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On $K_{p,q}$-FACTORIZATION OF COMPLETE BIPARTITE MULTIGRAPHS[1]

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Let $\lambda K_{m,n}$ be a complete bipartite multigraph with two partite sets having m and n vertices, respectively. A $K_{p,q}$-factorization of $\lambda K_{m,n}$ is a set of edge-disjoint $K_{p,q}$-factors of $\lambda K_{m,n}$ which partition the set of edges of $\lambda K_{m,n}$. When $p = 1$ and q is a prime number, Wang, in his paper [On $K_{1,q}$-factorization of complete bipartite graph, *Discrete Math*., **126**: (1994), 359-364], investigated the $K_{1,q}$-factorization of $K_{m,n}$ and gave a sufficient condition for such a factorization to exist. In papers [$K_{1,k}$-factorization of complete bipartite graphs, *Discrete Math*., **259**: 301-306 (2002), ; $K_{p,q}$-factorization of complete bipartite graphs, *Sci. China Ser. A-Math*., **47**: (2004), 473-479], Du and Wang extended Wang's result to the case that p and q are any positive integers. In this paper, we give a sufficient condition for $\lambda K_{m,n}$ to have a $K_{p,q}$-factorization. As a special case, it is shown that the necessary condition for the $K_{p,q}$-factorization of $\lambda K_{m,n}$ is always sufficient when $p : q = k : (k + 1)$ for any positive integer k.

Key words : Complete bipartite multigraph; factor; factorization.

1. INTRODUCTION

Let $K_{m,n}$ be a complete bipartite graph with two partite sets having m and n vertices, respectively. $\lambda K_{m,n}$ is the complete bipartite multigraph formed by replacing each edge of $K_{m,n}$ with λ edges. A subgraph F of $\lambda K_{m,n}$ is called a spanning subgraph of $\lambda K_{m,n}$ if F contains all the vertices of $\lambda K_{m,n}$. A G-factor of $\lambda K_{m,n}$ is a spanning subgraph F of $\lambda K_{m,n}$ such that every component of F is isomorphic to G. A G-factorization of $\lambda K_{m,n}$ is a set of edge-disjoint G-factors of $\lambda K_{m,n}$ which is a partition of the set of edges of $\lambda K_{m,n}$. The graph $\lambda K_{m,n}$ is called G-factorizable whenever it has

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a G-factorization. In paper [1] a G-factorization of $\lambda K_{m,n}$ is defined as a resolvable $(m, n, p+q, \lambda)$ G-design. For graph theoretical terms, see [2].

The $K_{p,q}$-factorizations and P_v-factorizations of $\lambda K_{m,n}$ have been studied by many researchers and found to have a number of applications. Especially, Yamamoto *et al*. [3, 4] have given some applications in $HUBFS_2$ and $HUBMFS_2$ schemes of database systems. There are also some known results on the existence of the $K_{p,q}$-factorizations and P_v-factorizations of $\lambda K_{m,n}$. In 1988, the existence of P_3-factorizations of $K_{m,n}$ has been completely solved by Ushio [5]. Notice that a P_3 is also $K_{1,2}$. Since then Ushio, Martin, Du, Wang *et al*. have some further researches in $K_{p,q}$-factorizations and P_v-factorizations of $\lambda K_{m,n}$. When v is an even number, Ushio [1], Wang [6] and Du [7] gave a necessary and sufficient condition for their existence. When v is an odd number, we in a series of papers [8-11] gave a necessary and sufficient condition for such factorizations to exist and completely solved the spectrum for existence of P_v-factorizations of $\lambda K_{m,n}$. For the $K_{p,q}$-factorizations of $\lambda K_{m,n}$, when $p = 1$ and $q = 3$, Martin in paper [12, 13] gave the necessary and sufficient condition for $K_{m,n}$ to have a $K_{1,3}$-factorization. When $p = 1$ and q is a prime number, Wang in [14] investigated the $K_{1,q}$-factorization of $K_{m,n}$ and gave a sufficient condition for such a factorization to exist. In papers [15-17], Du and Wang extended Wang's result to the case that p and q are any positive integers.

