Indian J. Pure Appl. Math., **48**(2): 245-257, June 2017

DOI: 10.1007/s13226-017-0226-7

MONOTONICITY OF STRATA IN THE STRATIFICATION OF THE CONE OF TOTALLY POSITIVE MATRICES

Amitava Ghosh

Durgapur Women's College, Durgapur, West Bengal, India
e-mail: amitava1729@gmail.com

(*Received* 2 *November* 2016; *accepted* 16 *November* 2016)

According to a theorem of Bjorner [5], there exists a stratified space whose strata are labeled by the elements of $[u, v]$ for every interval $[u, v]$ in the Bruhat order of a Coxeter group W, and each closed stratum (respectively the boundary of each stratum) has the homology of a ball (respectively of a sphere). In [6], Fomin and Shapiro suggest a natural geometric realization of these stratified spaces for a Weyl group W of a semi-simple Lie group G, and then prove its validity in the case of the symmetric group. The stratified spaces arise as links in the Bruhat decomposition of the totally non-negative part of the unipotent radical of G. In this article, we verify the topological regularity property of the strata formed as a result of Bruhat partial ordering on the elements of the Weyl group (of rank 4) of a semi-simple simply connected algebraic group G which is $SL(4, \mathbb{R})$ in our case here. The Weyl group here is the Coxeter group S_4.

Key words : Bruhat order; Weyl group; Coxeter group; strata; semi-monotone sets; monotone cells.

1. INTRODUCTION

Bjorner in [5] showed that every interval in the Bruhat order of a Coxeter group W is the *face poset* of some stratified space, in which each closed stratum (resp., the boundary of each stratum) has the homology of a ball (resp., of a sphere). Besides, he also proved a stronger result, namely, every interval in the Bruhat order is the face poset of a regular cell complex (i.e. closed strata actually are balls).

In [6], Fomin and Shapiro talked about a geometric construction for the case where W is the Weyl group of a semisimple Lie group G and proved that it holds in the case of the symmetric group. In the type A case, where W is the symmetric group and G the special linear group, they prove that their stratified spaces satisfy the required homological properties. The spaces that they construct are links of cells in the Bruhat decomposition of the totally nonnegative part of the unipotent radical of G.

Now let G be a semisimple, simply connected algebraic group defined and split over $\mathbb{R}$. Let B and B' be two opposite Borel subgroups of G, so that $H = B' \cap B$ is an $\mathbb{R}$-split maximal torus in G; we denote by N and N' the unipotent radicals of B and B', respectively.

While considering the type A_{n-1}, the group G is the real special linear group $SL(n, \mathbb{R})$; H, B and B' are the subgroups of diagonal, upper-triangular, and lower-triangular matrices, respectively; N and N' are the subgroups of B and B' that consist of matrices whose diagonal entries are equal to 1. We denote by Y the set of all totally nonnegative elements in N. In the case of the special linear group, Y consists of the upper-triangular unipotent matrices whose all minors are nonnegative. The general definition was first suggested by Lusztig (please look at [11] and references therein). In our current notation, Lusztig defined Y as the multiplicative submonoid of N generated by the elements $\exp(te_i)$, $t \geq 0$, where the $e_i's$ are the Chevalley generators of the Lie algebra of N. An alternative description in terms of nonnegativity of certain *generalized minors* was given in [8]. Let W be the Weyl group of G. The length of an element $w \in W$ is denoted by $l(w)$. The group W is partially ordered by the Bruhat order, defined geometrically by $u \leq v \iff B'uB' \subset \overline{B'vB'}$. The Bruhat decomposition $G = \bigcup_{w \in W} B'wB'$ induces the partition of Y into mutually disjoint totally positive varieties $Y_w^\circ = Y \cap B'wB'$, $w \in W$ (we are following the terminology as in [7]).

Let us denote $Y_w = \overline{Y_w^\circ}$. Lusztig was the first to study the varieties Y_w° in [10]. In this article, we verify some basic properties (topological regularity) of the strata formed by the semi-algebraic varieties which arise as a result of the Bruhat order on the elements of the Weyl group of rank 4. In our case here, the Weyl group is the finite Coxeter group S_4. We borrow and utilize ideas and concepts from [1] and [2] and show that the semi-algebraic sets or the strata are semi-monotone sets or monotone cells. The rank-3 case has been studied and verified in [6].

