Indian J. Pure Appl. Math., **48**(2): 259-269, June 2017

DOI: 10.1007/s13226-017-0219-6

ON COMMUTATIVITY OF NEAR-RINGS WITH GENERALIZED DERIVATIONS

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(*Received* 7 *May* 2014; *after final revision* 10 *July* 2016;
accepted 17 *November* 2016)

In this paper we prove some theorems of commutativity for near-rings with generalized derivations. As a consequence of the results obtained, we generalize some published results. Also, we give some examples to show that some conditions in some results obtained are not redundant.

Key words : Near-rings**;** generalized derivation; primeness; commutativity.

1. INTRODUCTION

Throughout this paper, R will be a left near-ring and $Z(R)$ is the multiplicative center of R. A near-ring R is called 3-prime if, for all $x, y \in R$, $xRy = \{0\}$ implies $x = 0$ or $y = 0$. A non-empty subset U of R is called a semigroup right (left) ideal of R, if U satisfies $UR \subseteq U$ ($RU \subseteq U$). A non-empty subset U of R is called a semigroup ideal if it is both a semigroup right and left ideal. A non-empty subset U of R is called a right (left) R-subgroup of R if U is a semigroup right (left) ideal of R and $(U, +)$ is a subgroup of $(R, +)$. A non-empty subset U of R is called a two-sided R-subgroup of R if it is both a right R-subgroup of R and a left R-subgroup of R. A map $d : R \to R$ is a derivation on R if d is an additive mapping and $d(xy) = xd(y) + d(x)y$ for all $x, y \in R$. A generalized derivation f on R is an additive mapping $f : R \to R$ such that $f(xy) = f(x)y + xd(y)$ for all $x, y \in R$, where d is a derivation on R. An element $x \in R$ is called a left (right) zero divisor in R if there exists a non-zero element $y \in R$ such that $xy = 0$ ($yx = 0$). A zero divisor is either a left or a right zero divisor. A near-ring R is called a zero-symmetric near-ring, if $0x = 0$ for all $x \in R$. By an integral near-ring, we mean a near-ring without non-zero divisors of zero. For further information about near-rings, see [13] and [14].

The study of commutativity of 3-prime near-rings by using derivations was initiated by Bell and Mason in 1987 in [4]. After that, Beidar, Fong and Wang generalize some results from rings to near-rings with derivations in [1]. Again, Bell generalizes several results of [4] by using one (two) sided semigroup ideal of the near-ring in his work in [2]. In 2006, Golbasi used in [8] and [9] the definition that a map on a near-ring is called a generalized derivation if it is both a left and a right generalized derivation on the near-ring. Golbasi used that definition to deduce some results on the class of 3-prime near-rings. In 2008, Bell investigated in [3] possible analogues of three results of [4] for generalized derivations on near-rings and he got only one extension. Recently, many papers studied the commutativity on near-rings by using derivations and generalized derivations, such as [6, 7, 10, 11]. In this paper, we generalize some results of [3] and [11] and prove some theorems of commutativity for 3-prime near-rings.

In Section 2 we give some well-known results and we add some auxiliary results which will be useful in the sequel.

In Section 3 we study the commutativity of a 3-prime near-ring R admitting a non-zero generalized derivation f under the conditions $f(U) \subseteq Z(R)$ and $f(uv) = f(vu)$, where $u, v \in U$ and U is a suitable subset of R. As a consequence of the results obtained in this section, we generalize some results of [3] and [11].

2. Preliminaries and Some Results

In this section, we have some well-known results which will be useful in the sequel. We will start with the following four results and we will use them in this paper without explicit mention.

Lemma 2.1 [11, Proposition 2.2] — A near-ring R is zero-symmetric if and only if R admits a generalized derivation.

Lemma 2.2 [15, Proposition 1] — Let R be a near-ring with a derivation d. Then $xd(y) + d(x)y = d(x)y + xd(y)$ for all $x, y \in R$.

From Lemma 2.2, observe that any derivation is a generalized derivation.

Lemma 2.3 [4, Lemma 1] — (The partial distributive law) Let R be a near-ring and d a derivation on R. For all $x, y, z \in R$ we have $(xd(y) + d(x)y)z = xd(y)z + d(x)yz$.