***Theorem* 1** — *Let p, q, m and n be positive integers with $p < q$. Assume $(1)\ pn \le qm,$ $(2)\ pm \le qn,$ $(3)\ qm - pn \equiv qn - pm \equiv 0\ (mod\ (q^2 - p^2)),$ $(4)\ (qm - pn)(qn - pm) \equiv 0\ (mod\ pq(q-p)(q^2 - p^2)(m+n))$. Then $K_{m,n}$ has a $K_{p,q}$-factorization.*

In this paper, we pay attention to the existence for the $K_{p,q}$-factorization of a complete bipartite multigraph $\lambda K_{m,n}$. For any positive integers p and q $(q > p)$, gcd(p, q) denote the greatest common divisor of p and q. We will give a sufficient condition for $\lambda K_{m,n}$ to have a $K_{p,q}$-factorization. As a special case, it is shown that the necessary conditions for the $K_{p,q}$-factorization of $\lambda K_{m,n}$ are always sufficient when $p : q = k : (k + 1)$ for any positive integer k. That is, we shall prove.

***Theorem* 2** — *Let m, n, p and q $(p < q)$ be positive integers with $pq > 1$. If $\lambda K_{m,n}$ has a $K_{p,q}$-factorization, then $(1)\ pn \le qm,$ $(2)\ pm \le qn,$ $(3)\ qm - pn \equiv qn - pm \equiv 0\ (mod\ (q^2 - p^2)),$ $(4)\ \lambda(qm - pn)(qn - pm) \equiv 0\ (mod\ pq(q^2 - p^2)(m+n)/d)$, where $d =$gcd(p, q).*

***Theorem* 3** — *Let p, q, m and n be positive integers with $p < q$. Assume $(1)\ pn \le qm,$ $(2)\ pm \le qn,$ $(3)\ qm - pn \equiv qn - pm \equiv 0\ (mod\ (q^2 - p^2)),$ $(4)\ \lambda(qm - pn)(qn - pm) \equiv 0\ (mod\ pq(q-p)(q^2 - p^2)(m+n)/d)$, where $d =$gcd(p, q). Then $\lambda K_{m,n}$ has a $K_{p,q}$-factorization.*

***Theorem* 4** — *The necessary conditions for $\lambda K_{m,n}$ to have a $K_{p,q}$-factorization is always sufficient when $p : q = k : (k + 1)$ for any positive integer k.*

2. PROOF OF THE NECESSARY CONDITION

We first give the proof of the necessary condition for $\lambda K_{m,n}$ to have a $K_{p,q}$-factorization.

***Theorem* 2** — *Let m, n, p and q $(p < q)$ be positive integers with $pq > 1$. If $\lambda K_{m,n}$ has a $K_{p,q}$-factorization, then* $(1)\ pn \le qm,\ (2)\ pm \le qn,\ (3)\ qm - pn \equiv qn - pm \equiv 0\ (mod\ (q^2 - p^2)),\ (4)\ \lambda(qm - pn)(qn - pm) \equiv 0\ (mod\ pq(q^2 - p^2)(m + n)/d)$, *where d=gcd(p, q).*

PROOF : Let X and Y be the two partite sets of $\lambda K_{m,n}$ with $|X| = m$ and $|Y| = n$. Let $\{F_1, F_2, \cdots, F_r\}$ be a $K_{p,q}$-factorization of $\lambda K_{m,n}$. In a particular $K_{p,q}$-factor, let a copies of $K_{p,q}$ with its partite set of size p in X and b copies with it in Y. Then we have $ap+bq = m$ and $aq+bp = n$. Thus,

$$a = \frac{qn - pm}{q^2 - p^2},\ b = \frac{qm - pn}{q^2 - p^2}.$$

Since $q > p$ and a and b are nonnegative integers, conditions (1), (2) and (3) are necessary. Let