2. Stratification of the Semi-algebraic Set T (As Described Below) Induced by the Bruhat Partial Order on Symmetric Group S_4

Let T be the set of the upper-triangular unipotent matrices all of whose minors are non-negative i.e.
$$T = \left\{ \alpha = \begin{bmatrix} 1 & x & y & z \\ 0 & 1 & u & v \\ 0 & 0 & 1 & w \\ 0 & 0 & 0 & 1 \end{bmatrix} \mid ux - y \geq 0, vy - uz \geq 0, uw - v \geq 0, (x, y, z, u, v, w) \in \mathbb{R}^6_{\geq 0} \right\}$$

The semi-algebraic set T decomposes or partitions into 24 (i.e. 4-factorial) disjoint strata or semi-algebraic sets. The partition is induced by the Bruhat partial ordering on the elements of the symmetric group S_4 such that each stratum is homeomorphic to an open ball of the same dimension as itself. Furthermore, the boundary of a stratum is a union of lower dimension strata. Now there exists a matrix corresponding to each element of the Coxeter group S_4 and the reduced wordlength of a word in the Coxeter group is precisely the dimension of the corresponding stratum. In this article we aim to show that the strata are topologically regular via a route different from that undertaken by Hersh in [9]. From the two papers [1] and [2] by Basu, Gabrielov and Vorobjov, we know that semi-monotone sets and monotone cells are regular. We urge the readers to go through [1] and [2] to get acquainted with the concepts of semi-monotone sets and monotone cells. Here we shall show that the open and locally closed strata here are either semi-monotone sets or monotone cells. In the following sections, we describe what the matrix representations are corresponding to the elements of the Coxeter group. We also enlist the different strata varying in dimension from one to six. The identity element corresponds to the zero-dimensional stratum which is the origin in this case. The full 6-dimensional is the open semi-algebraic set T_0 which we discuss in the next section.

3. The Full 6-dimensional and Zero-dimensional Strata

Let $T_0 = \{(x, y, z, u, v, w) \in \mathbb{R}^6_{>0} \mid ux - y > 0, vy - uz > 0, uw - v > 0$ and $x, y, z, u, v, w \in \mathbb{R}^+\}$

The semi-algebraic set T_0 defined above is the 6-dimensional strata because it is an open connected set. We can make it bounded if we intersect it with an open ball of finite diameter. We will show that it is a semi-monotone set. The empty space is the zero-dimensional stratum. There are sub-strata of T and we need to figure out what they are. All the sub-strata except the full 6-dimensional stratum are not semi-monotone. They are infact locally closed monotone cells. A set is locally closed if it is an intersection of a closed subset of $\mathbb{R}^n$ with an open subset of $\mathbb{R}^n$. These substrata are monotone cells if their intersection with any coordinate subspace of $\mathbb{R}^6$ is connected if non-empty and they are graphs of quasi-affine maps respectively. The zero-dimensional stratum arises from the identity element from the permutation group S_n. There is nothing to be shown regarding connectivity of the zero-dimensional strata as it is empty and hence its intersection with all the affine subspaces of $\mathbb{R}^6$ will be empty too. Before we figure out the other sub-strata of T, let us try to show that the semi-algebraic set T_0(as defined above), intersected with coordinate affine sub-spaces of $\mathbb{R}^6$ is connected if non-empty. show this by considering different cases.

Case 1 : Let us fix a coordinate line in $\mathbb{R}^6$, i.e. let x, y, z, u, v be constant real numbers. When we intersect this with T, we get $w > v/u$, provided the other inequalities are satisfied i.e. $ux - y > 0$ and $vy - uz > 0$. The resulting intersection is $w > v/u$, a half-space, and hence connected in $\mathbb{R}^6$.

Case 2 : If we fix a plane now, (i.e. let x, y, z, u be constants) and intersect this plane with T, we also get a half-space contained inside $\mathbb{R}^6$, and hence is connected. We get the same result if we fix hyperplanes and intersect them respectively with T.