Lemma 2.4 [8, Lemma 2.3(i)] — Let R be a near-ring with a generalized derivation f associated with a derivation d. Then $(f(x)y + xd(y))z = f(x)yz + xd(y)z$ for all $x, y, z \in R$.

The following two results are very useful in the sequel.

Lemma 2.5 (i) [15, Lemma 2] — Let R be a near-ring with a derivation d. If $x \in Z(R)$ then $d(x) \in Z(R)$.

(ii) [4, Lemma 3(i)] Let R be a 3-prime near-ring and $z \in Z(R) - \{0\}$. Then, z is not a zero divisor.

(iii) [2, Lemma 1.2(iii)] Let R be a 3-prime near-ring and $x \in Z(R) - \{0\}$. If either yx or xy is in $Z(R)$, then $y \in Z(R)$.

Lemma 2.6 [2, Lemma 1.4] — Let R be a 3-prime near-ring and U a non-zero semigroup ideal of R. Suppose d is a non-zero derivation on R and $x \in R$ such that $d(U)x = \{0\}$ ($xd(U) = \{0\}$), so $x = 0$.

Lemma 2.7 (i) [2, Theorem 2.1] — Let R be a 3-prime near-ring with a non-zero semigroup right (left) ideal U. If R admits a non-zero derivation d such that $d(U) \subseteq Z(R)$, then R is a commutative ring.

(ii) [2, Theorem 3.1] Let R be a 3-prime near-ring with a non-zero semigroup ideal U. If R admits a derivation d such that $d^2 \neq 0$ and $[d(U), d(U)] = \{0\}$, then R is a commutative ring.

Lemma 2.8 [2, Lemma 1.3(ii)] — Let R be a 3-prime near-ring with a non-zero derivation d. If U is a non-zero semigroup right (left) ideal of R, then $d(U) \neq \{0\}$.

Lemma 2.9 [11, Lemma 4.6] — Let R be a 3-prime near-ring with a non-zero semigroup ideal U and a non-zero generalized derivation f.

(i) If f is associated with a non-zero derivation and $af(U) = \{0\}$ for some $a \in R$, then $a = 0$.

(ii) If f is associated with the zero derivation and $f(U)a = \{0\}$ for some $a \in R$, then $a = 0$.

Lemma 2.10 [2, Lemma 1.3(iii)] — Let R be a 3-prime near-ring with a non-zero semigroup right ideal U. If x is an element of R which centralizes U, then $x \in Z(R)$.

Lemma 2.11 [2, Lemma 1.5] — Let R be a 3-prime near-ring with a non-zero semigroup right (left) ideal U. If $U \subseteq Z(R)$, then R is a commutative ring.

Lemma 2.12 [12, Lemma 2.6(i)] — Let R be a non-zero near-ring. If R has no non-zero zero divisors, then R is 3-prime.

Lemma 2.13 [11, Theorem 3.2] — Let R be a 3-prime near-ring with a non-zero generalized derivation f such that $f(U)$ centralizes U, where U is a non-zero semigroup ideal of R. Then R is a commutative ring.

The following result generalizes Lemma 3(i) of [4].

Lemma 2.14 — Let R be a 3-prime zero-symmetric near-ring such that:

(i) $x_1x_2\dots x_n \in Z(R)-\{0\}$ for some $x_1, x_2, \dots, x_n \in R$, where $n \in \mathbb{N}$. Then x_n is not a left zero divisor and x_1 is not a right zero divisor.

(ii) $x^n \in Z(R)-\{0\}$ for some $x \in R, n \in \mathbb{N}$. Then x is not a zero divisor.

(iii) The product of every n elements of a subset $S \subseteq R$ is in $Z(R)$. If $x_1x_2\dots x_n \neq 0$ for some $x_1, x_2, \dots, x_n \in S$, then each x_i is not a zero divisor, where $i \in \{1, 2, \dots, n\}, n \in \mathbb{N}$. Moreover, $st = ts$ for all $s, t \in S$.

PROOF : (i) Suppose $x_ny = 0\,(yx_1 = 0)$ for some $y \in R$. Then $x_1x_2\dots x_ny = 0\,(yx_1x_2\dots x_n = 0)$ and hence $y = 0$ by Lemma 2.5(ii).