$$c = \frac{\lambda(qm - pn)(qn - pm)}{pq(p + q)(m + n)}.$$

Then

$$r = \frac{\lambda(p + q)mn}{pq(m + n)} = \lambda(a + b) + c.$$

Thus c is an integer. Let $u \in X$. Suppose that there are only r' F_i's, each of which contains u contributing q edges. Then $qr' + p(r - r') = \lambda n$, i.e. $(q - p)r' + p(\lambda a + \lambda b + c) = (\lambda aq + \lambda bp)$. Therefore $c \equiv 0\ (mod\ (q - p)/d)$, where d =gcd(p, q), i.e. $\lambda(qm - pn)(qn - pm) \equiv 0\ (mod\ pq$ $(q^2 - p^2)(m + n)/d)$. Therefore the condition (4) is necessary. This proves Theorem 2.

When p and q are coprime, we have the following condition.

Corollary 1 — Let p and q $(q > p)$ be a coprime pair of positive integers. If $\lambda K_{m,n}$ has a $K_{p,q}$-factorization, then $(1)\ pn \le qm,\ (2)\ pm \le qn,\ (3)\ qm - pn \equiv qn - pm \equiv 0\ (mod\ (q^2 - p^2)),\ (4)\ \lambda(qm - pn)(qn - pm) \equiv 0\ (mod\ pq(q^2 - p^2)(m + n))$.

3. MAIN RESULT

In this section, we prove the following main result.

***Theorem* 5** — *Let p and q $(q > p)$ be a coprime pair of positive integers. Assume* $(1)\ pn \le qm,\ (2)\ pm \le qn,\ (3)\ qm - pn \equiv qn - pm \equiv 0\ (mod\ (q^2 - p^2)),\ (4)\ \lambda(qm - pn)(qn - pm) \equiv 0\ (mod\ pq(q - p)(q^2 - p^2)(m + n))$. *Then $\lambda K_{m,n}$ has a $K_{p,q}$-factorization.*

The proof of Theorem 5 consists of some lemmas. The following two lemmas are obvious.

Lemma 1 — Let $u,\ v,\ x$ and y be positive integers. If gcd$(ux, vy) = 1$, then gcd$(uv, ux + vy)$ $= 1$.

Lemma 2 — If $\lambda K_{m,n}$ has a $K_{p,q}$-factorization, then $s\lambda K_{m,n}$ has a $K_{p,q}$-factorization for any positive integer s.

Lemma 3 — If $\lambda K_{m,n}$ has a $K_{p,q}$-factorization, then $\lambda K_{ms,ns}$ has a $K_{p,q}$-factorization for any positive integer s.

PROOF : Let $\{F_i : 1 \leq i \leq s\}$ be a 1-factorization of $K_{s,s}$ (whose existence see [2]). For each $1 \leq i \leq s$, replace every edge of F_i by a $\lambda K_{m,n}$ to get a factor G_i of $\lambda K_{ms,ns}$ such that the graph G_i are pairwise edge-disjoint and their union is $\lambda K_{ms,ns}$. Since $\lambda K_{m,n}$ has a $K_{p,q}$-factorization, it is clear that the graph G_i, too, has a $K_{p,q}$-factorization. Consequently, $\lambda K_{ms,ns}$ has a $K_{p,q}$-factorization.

A corollary of Lemma 3 is as follows.

Corollary 2 — $\lambda K_{ps,qs}$ has a $K_{p,q}$-factorization for any positive integer s.