4. Bruhat Order, Coxeter Words and the Corresponding Sub-strata

The Weyl group in this context is S_4, the permutation group on four letters. The Bruhat order is on the elements of S_4 in terms of their reduced wordlengths. The dimension of the stratum corresponding to an element of the Coxeter group equals its reduced wordlength. If we refer to the Coxeter diagram of S_4, we can observe that there are three cycles of length 1, five of length 2, six of length 3, five of length 4 and three cycles of length 5. These correspond to 1-dimensional, 2-dimensional, 3-dimensional, 4-dimensional and 5-dimensional strata respectively. All these strata are respectively homeomorphic to open balls in $\mathbb{R}^6$ of that particular dimension. Besides these, we of course have one cycle of length 6 which corresponds to the full 6-dimensional strata which is T and the empty substratum which is the origin, from the identity element of S_4. All elements of S_4 can be expressed as product of three transpositions s_1=(12), s_2=(23) and s_3=(34). The 1-dimensional sub-strata correspond to these transpositions. The reduced word-length in terms of the just-mentioned three transpositions, of a specific element of S_4 corresponds to the sub-stratum described by it.

Now, s_i corresponds to $X_i(t) = I + tE_{i,i+1}$, where $E_{i,i+1}$ is the 4×4 matrix with the $(i, i+1)^{th}$ entry being 1 and zero elsewhere, I is the 4×4 identity matrix and t is a parameter and $t \in \mathbb{R}_+$. If we consider an element say $\pi \in S_4$ and let $\pi = s_{i_1}s_{i_2}....s_{i_m}$, where m is reduced length of the word in terms of product of transpositions, then the product of matrices $X_{i_1}(t_1)X_{i_2}(t_2)...X_{i_m}(t_m)$ as a function of m variables t_1, t_2 .... t_m ($t_i \in \mathbb{R}_+$) describes the sub-strata corresponding to π. The entries after matrix multiplication are x, y, z, u, v and w in that order and each of these are functions of the t_i's and looks as follows:

$$\begin{bmatrix} 1 & x & y & z \\ 0 & 1 & u & v \\ 0 & 0 & 1 & w \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Therefore we have $\phi_\pi : (t_1, ..., t_m) \to X_{i_1}(t_1)X_{i_2}(t_2)....X_{i_m}(t_m)$.

Hence $\phi_\pi : \mathbb{R}^m_{>0} \to T$.

Therefore, the stratum corresponding to the Coxeter element π of symmetric group S_n is $T_\pi = Im(\phi_\pi)$. When wordlength or m is zero, then it corresponds to the zero-dimensional stratum which is the origin.

5. 1-dimensional Sub-strata

The 1-dimensional sub-strata arise from the transpositions themselves i.e. from $s_1 = (12)$, $s_2 = (23)$ and $s_3 = (34)$. And after matrix multiplication, we can see that the strata corresponding to $s_1 = (12)$

is described by $T_{1\alpha}$ (say), where $T_{1\alpha} = \{(x, y, z, u, v, w) \in \mathbb{R}^6_{\geq 0} \mid x > 0, y = z = u = v = w = 0\}$ and the matrix representation corresponding to ϕ_{s_1} is :

$$\begin{bmatrix} 1 & t_1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The sub-stratum corresponding to the transposition $s_2 = (23)$ is $T_{1\beta}$ (say), where $T_{1\beta} = \{(x, y, z, u, v, w) \in \mathbb{R}^6_{\geq 0} \mid u > 0, x = y = z = v = w = 0\}$ and the matrix representation corresponding to ϕ_{s_2} is :

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & t_2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

And the sub-stratum corresponding to the transposition $s_3 = (34)$ is $T_{1\gamma}$ (say), where $T_{1\gamma} = \{(x, y, z, u, v, w) \in \mathbb{R}^6_{\geq 0} \mid w > 0, x = y = z = u = v = 0\}$ and the matrix representation corresponding to ϕ_{s_3} is :

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & t_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

And we if we intersect these three 1-dimensional sub-strata with affine subspaces of $\mathbb{R}^6$, it is obvious that they are connected if non-empty. We can check this as we did in the full 6-dimensional case, by fixing a line first, then a plane, then a hyper-plane and so on.

6. 2-DIMENSIONAL SUB-STRATA

The elements of the pemutation group S_4 corresponding to the 2-dimensional sub-strata have Coxeter wordlengths two i.e. these elements from S_4 are product of only two of the transpositions s_1, s_2 and s_3. For example, (234) has Coxeter length two because it is product of s_2 and s_3. Infact, (243), (12)(34), (312) and (231) are the rest of the elements of S_4 which have length two.