(ii) The proof is clear by (i).

(iii) Observe that

$$x_i\dots x_nx_1\dots x_{i-1}x_i\dots x_n = x_i\dots x_nx_i\dots x_nx_1\dots x_{i-1}$$

for x_i and $i \in \{2, \dots, n-1\}$. So $x_i\dots x_n(x_1\dots x_{i-1}x_i\dots x_n - x_i\dots x_nx_1\dots x_{i-1}) = 0$ and then

$$x_1x_2\dots x_n(x_1\dots x_{i-1}x_i\dots x_n - x_i\dots x_nx_1\dots x_{i-1}) = 0.$$

Using Lemma 2.5(ii), we have

$$x_1\dots x_{i-1}x_i\dots x_n = x_i\dots x_nx_1\dots x_{i-1} \tag{2.1}$$

and by the same way we have

$$x_1\dots x_ix_{i+1}\dots x_n = x_{i+1}\dots x_nx_1\dots x_i \tag{2.2}$$

for x_{i+1}. Using (i), (2.1) and (2.2), we get that x_i is not a zero divisor for all $i \in \{2, \dots, n-1\}$. For x_1, putting $i = 2$, equation (2.1) will be

$$x_1\dots x_n = x_2\dots x_nx_1$$

and x_1 is not a zero divisor by (i). For x_n, putting $i = n-1$, equation (2.2) will be

$$x_1\dots x_n = x_nx_1\dots x_{n-1}$$

and x_n is not a zero divisor by (i). Now, $(x_i)^n s = (x_i)^{n-1} s x_i$ for all $s \in S$. Thus, $(x_i)^{n-1}(x_i s - s x_i) = 0$ and hence $x_i s = s x_i$ for all $s \in S$. By the same way above, we have $(x_i)^{n-1} ts = t(x_i)^{n-1} s = (x_i)^{n-1} st$ which implies that $st = ts$ for all $s, t \in S$. □

Corollary 2.15 — Let R be a 3-prime zero-symmetric near-ring with a non-zero semigroup right ideal U of R such that $u_1 u_2 \dots u_n \in Z(R)$ for all $u_1, u_2, \dots, u_n \in U, n \in \mathbb{N}$. If $v_1 v_2 \dots v_n \in Z(R) - \{0\}$ for some $v_1, v_2, \dots, v_n \in U$, then R is a commutative ring.

PROOF : By Lemma 2.14(iii), U centralizes itself. Thus, $U \subseteq Z(R)$ by Lemma 2.10. Therefore R is a commutative ring by Lemma 2.11. □

Lemma 2.16 — Every non-zero semigroup right ideal U of a near-ring R contains vR which is a right R-subgroup of R, where $0 \neq v \in U$.

PROOF : Suppose U is a non-zero semigroup right ideal of R. It is easy to check that $vx \pm vy, vxy \in vR \subseteq U$ for all $x, y \in R$. So vR is a right R-subgroup of R. □

3. MAIN RESULTS

In this section we will study the commutativity of near-rings, each admitting a generalized derivation f satisfies the conditions $f(xy) = f(yx)$ or $f(y)x = xf(y)$.

The next result is a generalization of Theorem 2.1 of [3].

***Theorem* 3.1** — *Let R be a 3-prime near-ring with a non-zero generalized derivation f on R associated with a derivation d. If $f(x_1)f(x_2)\dots f(x_n) \in Z(R)$ for all $x_1, x_2, \dots, x_n \in R$, where $n \in \mathbb{N}$, then R is a commutative ring.*