Corollary 2 implies that we only need to treat the case $qn > pm$ and $qm > pn$. Let

$$a = \frac{qn - pm}{q^2 - p^2},\ \ b = \frac{qm - pn}{q^2 - p^2},\ = \frac{\lambda(qm - pn)(qn - pm)}{pq(p + q)(m + n)}\ , r = \frac{\lambda(p+q)mn}{pq(m+n)}.$$

From conditions (1)-(4) in Theorem 5, $a,\ b,\ r$ are positive integers. Note that $ap + bq = m$, $aq + bp = n$ and $r = \lambda(a + b) + c$. This implies that $a < m$, $b < n$ and r is an integer. And let $gcd(ap, bq) = d$ with $ap = de$, $bq = dk$ and $gcd(e, k) = 1$. Since $c \equiv 0\ (mod\ (q - p)^2)$. Let $z = c/(q - p)^2$. Then these equalities imply the following equalities:

$$d = \frac{pq(qe + pk)z}{\lambda ek},\ \ r = \frac{(e + k)(q^2e + p^2k)z}{ek},\ \ m = \frac{pq(e + k)(qe + pk)z}{\lambda ek},$$

$$n = \frac{(q^2e + p^2k)(qe + pk)z}{\lambda ek},\ \ a = \frac{qe(qe + pk)z}{\lambda ek},\ \ b = \frac{pk(qe + pk)z}{\lambda ek}.$$

Let $q = {q_1}^{k_1}{q_2}^{k_2} \cdots {q_\gamma}^{k_\gamma}$, where $q_1,\ q_2,\ \cdots,\ q_\gamma$ are distinct prime numbers and $k_1,\ k_2, \cdots,\ k_\gamma$ are positive integers, and $p = {p_1}^{h_1}{p_2}^{h_2} \cdots {p_\omega}^{h_\omega}$, where $p_1,\ p_2,\ \cdots,\ p_\omega$ are distinct prime numbers and $h_1,\ h_2,\ \cdots,\ h_\omega$ are positive integers.

If gcd$(k, q^2) = {q_1}^{i_1}{q_2}^{i_2} \cdots {q_\alpha}^{i_\alpha}{q_{\alpha+1}}^{2k_{\alpha+1}-i_{\alpha+1}}\ {q_{\alpha+2}}^{2k_{\alpha+2}-i_{\alpha+2}} \cdots {q_\beta}^{2k_\beta - i_\beta}{q_{\beta+1}}^{2k_{\beta+1}}$ ${q_{\beta+2}}^{2k_{\beta+2}} \cdots {q_\gamma}^{2k_\gamma}$, where $1 \leq \alpha \leq \beta \leq \gamma,\ 0 \leq i_j \leq k_j$ (when $1 \leq j \leq \alpha$) or $0 < i_j < k_j$ (when $\alpha + 1 \leq j \leq \beta$). And gcd$(e, p^2) = {p_1}^{j_1}{p_2}^{j_2} \cdots {p_\mu}^{j_\mu}{p_{\mu+1}}^{2h_{\mu+1}-j_{\mu+1}}\ {p_{\mu+2}}^{2h_{\mu+2}-j_{\mu+2}} \cdots {p_\nu}^{2h_\nu - j_\nu}$

${p_{\nu+1}}^{2h_{\nu+1}}{p_{\nu+2}}^{2h_{\nu+2}}\cdots{p_\omega}^{2h_\omega}$, where $1 \le \mu \le \nu \le \omega$, $0 \le j_i \le h_i$ (when $1 \le i \le \mu$) or $0 < j_i < h_i$ (when $\mu+1 \le i \le \nu$). Let

$$s = {q_1}^{i_1}{q_2}^{i_2}\cdots{q_\alpha}^{i_\alpha},\ \ t = {q_1}^{k_1-i_1}{q_2}^{k_2-i_2}\cdots{q_\alpha}^{k_\alpha-i_\alpha},\ \ u = {q_{\alpha+1}}^{i_{\alpha+1}}{q_{\alpha+2}}^{i_{\alpha+2}}\cdots{q_\beta}^{i_\beta},$$

$$v = {q_{\alpha+1}}^{k_{\alpha+1}-i_{\alpha+1}}{q_{\alpha+2}}^{k_{\alpha+2}-i_{\alpha+2}}\cdots{q_\beta}^{k_\beta-i_\beta},\ \ w = {q_{\beta+1}}^{k_{\beta+1}}{q_{\beta+2}}^{k_{\beta+2}}\cdots{q_\gamma}^{k_\gamma}.$$