The sub-stratum corresponding to the cycle (231) = $s_1 s_2$ is $T_{2\alpha}$ (say), where $T_{2\alpha} = \{(x, y, z, u, v, w)$

$\in \mathbb{R}^6_+ \mid y = ux, z = v = w = 0\}$ and the corresponding matrix representation is :

$$\begin{bmatrix} 1 & t_1 & t_1t_2 & 0 \\ 0 & 1 & t_2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The sub-stratum corresponding to the cycle (213) = s_2s_1 is $T_{2\beta}$ (say), where $T_{2\beta} = \{(x, y, z, u, v, w) \in \mathbb{R}^6_+ \mid x > 0, u > 0, y = z = v = w = 0\}$ and the corresponding matrix representation is :

$$\begin{bmatrix} 1 & t_2 & 0 & 0 \\ 0 & 1 & t_1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The sub-stratum corresponding to the cycle (12)(34) = s_1s_3 is $T_{2\gamma}$ (say), where $T_{2\gamma} = \{(x, y, z, u, v, w) \in \mathbb{R}^6_+ \mid x > 0, w > 0, y = z = u = v = 0\}$ and the corresponding matrix representation is :

$$\begin{bmatrix} 1 & t_1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & t_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The sub-stratum corresponding to the cycle (234) = s_2s_3 is $T_{2\delta}$ (say), where $T_{2\delta} = \{(x, y, z, u, v, w) \in \mathbb{R}^6_+ \mid v = uw, x = y = z = 0\}$ and the corresponding matrix representation is :

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & t_1 & t_1t_2 \\ 0 & 0 & 1 & t_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The sub-stratum corresponding to the cycle (243) = s_3s_2 is $T_{2\theta}$ (say), where $T_{2\theta} = \{(x, y, z, u, v, w) \in \mathbb{R}^6_+ \mid u > 0, w > 0, x = y = z = v = 0\}$ and the corresponding matrix representation is :

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & t_2 & 0 \\ 0 & 0 & 1 & t_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

So, these were the sub-strata of the semi-algebraic set T and these all have dimension 2. Let us check connectivity of this strata. If we fix u or w, the intersection is an open interval on the w and u coordinate lines and they are connected. If we don't fix anything, we get a half-space in uw-coordinate plane and this is connected. The rest of these are very similar and all of these, when intersected with affine subspaces of $\mathbb{R}^6$ give rise to semi-algebraic sets which are connected when non-empty.

7. 3-DIMENSIONAL SUB-STRATA

From the *Bruhat order of* S_4 diagram on page no. 31 of the book *Combinatorics of Coxeter Groups* by Bjorner and Brenti, we get to know the reduced Coxeter-word decomposition of various elements of the permutation groups S_4. The six elements of S_4 which belong to this category are (1234), (1243), (1342), (1432), (13) and (24).

The sub-strata corresponding to the cycle (1234) = $s_1 s_2 s_3$ is $T_{3\alpha}$ (say), where $T_{3\alpha} = \{(x, y, z, u, v, w) \in \mathbb{R}^6_+ \mid y = ux, v = uw, z = uxw\}$. And the corresponding matrix product is :

$$\begin{bmatrix} 1 & t_1 & t_1 t_2 & t_1 t_2 t_3 \\ 0 & 1 & t_2 & t_2 t_3 \\ 0 & 0 & 1 & t_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

To prove connectivity of this strata, we fix a line in $\mathbb{R}^6$ first i.e. we fix all the variables except z and what we get is a point in $\mathbb{R}^6$. Same is the case when we fix four variables. If we fix three variables, we get intersection of two lines in $\mathbb{R}^6$ and that is connected, since all the variables are positive reals. If we fix two variables say x and y, we see that u is also fixed and v and z are both "a constant times w" respectively. Therefore, the resulting intersection is $z = k.v$ (k is a constant), which is a 3-dimensional hyperplane in $\mathbb{R}^6$ and hence is connected.

And when we fix one variable, say $x = k$, we get $y = ku$ and $z = kv$. The resulting set is intersection of two hyperplanes in the positive space (where all variables are greater than zero) and it is connected because intersection of two convex sets is connected. I am assuming here that a hyperplane is convex. The other 3-dimenional strata have relatively simpler equations describing them, and hence to show connectivity of their intersections with affine sub-spaces respectively is possible.

The sub-stratum corresponding to the cycle (1243) = $s_1 s_3 s_2$ is $T_{3\beta}$ (say), where $T_{3\beta} = \{(x, y, z, u, v, w) \in \mathbb{R}^6_+ \mid y = ux, x > 0, u > 0, w > 0, z = 0, v = 0\}$.