PROOF : If $f(x_1)f(x_2)\dots f(x_n) = 0$ for all $x_1, x_2, \dots, x_n \in R$, then $f(R) = \{0\}$ (by Lemma 2.9(i) if $d \neq 0$ and Lemma 2.9(ii) if $d = 0$), a contradiction. So $f(a_1)f(a_2)\dots f(a_n) \in Z(R) - \{0\}$ for some $a_1, a_2, \dots, a_n \in R$. If $d = 0$, then $f(a_1)f(a_2)\dots f(a_n x) \in Z(R)$ for all $x \in R$ which implies that $x \in Z(R)$ by lemma 2.5(iii). Therefore, R is a commutative near-ring and hence a commutative ring by Lemma 2.11. Now, suppose $d \neq 0$. Using Lemma 2.14(iii), we have $f(x)f(y) = f(y)f(x)$ for all $x, y \in R$. Thus, $f(xz)f(y) = f(y)f(xz)$ and then $xd(z)f(y) = f(y)xd(z)$ for all $x, y \in R, z \in Z(R)$. Using Lemma 2.5(i), we get that $d(z)(xf(y) - f(y)x) = 0$ for all $x, y \in R, z \in Z(R)$. If $d(z) \neq \{0\}$ for some $z \in Z(R)$, then $f(R) \subseteq Z(R)$ by Lemma 2.5(ii) and R is a commutative ring by Lemma 2.13. If $d(Z(R)) = \{0\}$, then

$$
\begin{aligned}
0 &= d((f(a_1))^{n-1} f(f(x)y) = d((f(a_1))^{n-1}(f(f(x))y + f(x)d(y)) \\
&= d((f(a_1))^{n-1} f(f(x))y + (f(a_1))^{n-1} f(x)d(y))
\end{aligned}
$$

$$\begin{aligned} &= (f(a_1))^{n-1}f(f(x))d(y) + (f(a_1))^{n-1}f(x)d^2(y) \\ &= (f(a_1))^{n-1}(f(f(x))d(y) + f(x)d^2(y)) = (f(a_1))^{n-1}f(f(x)d(y)) \end{aligned}$$

for all $x, y \in R$. But $f(a_1)$ is not a zero divisor by Lemma 2.14(iii), so $f(f(x)d(y)) = 0$ for all $x, y \in R$. Now, $(f(a_1))^{n-1}f(f(a_1)d(y)r) \in Z(R)$ and then $(f(a_1))^n d(y)d(r) \in Z(R)$ for all $y, r \in R$. So $d(y)d(r) \in Z(R)$ for all $y, r \in R$ by Lemma 2.5(iii). Since $d(R)d(R) \neq \{0\}$ by Lemma 2.6, there exist $a, b \in R$ such that $d(a)d(b) \in Z(R) - \{0\}$. Using Lemma 2.14(iii), we have that $d(x)d(y) = d(y)d(x)$ for all $x, y \in R$. If $d^2 \neq 0$, then R is a commutative ring by Lemma 2.7(ii). If $d^2 = 0$, then $f(a_1)f(a_2)\dots f(a_n d(x)) \in Z(R)$ for all $x \in R$, and hence $d(R) \subseteq Z(R)$ by Lemma 2.5(iii). Therefore, R is a commutative ring by Lemma 2.7(i). □

The following result generalizes Theorem 2.1 of [3] and Theorem 3.2 of [11].

***Theorem* 3.2** — *Let R be a 3-prime near-ring with a non-zero generalized derivation f such that $f(U)$ centralizes U, where U is a non-zero semigroup right ideal of R. Then either $f(U) = \{0\}$ or R is a commutative ring.*

PROOF : $f(U)$ centralizes U implies $f(U) \subseteq Z(R)$ by Lemma 2.10. If $f(U) \neq 0$, then there exists $a \in U$ such that $0 \neq f(a) \in Z(R)$.

If $d = 0$, then $f(uy) = f(u)y \in Z(R)$ for all $u \in U, y \in R$. So $f(a)y \in Z(R)$ and we get that $y \in Z(R)$ by Lemma 2.5(iii). Hence, R is a commutative ring by Lemma 2.11. So suppose $d \neq 0$. Now, take $z \in Z(R)$, so $f(uz) = f(u)z + ud(z) \in Z(R)$ for all $u \in U$. For all $x \in R$, we have $xf(u)z + xud(z) = f(u)zx + ud(z)x$. Using Lemma 2.5(i), we get that $d(z)(xu - ux) = 0$ for all $x \in R, u \in U, z \in Z(R)$. If $d(Z(R)) \neq \{0\}$, then R is a commutative ring by Lemma 2.5(ii) and Lemma 2.11. So suppose $d(Z(R)) = \{0\}$. Then $d(f(U)) = \{0\}$. For all $u \in U, y \in R$, we have

$$\begin{aligned} 0 &= d(f(uy)) = d(f(u)y + ud(y)) = d(f(u)y) + d(ud(y)) \\ &= d(f(u))y + f(u)d(y) + ud^2(y) + d(u)d(y) \\ &= f(ud(y)) + d(u)d(y). \end{aligned}$$