$$s' = {p_1}^{j_1}{p_2}^{j_2}\cdots{p_\mu}^{j_\mu},\ \ t' = {p_1}^{h_1-j_1}{p_2}^{h_2-j_2}\cdots{p_\mu}^{h_\mu-j_\mu},\ \ u' = {p_{\mu+1}}^{j_{\mu+1}}{p_{\mu+2}}^{j_{\mu+2}}\cdots{p_\nu}^{j_\nu},$$

$$v' = {p_{\mu+1}}^{h_{\mu+1}-j_{\mu+1}}{p_{\mu+2}}^{h_{\mu+2}-j_{\mu+2}}\cdots{p_\nu}^{h_\nu-j_\nu},\ \ w' = {p_{\nu+1}}^{h_{\nu+1}}{p_{\nu+2}}^{h_{\nu+2}}\cdots{p_\omega}^{h_\omega}.$$

Then gcd$(k,q^2) = suv^2w^2$ and gcd$(e,p^2) = s'u'{v'}^2{w'}^2$.

Recall p and q are coprime, we can establish the following lemma.

Lemma 4 — If gcd$(k,q^2) = suv^2w^2$, and gcd$(e,p^2) = s'u'{v'}^2{w'}^2$. Let $k = suv^2w^2k'$, $e = s'u'{v'}^2{w'}^2e'$ and gcd$(sus'u'(tv'w'e'+vwt'k'),\lambda) = \lambda'$. Then

$$m = \frac{stus't'u'(s'u'{v'}^2{w'}^2e' + suv^2w^2k')(tv'w'e'+vwt'k')z'}{\lambda'},$$

$$n = \frac{suvws'u'v'w'(st^2ue' + s'{t'}^2u'k')(tv'w'e'+vwt'k')z'}{\lambda'},$$

$$a = \frac{stus'u'v'w'e'(tv'w'e'+vwt'k')z'}{\lambda'},\ \ b = \frac{suvws't'u'k'(tv'w'e'+vwt'k')z'}{\lambda'},$$

$$r = \frac{(s'u'{v'}^2{w'}^2e' + suv^2w^2k')(st^2ue' + s'{t'}^2u'k')z'\lambda}{\lambda'},\ \ d = \frac{stus't'u'(tv'w'e'+vwt'k')z'}{\lambda'},$$

for some positive integer z'.

PROOF : We assume that the gcd$(k,q^2) = suv^2w^2$, $gcd(e,p^2) = s'u'{v'}^2{w'}^2$ and $k = suv^2w^2k'$, $e = s'u'{v'}^2{w'}^2e'$ hold. Since gcd$(e,k) = 1$, we have gcd$(e',k') = 1$, gcd$(s'u'{v'}^2\ {w'}^2e', suv^2w^2k') = 1$, gcd$(st^2ue', s'{t'}^2u'k') = 1$. It is easy to see that

$$r = \frac{(s'u'{v'}^2{w'}^2e' + suv^2w^2k')(st^2ue' + s'{t'}^2u'k')z}{e'k'}.$$

By Lemma 1, we see that gcd$(e'k', s'u'{v'}^2{w'}^2e' + suv^2w^2k') = 1$ and gcd$(e'k', st^2ue' + s'{t'}^2u'k')$ $= 1$. Since r is an integer, therefore,

$$\frac{z}{e'k'}$$

must be an integer. Let

$$z_1 = \frac{z}{e'k'},$$

and let λ_1 =gcd$(\lambda, stus'u'v'w'e'(tv'w'e'+vwt'k'))$ and λ_2 =gcd$(\lambda, suvws't'u'k'(tv'w'e'+vwt'k'))$. By

$$a = \frac{stus'u'v'w'e'(tv'w'e' + vwt'k')z_1}{\lambda}$$

and

$$b = \frac{suvws't'u'k'(tv'w'e' + vwt'k')z_1}{\lambda},$$

we see that

$$\frac{\lambda_1 z_1}{\lambda}$$

and

$$\frac{\lambda_2 z_1}{\lambda}$$

must be integers. Since gcd $(tv'w'e', vwt'k') = 1$, so we have

$$\frac{z_1\lambda'}{\lambda}$$

must be an integer, where $gcd(sus'u'(tv'w'e' + vwt'k'), \lambda) = \lambda'$. Let

$$z' = \frac{z_1\lambda'}{\lambda},$$

then the equalities hold.