And the corresponding matrix product is :

$$\begin{bmatrix} 1 & t_1 & t_1t_2 & 0 \\ 0 & 1 & t_2 & 0 \\ 0 & 0 & 1 & t_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The sub-stratum corresponding to the cycle (1342) = $s_2s_1s_3$ is $T_{3\gamma}$ (say), where $T_{3\gamma} = \{(x, y, z, u, v, w) \in \mathbb{R}^6_+ \mid v = uw, x > 0, u > 0, w > 0, y = 0, z = 0\}$. And the corresponding matrix product is :

$$\begin{bmatrix} 1 & t_1 & 0 & 0 \\ 0 & 1 & t_2 & t_2t_3 \\ 0 & 0 & 1 & t_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The sub-stratum corresponding to the cycle (1432) = $s_3s_2s_1$ is $T_{3\delta}$ (say), where $T_{3\delta} = \{(x, y, z, u, v, w) \in \mathbb{R}^6_+ \mid x > 0, u > 0, w > 0, y = 0, z = 0, v = 0\}$. And the corresponding matrix product is :

$$\begin{bmatrix} 1 & t_1 & 0 & 0 \\ 0 & 1 & t_2 & 0 \\ 0 & 0 & 1 & t_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The sub-stratum corresponding to the cycle (13) = $s_1s_2s_1$ is $T_{3\theta}$ (say), where $T_{3\theta} = \{(x, y, z, u, v, w) \in \mathbb{R}^6_+ \mid z = 0, v = 0, w = 0, x - \frac{y}{u} > 0\}$. And the corresponding matrix product is :

$$\begin{bmatrix} 1 & t_1 + t_3 & t_1t_2 & 0 \\ 0 & 1 & t_2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The sub-stratum corresponding to the cycle (24) = $s_2s_3s_2$ is $T_{3\zeta}$ (say), where $T_{3\zeta} = \{(x, y, z, u, v, w) \in \mathbb{R}^6_+ \mid x = 0, y = 0, z = 0, u - \frac{v}{w} > 0\}$. And the corresponding matrix product is :

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & t_1 + t_3 & t_1t_2 \\ 0 & 0 & 1 & t_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

It can be checked that all the above 3-dimensional sub-strata when intersected with any affine subspace of $\mathbb{R}^6$ are connected, if non-empty. In order to check this, we follow the method we did for the full-dimensional space.

8. 4-DIMENSIONAL SUB-STRATA

The 4-dimensional sub-strata have Coxeter reduced length 4, i.e. these elements of S_4 can be expressed as product of 4 words, where each word is one of the three transpositions s_1, s_2 and s_3. The five elements of S_4 which belong to this category are (142), (124), (143), (134) and (13)(24).

The sub-stratum corresponding to the cycle (142) = $s_2s_3s_2s_1$ is $T_{4\alpha}$ (say), where $T_{4\alpha} = \{(x,y,z,u,v,w) \in \mathbb{R}^6_+ \mid y = 0, z = 0, u - \frac{v}{w} > 0\}$. And the corresponding matrix product is :

$$\begin{bmatrix} 1 & t_4 & 0 & 0 \\ 0 & 1 & t_1+t_3 & t_1t_2 \\ 0 & 0 & 1 & t_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

To prove connectivity, we fix a line, plane, hyperplane and so on. But since the equations governing the stratum are naive, connectivity is quite easy to show. Similar are the cases with the strata $T_{4\gamma}$ and $T_{4\zeta}$.

The sub-stratum corresponding to the cycle (124) = $s_1s_2s_3s_2$ is $T_{4\beta}$ (say), where $T_{4\beta} = \{(x,y,z,u,v,w) \in \mathbb{R}^6_+ \mid x = \frac{z}{v}, y = ux, u - \frac{v}{w} > 0\}$. And the corresponding matrix product is :

$$\begin{bmatrix} 1 & t_1 & t_1(t_2+t_4) & t_1t_2t_3 \\ 0 & 1 & t_2+t_4 & t_2t_3 \\ 0 & 0 & 1 & t_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The cases $T_{4\beta}$ and $T_{4\delta}$ are similar. So we are going to check the connectivity of $T_{4\delta}$ only.

