2k-1)(2k+1)}{(2k)^2}.$$

Then for $n = 0, 1, 2, 3, \cdots$ we have:

$$(4n+1) + \cfrac{1^2}{2(4n+1) + \cfrac{3^2}{2(4n+1) + \cfrac{5^2}{2(4n+1) + \cdots}}} = \frac{(2n+1)}{P_n}\frac{4}{\pi},$$

$$(4n+3) + \cfrac{1^2}{2(4n+3) + \cfrac{3^2}{2(4n+3) + \cfrac{5^2}{2(4n+3) + \cdots}}} = (2n+1)P_n\pi.$$

These formulas are special cases of the known formula

$$\frac{4\,\Gamma\left(\dfrac{x+3+y}{4}\right)\Gamma\left(\dfrac{x+3-y}{4}\right)}{\Gamma\left(\dfrac{x+1+y}{4}\right)\Gamma\left(\dfrac{x+1-y}{4}\right)} = x + \cfrac{1^2-y^2}{2x + \cfrac{3^2-y^2}{2x + \cfrac{5^2-y^2}{2x+\cdots}}}$$

valid for either y an odd integer and x any complex number or y any complex number and $\Re(x) > 0$. Originally due to Euler [6, §67], it occurs in an inverted form in [1, p. 140].

4. Series with Quadratic Factors and Continued Fractions

Ramanujan's Notebook II [1, pp. 151, 153] contains these two continued fractions for Catalan's constant $G = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^2} = 0.915965594177\ldots$:

$$2G = 2 - \cfrac{1}{3 + \cfrac{2^2}{1 + \cfrac{2^2}{3 + \cfrac{4^2}{1 + \cfrac{4^2}{3+\cdots}}}}}$$

and

$$2G = 1 + \cfrac{1^2}{\frac{1}{2} + \cfrac{1\cdot 2}{\frac{1}{2} + \cfrac{2^2}{\frac{1}{2} + \cfrac{2\cdot 3}{\frac{1}{2} + \cfrac{3^2}{\frac{1}{2} + \cdots}}}}}.$$

We find in Ramanujan's Manuscript Book 1 [18, Ch. XIV, p. 107, entry 14] (and in [1, p. 123, entry 16]) this formula with $m, n \notin \mathbb{Z}^-$:

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(m+k)(n+k)} = \cfrac{1}{m+n+1+mn + \cfrac{(m+1)^2(n+1)^2}{m+n+3 + \cfrac{(m+2)^2(n+2)^2}{m+n+5 + \cfrac{(m+3)^2(n+3)^2}{m+n+7+\cdots}}}}$$

Setting $m = n = -\frac{1}{2}$ and doing a little manipulation, the last formula yields a fraction given in [2]:

$$G = \cfrac{1}{1 + \cfrac{1^4}{8 + \cfrac{3^4}{16 + \cfrac{5^4}{24 + \cdots}}}}. \tag{4}$$

Using a generalisation of the series defining G, we will obtain new continued fractions for this constant and for π. In the *Appendix* we prove that y_k defined by

$$y_k = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^2(2n+1)^2(2n+3)^2 \cdots (2n+2k-3)^2}$$

satisfies the recurrence

$$y_k = \frac{10k^2+8k+1}{2(2k+1)!!^2} - 4k(k+1)^3 y_{k+2}.$$

Since $y_1 = G$ and $y_2 = \frac{1}{2} - \frac{\pi}{8}$ (which can be proved using a partial fraction expansion and telescoping), the previous recurrence gives for y_4, y_6 and y_8:

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^2(2n+1)^2(2n+3)^2(2n+5)^2}=\frac{2!^3}{4!^3 1!}\pi-\frac{7}{4050}$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^2(2n+1)^2\cdots(2n+9)^2}=-\frac{3!^3}{6!^3 2!}\pi+\frac{41}{44651250}$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^2(2n+1)^2\cdots(2n+13)^2}=\frac{4!^3}{8!^3 3!}\pi-\frac{14789}{134221791453750}$$

and for y_3, y_5 and y_7:

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^2(2n+1)^2(2n+3)^3}=-\frac{1!!^3 1!}{2!^4 2^1}G+\frac{19}{576}$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^2(2n+1)^2(2n+3)^2(2n+5)^2(2n+7)^3}=\frac{3!!^3 2!}{4!^4 2^2}G-\frac{3919}{108380160}$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^2(2n+1)^2\cdots(2n+11)^2}=-\frac{5!!^3 3!}{6!^4 2^3}G+\frac{22133579}{2549361475584000}.$$

Note that an *odd number* of consecutive odd squares in the denominator gives a series for G while an *even number* of these squares gives a series for π.