To complete the proof of Theorem 5, we need the following direct construction.

Lemma 5 — For any positive integers $s,\ t,\ u,\ v,\ w,\ s',\ t',\ u',\ v',\ w',\ e,\ k$ and λ, if

$$\frac{sus'u'(tv'w'e' + vwt'k')}{\lambda}$$

is an integer and

$$m = \frac{stus't'u'(s'u'v'^2w'^2e + suv^2w^2k)(tv'w'e + vwt'k)}{\lambda},$$

$$n = \frac{suvws'u'v'w'(st^2ue + s't'^2u'k)(tv'w'e + vwt'k)}{\lambda},$$

then $\lambda K_{m,n}$ has a $K_{s't'u'v'w',stuvw}$-factorization.

PROOF : Let

$$a = \frac{stus'u'v'w'e(tv'w'e+vwt'k)}{\lambda},\ b = \frac{suvws't'u'k(tv'w'e+vwt'k)}{\lambda},$$

$r = (s'u'v'^2w'^2e + suv^2w^2k)(st^2ue + s't'^2u'k),\ r_1 = s'u'v'^2w'^2e + suv^2w^2k$ and $r_2 = st^2ue + s't'^2u'k$. Let X and Y be two partite sets of $\lambda K_{m,n}$,

$$X = \{x_{i,j} :\ 1 \le i \le r_1;\ 1 \le j \le \tfrac{stus't'u'(tv'w'e+vwt'k)}{\lambda}\},$$

$$Y = \{y_{i,j} :\ 1 \le i \le r_2;\ 1 \le j \le \tfrac{suvws'u'v'w'(tv'w'e+vwt'k)}{\lambda}\}.$$

We will construct a $K_{s't'u'v'w',stuvw}$-factorization of $\lambda K_{m,n}$. We remark in advance that the additions in the first subscripts of $x_{i,j}$ and $y_{i,j}$ are taken modulo r_1 and r_2 in $\{1,\ 2,\ \cdots,\ r_1\}$ and $\{1,\ 2,\ \cdots,\ r_2\}$, respectively, and the additions in the second subscripts of $x_{i,j}$'s and $y_{i,j}$'s are taken modulo

$$\frac{stus't'u'(tv'w'e+vwt'k)}{\lambda}$$

and

$$\frac{suvws'u'v'w'(tv'w'e+vwt'k)}{\lambda}$$

in

$$\{1,\ 2,\ \cdots,\ \frac{stus't'u'(tv'w'e+vwt'k)}{\lambda}\}$$

and

$$\{1,\ 2,\ \cdots,\ \frac{suvws'u'v'w'(tv'w'e+vwt'k)}{\lambda}\},$$

respectively.

For each $i,\ x,\ x',\ y,\ y',\ z$ and $z',\ 1 \le i \le e,\ 1 \le x \le stu,\ 1 \le x' \le s'u'v'w',\ 1 \le y \le vw,\ 1 \le y' \le v'w',\ 1 \le z \le t$ and $1 \le z' \le t'$, let

$$f(z,z') = \frac{sus'u'(tv'w'e+vwt'k)(z'-1)}{\lambda} + \frac{sus't'u'(tv'w'e+vwt'k)(z-1)}{\lambda},$$