The sub-stratum corresponding to the cycle (143) = $s_1s_3s_2s_1$ is $T_{4\gamma}$ (say), where $T_{4\gamma} = \{(x,y,z,u,v,w) \in \mathbb{R}^6_+ \mid z = 0, v = 0, x - \frac{y}{u} > 0\}$. And the corresponding matrix product is :

$$\begin{bmatrix} 1 & t_1+t_4 & t_1t_3 & 0 \\ 0 & 1 & t_3 & 0 \\ 0 & 0 & 1 & t_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The sub-stratum corresponding to the cycle (134) = $s_1 s_2 s_3 s_1$ is $T_{4\delta}$ (say), where $T_{4\delta} = \{(x, y, z, u, v, w) \in \mathbb{R}^6_+ \mid z = yw, v = uw, x - \frac{y}{u} > 0\}$. And the corresponding matrix product is :

$$\begin{bmatrix} 1 & t_1 + t_4 & t_1 t_2 & t_1 t_2 t_3 \\ 0 & 1 & t_2 & t_2 t_3 \\ 0 & 0 & 1 & t_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Let us show connectivity of the stratum when it is intersected with the coordinate subspaces. When we fix 5 variables u, v, w, y, z the intersection is empty in $\mathbb{R}^6$, and hence nothing is to be shown. If we fix 4 variables say u, v, w, y, we get a point which is connected in $\mathbb{R}^6$. When we fix three variables say u, v, w, we get $z = (constant).y$ which is a hyperplane in $\mathbb{R}^6$. If we fix two variables say u, v, the intersection is $z = (constant).w$ which is again a hyperplane in $\mathbb{R}^6$. If we fix one variable say w, we get $z = (constant).y$ and $v = (constant).u$. Thus the resulting intersection is overlapping of two hyperplanes in the positive 4-space in $\mathbb{R}^6$. Hyperplanes are convex sets and their intersections are also convex and hence connectivity is proved.

The sub-stratum corresponding to the cycle (13)(24) = $s_3 s_2 s_3 s_1$ is $T_{4\zeta}$ (say), where $T_{4\zeta} = \{(x, y, z, u, v, w) \in \mathbb{R}^6_+ \mid y = 0, z = 0, w - \frac{v}{u} > 0\}$. And the corresponding matrix product is :

$$\begin{bmatrix} 1 & t_4 & 0 & 0 \\ 0 & 1 & t_2 & t_2 t_3 \\ 0 & 0 & 1 & t_1 + t_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

9. 5-DIMENSIONAL SUB-STRATA

The elements of S_4 corresponding to this set of sub-strata are (1423), (14) and (1324) and each of these have reduced Coxeter length 5, i.e. they can be expressed as product of 5 words, where each word is one of the three transpositions s_1, s_2 and s_3.

The sub-stratum corresponding to the cycle (1324) = $s_2 s_1 s_2 s_3 s_2$ is $T_{5\alpha}$ (say), where $T_{5\alpha} = \{(x, y, z, u, v, w) \in \mathbb{R}^6_+ \mid z = vx - uxw + yw, u - \frac{v}{w} - \frac{y}{x} > 0\}$. And the corresponding matrix product is :

$$\begin{bmatrix} 1 & t_2 & t_2(t_3+t_5) & t_2t_3t_4 \\ 0 & 1 & t_1+t_3+t_5 & t_4(t_1+t_3) \\ 0 & 0 & 1 & t_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

To prove that the intersection of $T_{5\alpha}$ with affine subspaces of $\mathbb{R}^6$ is connected if non-empty, fix a line in $\mathbb{R}^6$ i.e. let x, y, z, u, v be constant real numbers. Then what we get is $z = c$, which is connected. If we fix 4 variables, we get a line and if we fix three, we get a plane. When we fix two variables u and v, we get a quadric surface in the positive space and it can be shown that it is connected. If we fix one variable say z, then we get a cubic surface in positive space and it can be again shown that it is connected.

The sub-stratum corresponding to the cycle (1423) = $s_2s_3s_2s_1s_2$ is $T_{5\beta}$ (say), where $T_{5\beta} = \{(x, y, z, u, v, w) \in \mathbb{R}^6_+ \mid z = 0, u - \frac{v}{w} - \frac{y}{x} > 0\}$. And the corresponding matrix product is :

$$\begin{bmatrix} 1 & t_4 & t_4t_5 & 0 \\ 0 & 1 & t_1+t_3+t_5 & t_1t_2 \\ 0 & 0 & 1 & t_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This case is easy because the equation describing the stratum is only $z = 0$.