In general we found the following sums for the series related to π: $(k=0,1,2,\cdots)$

$$\sum_{n=1-k}^{\infty}\frac{(-1)^{n-1}}{\prod_{j=0}^{2k+1}(2n+2j-1)^2}=(-1)^{k+1}\left(\frac{(k+1)!^3}{(2k+2)!^3 k!}\pi-\frac{1}{2(2k+1)!!^4}\right).$$

For the series above the corresponding continued fractions are given in the following theorem:

***Theorem* 2** — *Let*

$$S=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^2(2n+1)^2(2n+3)^2\cdots(2n+(2k-1))^2}.$$

Then,

$$\frac{1}{(2k-1)!!^2\,S}=(2k+1)^2+\cfrac{1^2(2k+1)^2}{(2k+3)^2-1^2+\cfrac{3^2(2k+3)^2}{(2k+5)^2-3^2+\cfrac{5^2(2k+5)^2}{(2k+7)^2-5^2+\cdots}}}.$$

Using the above formulas and the theorem we get:

4.1 *Continued fractions for π*

The values $k = 1,\ 3,\ 5$ give

$$\frac{8}{4-\pi} = 3^2 + \cfrac{1^2 3^2}{(5^2-1^2) + \cfrac{3^2 5^2}{(7^2-3^2) + \cfrac{5^2 7^2}{(9^2-5^2) + \cfrac{7^2 9^2}{(11^2-7^2) + \cdots}}}}\,,$$

$$\frac{576}{75\pi - 224} = 7^2 + \cfrac{1^2 7^2}{(9^2-1^2) + \cfrac{3^2 9^2}{(11^2-3^2) + \cfrac{5^2 11^2}{(13^2-5^2) + \cfrac{7^2 13^2}{(11^2-7^2) + \cdots}}}}$$

and

$$\frac{25600}{20992 - 6615\pi} = 11^2 + \cfrac{1^2 11^2}{(13^2-1^2) + \cfrac{3^2 13^2}{(15^2-3^2) + \cfrac{5^2 15^2}{(17^2-5^2) + \cfrac{7^2 17^2}{(19^2-7^2) + \cdots}}}}.$$

4.2 *Continued fractions for G*

The values $k = 0,\ 2,\ 4$ give

$$\frac{1}{G} = 1^2 + \cfrac{1^4}{(3^2-1^2) + \cfrac{3^4}{(5^2-3^2) + \cfrac{5^4}{(7^2-5^2) + \cfrac{7^4}{(9^2-7^2) + \cdots}}}}\,,$$

(this is the same one as (4))

$$\frac{64}{19-18G} = 5^2 + \cfrac{1^2 5^2}{(7^2-1^2) + \cfrac{3^2 7^2}{(9^2-3^2) + \cfrac{5^2 9^2}{(11^2-5^2) + \cfrac{7^2 11^2}{(13^2-7^2)+\cdots}}}}$$

and

$$\frac{49152}{22050G-19595} = 9^2 + \cfrac{1^2 9^2}{(11^2-1^2) + \cfrac{3^2 11^2}{(13^2-3^2) + \cfrac{5^2 13^2}{(15^2-5^2) + \cfrac{7^2 15^2}{(17^2-7^2)+\cdots}}}}.$$

5. Other Continued Fractions for G

While looking for continued fractions for G related to (4) in the same way that Lange's continued fraction for π is related to Brouncker's, we found these:

$$\frac{2^5}{6G-1} = 7 + 3\cdot\cfrac{1^4}{3(3^2-1^2) + \cfrac{3^4}{3(5^2-3^2) + \cfrac{5^4}{3(7^2-5^2) + \cfrac{7^4}{3(9^2-7^2)+\cdots}}}} \tag{5}$$

$$\frac{2^{13}}{82G-19} = 145 + 41\cdot\cfrac{1^4}{5(3^2-1^2) + \cfrac{3^4}{5(5^2-3^2) + \cfrac{5^4}{5(7^2-5^2) + \cfrac{7^4}{5(9^2-7^2)+\cdots}}}}$$

$$\frac{2^{17}}{882G-\frac{713}{3}} = 229 + 49\cdot\cfrac{1^4}{7(3^2-1^2) + \cfrac{3^4}{7(5^2-3^2) + \cfrac{5^4}{7(7^2-5^2) + \cfrac{7^4}{7(9^2-7^2)+\cdots}}}}$$

and it goes on like this. We'll prove (5). This continued fraction is related to the following series:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(4(n-1)^2+3)(4n^2+3)(2n-1)^2}$$

as can be checked using Euler's method to transform a series into a continued fraction. Now the sum of this series can be calculated like this:

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(4(n-1)^2+3)(4n^2+3)(2n-1)^2} \\
&= \sum_{n=1}^{\infty} (-1)^{n-1} \left(-\frac{1}{32(4(n-1)^2+3)} - \frac{1}{32(4n^2+3)} + \frac{1}{16(2n-1)^2} \right) \\
&= -\frac{1}{32} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{4(n-1)^2+3} - \frac{1}{32} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{4n^2+3} + \frac{1}{16} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^2} \\
&= -\frac{1}{3 \cdot 32} + \frac{1}{16} G
\end{aligned}$$

as a consequence of telescoping and the definition of G.

The series needed to prove the value of the other 2 continued fractions for G are:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{r(n-1)r(n)(2n-1)^2}$$

with $r(n) = 16n^4 + 88n^2 + 41$ and $r(n) = 64n^6 + 1168n^4 + 3628n^2 + 1323$ respectively.

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APPENDIX

The series y_m ($m = 1, 2, 3, \ldots$) defined by

$$y_m = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^2(2n+1)^2(2n+3)^2 \cdots (2n+2m-3)^2}$$

satisfy the recurrence relation:

$$y_m = \frac{10m^2+8m+1}{2(2m+1)!!^2} - 4m(m+1)^3 y_{m+2}.$$

Proof. By manipulating the terms of the series, we immediately get the result. We have:

$$\begin{aligned}
y_{m+2} &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^2(2n+1)^2 \cdots (2n+2m-1)^2(2n+2m+1)^2} \\
&= \frac{1}{2m+2} \sum_{n=1}^{\infty} (-1)^{n-1} \left[\frac{1}{(2n-1)^2(2n+1)^2 \cdots (2n+2m-1)^2(2n+2m+1)} \right. \\
&\qquad \left. - \frac{1}{(2n-1)(2n+1)^2 \cdots (2n+2m+1)^2} \right] \\
&= \frac{1}{(2m+2)^2} \sum_{n=1}^{\infty} (-1)^{n-1} \left[\frac{1}{(2n-1)^2(2n+1)^2 \cdots (2n+2m-1)^2} \right. \\
&\qquad - \frac{1}{(2n-1)(2n+1)^2 \cdots (2n+2m-1)^2(2n+2m+1)} \\
&\qquad - \frac{1}{(2n-1)(2n+1)^2 \cdots (2n+2m-1)^2(2n+2m+1)} \\
&\qquad \left. + \frac{1}{(2n+1)^2(2n+3)^2 \cdots (2n+2m+1)^2} \right]
\end{aligned}$$

or

$$\begin{aligned}
(2m+2)^2 y_{m+2} &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^2(2n+1)^2 \cdots (2n+2m-1)^2} \\
&\quad - 2 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)(2n+1)^2 \cdots (2n+2m-1)^2(2n+2m+1)} \\
&\quad - \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^2(2n+1)^2 \cdots (2n+2m-1)^2}
\end{aligned}$$

where we have shifted the summation index of the last series. Now the first and last sum cancel out, leaving one term, and the previous equation reduces to:

$$(2m+2)^2 y_{m+2} = \frac{1}{(2m+1)!!^2} - 2 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)(2n+1)^2 \cdots (2n+2m-1)^2(2n+2m+1)}.$$

We now repeat the two steps we used at the beginning of the proof to rewrite the remaining series. We get:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)(2n+1)^2 \cdots (2n+2m-1)^2(2n+2m+1)}$$
$$= \frac{1}{2m(2m+2)} \left[\frac{1}{(2m-1)!!^2(2m+1)} - 2\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n+1)^2(2n+3)^2 \cdots (2n+2m-1)^2} \right]$$
$$= \frac{1}{2m(2m+2)} \left[\frac{1}{(2m-1)!!^2(2m+1)} + 2y_m - \frac{2}{(2m-1)!!^2} \right]$$

where again we have shifted the summation index in the last step. Combining everything leads to the desired result. □

Note: E. Fabry (*Théorie des Séries à Termes Constants: Applications aux Calculs Numériques*, A. Hermann & fils, Paris, 1910, p.135) proved this recurrence using Kummer's transformation. Our proof is more straightforward.