$g(i,x,z) = st^2u(i-1) + t(x-1) + z$ and $h(i,x,x',y,y') = stu(i-1)+$

$$stu(x'-1)e+\frac{suvws'u'(tv'w'e+vwt'k)(y'-1)}{\lambda}+\frac{sus'u'(tv'w'e+vwt'k)(y-1)}{\lambda}+x,$$

and let

$$E_i=\{x_{s'u'v'^2w'^2(i-1)+s'u'v'w'(y'-1)+x',f(z,z')+j}y_{g(i,x,z),h(i,x,x',y,y')+j}:$$
$$1\le j\le\frac{sus'u'(tv'w'e+vwt'k)}{\lambda},\ 1\le x\le stu,\ 1\le x'\le s'u'v'w',$$
$$1\le y\le vw,\ 1\le y'\le v'w',\ 1\le z\le t,\ 1\le z'\le t'\}.$$

For each $i,\ x,\ x',\ y,\ y'$ and $z,\ 1\le i\le k,\ 1\le x\le suvw,\ 1\le x'\le v'w',\ 1\le y\le vw,\ 1\le y'\le s't'u'$ and $1\le z\le t'$, let $\varphi(i,x,y)=s'u'v'^2w'^2e+suv^2w^2(i-1)+vw(x-1)+y,\ \theta(i,y',z)=st^2ue+s't'^2u'(i-1)+s't'u'(z-1)+y'$ and $\psi(i,x,x',y,y')=stus'u'v'w'e+$

$$\frac{suvws'u'(tv'w'e+vwt'k)(x'-1)}{\lambda}+\frac{sus'u'(tv'w'e+vwt'k)(y-1)}{\lambda}+suvwk(y'-1)+$$
$$suvw(i-1)+x,$$

and let

$$E_{e+i}=\{x_{\varphi(i,x,y),\frac{stus'u'(tv'w'e+vwt'k)(z-1)}{\lambda}+j}y_{\theta(i,y',z),\psi(i,x,x',y,y')+j}:$$
$$1\le j\le\tfrac{stus'u'(tv'w'e+vwt'k)}{\lambda},\ 1\le x\le suvw,\ 1\le x'\le v'w',$$
$$1\le y\le vw,\ 1\le y'\le s't'u',\ 1\le z\le t'\}.$$

Let $F=\cup_{1\le i\le e+k}E_i$. Then the graph F is a $K_{s't'u'v'w',stuvw}$-factor of $\lambda K_{m,n}$. Define a bijection σ from $X\cup Y$ onto $X\cup Y$ in such a way that $\sigma(x_{i,j})=x_{i+1,j}$, $\sigma(y_{i,j})=y_{i+1,j}$. For each $i\in\{1,\ 2,\ \cdots,\ r_1\}$ and each $j\in\{1,\ 2,\ \cdots,\ r_2\}$, let

$$F_{i,j}=\{\sigma^i(x)\sigma^j(y)|\ x\in X,\ y\in Y,\ xy\in F\}.$$

It is easy to show that the graphs $F_{i,j}$ $(1\le i\le r_1,\ 1\le j\le r_2)$ are $K_{s't'u'v'w',stuvw}$-factors of $\lambda K_{m,n}$ and their union is $\lambda K_{m,n}$. Thus, $\{F_{i,j}|\ 1\le i\le r_1,\ 1\le j\le r_2\}$ is a $K_{s't'u'v'w',stuvw}$-factorization of $\lambda K_{m,n}$.

PROOF OF THEOREM 5 : Applying Lemma 2 to Lemma 5, we see that for the parameters m and n satisfying the conditions in Theorem 5, $\lambda K_{m,n}$ has a $K_{p,q}$-factorization.

4. PROOF OF THE SUFFICIENT CONDITION

In this section, we shall give the proof of the sufficient condition for $\lambda K_{m,n}$ to have a $K_{p,q}$-factorization. We first give the following lemmas.

Lemma 6 — If $\lambda K_{m,n}$ has a $K_{p,q}$-factorization, then $\lambda K_{ms,ns}$ has a $K_{ps,qs}$-factorization for any positive integer s.