The sub-stratum corresponding to the cycle (14) = $s_1s_2s_1s_3s_2$ is $T_{5\gamma}$ (say), where $T_{5\gamma} = \{(x, y, z, u, v, w) \in \mathbb{R}^6_+ \mid y = ux + \frac{1}{w}(z - vx), x - \frac{z}{v} > 0, u - \frac{v}{w} > 0\}$. And the corresponding matrix product is :

$$\begin{bmatrix} 1 & t_1+t_3 & t_1t_2+t_5(t_1+t_3) & t_1t_2t_4 \\ 0 & 1 & t_2+t_5 & t_2t_4 \\ 0 & 0 & 1 & t_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This case is similar to that of $T_{5\alpha}$. To prove connectivity, we can observe that y is a continuous function of the other 5 variables in the positive space and hence the graph is connected because graph of a continuous map is homeomorphic to the domain. It stays the same when we fix variables and look at the equation.

To prove that $T_{5\gamma}$ is graph of a quasi-affine map, let us look at the following again: $y = ux + \frac{1}{w}(z - vx)$. Multiplying by w throughout, we get the following : $wy = uwx + z - vx$. This is linear in each variable and therefore any variable can be expressed as a function of the other 5 variables, provided some conditions are satisfied and I have checked that they do hold. Hence, quasi-affinity holds in this case. So $T_{5\gamma}$ is graph of a quasi-affine map and its intersection with coordinate subspaces are connected if non-empty.

10. 6-DIMENSIONAL OR THE FULL STRATA

The element of S_4 corresponding to this set of sub-strata is (14)(23) and it has reduced Coxeter word-length 6, i.e. they can be expressed as product of 6 words, where each word is one of the three transpositions s_1, s_2 and s_3. The semi-algebraic set corresponding to the above stratum is T_0, which was described at the beginning of this article in Section 2.

11. CONCLUSION

Thus we have verified the topological regularity property of the strata formed as a result of Bruhat partial ordering on the elements of the Weyl group (of rank 4) of a semi-simple simply connected algebraic group G which is $SL(4, \mathbb{R})$ in our case here. The Weyl group here is the finite Coxeter group S_4. There is further scope of research on verifying regularity properties of strata formed due to Bruhat partial ordering on the elements of the Weyl group of rank greater than or equal to 5.

ACKNOWLEDGEMENT

I sincerely thank Professor Saugata Basu for his immense help and encouragement. I am also thankful to Professor Paolo Mantero with whom I had useful discussions.

REFERENCES

1. S. Basu, A. Gabrielov and N. Vorobjov, Semi-monotone sets, *J. Eur. Math. Soc.*, **15**(2) (2013), 635-657.
2. S. Basu, A. Gabrielov and N. Vorobjov, Monotone functions and maps, Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales, *Serie A. Matematicas*, **107**(1), March (2013), 5-33. (Special Issue: 80-th birthday of Heisuke Hironaka).
3. A. Berenstein, S. Fomin, and A. Zelevinsky, Parametrizations of canonical bases and totally positive matrices, *Adv. Math.*, **122** (1996), 49-149.
4. A. Berenstein and A. Zelevinsky, Total positivity in Schubert varieties, *Comment. Math. Helv.*, **72** (1997), 128-166.
5. A. Bjorner, Posets, regular CW complexes and Bruhat order, *Europ. J. Combin.*, **5** (1984), 7-16.

6. Sergey Fomin and Michael Shapiro, Stratified spaces formed by totally positive varieties, *Michigan Mathematical Journal (special volume dedicated to William Fulton)*, **48** (2000), 253-270.

7. S. Fomin and A. Zelevinsky, Double Bruhat cells and total positivity, *J. Amer. Math. Soc.*, **12** (1999), 335-380.

8. S. Fomin and A. Zelevinsky, Totally nonnegative and oscillatory elements in semisimple groups, *Proc. Amer. Math. Soc.*, **128**(12) (2000), 3749-3759.

9. P. Hersh, Regular cell complexes in total positivity, *Inventiones Mathematicae*, **197**(1) (2014), 57-114.

10. G. Lusztig, *Total positivity in reductive groups in Lie theory and geometry: In honor of Bertram Kostant*, Progress in Mathematics 123, Birkhauser, 1994.

11. G. Lusztig, Introduction to total positivity, in: Positivity in Lie theory: Open problems, *de Gruyter Exp. Math.*, **26**, de Gruyter, Berlin, 1998, 133-145.