PROOF : Let $\{F_i : 1 \leq i \leq r\}$ be a $K_{p,q}$-factorization of $\lambda K_{m,n}$. For each $i \in \{1, 2, \cdots, r\}$, replace every vertex of F_i by s vertices and every edge of F_i by a $K_{s,s}$. Then we get a factor G_i of $\lambda K_{ms,ns}$ such that the graph G_i is pairwise edge-disjoint and their union is $\lambda K_{ms,ns}$. Therefore, $\lambda K_{ms,ns}$ has a $K_{ps,qs}$-factorization.

Lemma 7 — For any positive integer s, if m and n satisfy the necessary condition in Theorem 2 with $p = p_0 s$ and $q = q_0 s$, then there are positive integers m_0 and n_0 which satisfy the necessary condition in Theorem 2 with $p = p_0$ and $q = q_0$, such that $m = m_0 s$ and $n = n_0 s$.

PROOF : Suppose m and n satisfy the necessary condition in Theorem 2 with $p = p_0 s$ and $q = q_0 s$. Then we have

$$\frac{qn - pm}{q^2 - p^2} = \frac{q_0 sn - p_0 sm}{({q_0}^2 - {p_0}^2)s^2} = \frac{q_0(m+n)/[(q_0+p_0)s] - m/s}{q_0 - p_0},$$

$$\frac{qm - pn}{q^2 - p^2} = \frac{q_0 sm - p_0 sn}{({q_0}^2 - {p_0}^2)s^2} = \frac{q_0(m+n)/[(q_0+p_0)s] - n/s}{q_0 - p_0},$$

$$\frac{m+n}{(q_0+p_0)s} = \frac{qn - pm}{q^2 - p^2} + \frac{qm - pn}{q^2 - p^2}$$

are all integers. Hence m/s, n/s are integers. Let $m_0 = m/s$ and $n_0 = n/s$. It is easy to see that m_0 and n_0 satisfy the necessary conditions in Theorem 2 with $p = p_0$ and $q = q_0$.

The proof of the following lemma is similar as that of Lemma 7.

Lemma 8 — For any positive integer s, if m and n satisfy the sufficient conditions in Theorem 3 with $p = p_0 s$ and $q = q_0 s$, then there are positive integers m_0 and n_0 which satisfy the sufficient conditions in Theorem 3 with $p = p_0$ and $q = q_0$, such that $m = m_0 s$ and $n = n_0 s$.

PROOF OF THEOREM 3 : Applying Lemma 6, Lemma 8 and Theorem 5, we see that for the parameters m and n satisfying conditions in Theorem 3, $\lambda K_{m,n}$ has a $K_{p,q}$-factorization.

In the following, we give the proof of Theorem 4. When $p = k$ and $q = k + 1$, by Theorem 2, we have the following necessary condition for the $K_{k,k+1}$-factorization of $\lambda K_{m,n}$.

Lemma 9 — Let $\lambda,\ m,\ n$ and k be positive integers. If $\lambda K_{m,n}$ has a $K_{k,k+1}$-factorization, then $(1)\ kn \leq (k+1)m,\ (2)\ km \leq (k+1)n,\ (3)\ (k+1)m - kn \equiv (k+1)n - km \equiv 0\ (mod\ 2k+1),\ (4)\ \lambda((k+1)m-kn)((k+1)n-km) \equiv 0\ (mod\ k(k+1)(2k+1)(m+n))$.

When $p = k$ and $q = k+1$, by Theorem 3, the sufficient condition for $\lambda K_{m,n}$ to have a $K_{k,k+1}$-factorization as follows.

Lemma 10 — Let $\lambda,\ m,\ n$ and k be positive integers. Assume $(1)\ kn \leq (k+1)m,\ (2)\ km \leq (k+1)n,\ (3)\ (k+1)m - kn \equiv (k+1)n - km \equiv 0\ (mod\ 2k+1),\ (4)\ \lambda((k+1)m-kn)((k+1)n-km) \equiv 0\ (mod\ k(k+1)(2k+1)(m+n))$, then $\lambda K_{m,n}$ has a $K_{k,k+1}$-factorization.

PROOF OF THEOREM 4 : Combining the Lemma 6 to Lemma 10, we get the proof of Theorem 4.

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