Since $f(ud(y)) \in Z(R)$, we get that $d(u)d(-y) = d(u)d(r) \in Z(R)$ for all $u \in U, r \in R$. Now, for all $x, y, v \in U$, we have $d(x)d(yv) \in Z(R)$. Thus, $d(x)yd(v) + d(x)d(y)v \in Z(R)$. So for all $w \in U$, we deduce that

$$\begin{aligned} d(w)d(x)yd(v) + d(w)d(x)d(y)v &= d(x)yd(v)d(w) + d(x)d(y)vd(w) \\ &= d(v)d(w)d(x)y + vd(w)d(x)d(y) \\ &= d(w)d(x)d(v)y + d(w)d(x)vd(y) \end{aligned}$$

and hence $d(w)d(x)(yd(v)+d(y)v) = d(w)d(x)(d(v)y+vd(y))$ which means that $d(U)d(U)(d(yv) - d(vy)) = 0$. Since $d(U)d(U) \subseteq Z(R)$, either $d(U)d(U) = \{0\}$ or $d(yv) = d(vy)$ for all $y, v \in U$ by Lemma 2.5(ii). Now, we will show that $d(U)d(U) = \{0\}$ is impossible. We know that $d(u)d(r) \in Z(R)$ for all $u \in U, r \in R$. So if $d(U)d(R) = \{0\}$, then $d(U) = 0$ by Lemma 2.6 and from Lemma 2.8 we deduce that $d = 0$, a contradiction. Therefore, there exist $u_o \in U, r_o \in R$ such that $d(u_o)d(r_o) \in Z(R)-\{0\}$. If $d(U)d(U) = \{0\}$, then $d(u_o)d(u_o)d(r_o) = 0$. But $d(u_o)d(r_o)$ is not a zero divisor by Lemma 2.5(ii). Hence, $d(u_o) = 0$, a contradiction. Thus, $d(u_o)d(u_o) \in Z(R)-\{0\}$ which implies that $d(yv - vy) = 0$ for all $y, v \in U$. Replacing y by u_o and v by $u_o v$, we have

$$0 = d(u_o(u_o v - vu_o)) = d(u_o)(u_o v - vu_o).$$

But $d(u_o)$ is not a zero divisor by putting $S = d(U)$ and $n = 2$, and using Lemma 2.14(iii). Thus, the last equation implies that $u_o \in Z(R)$ by Lemma 2.10. So $d(u_o) = 0$ by using the assumption $d(Z(R)) = \{0\}$, a contradiction. So the assumption $d(Z(R)) = \{0\}$ is not true. □

***Theorem* 3.3** — *Let R be a 3-prime near-ring with a non-zero generalized derivation f such that $f(U) \subseteq Z(R)$, where U is a non-zero semigroup left ideal of R. If there exists $c \in Z(R) \cap U - \{0\}$, then either $f(U) = \{0\}$ or R is a commutative ring.*

PROOF : If $f(U) \neq 0$, then there exists $a \in U$ such that $0 \neq f(a) \in Z(R)$. If $d = 0$, then $f(yu) = f(y)u \in Z(R)$ for all $u \in U, y \in R$. So $f(a)u \in Z(R)$ for all $u \in U$ and we get that $u \in Z(R)$ by Lemma 2.5(iii). Hence, $U \subseteq Z(R)$ and R is a commutative ring by Lemma 2.11. So suppose $d \neq 0$. Now, $f(uc) = f(u)c + ud(c) \in Z(R)$ for all $u \in U$. Thus, $d(c)(xu - ux) = 0$ for all $x \in R, u \in U$ by the same way as in the proof of Theorem 3.2. If $d(c) \neq \{0\}$, then R is a commutative ring by Lemma 2.5(ii) and Lemma 2.11. If $d(c) = \{0\}$, then $f(xc) = f(x)c \in Z(R)$ for all $x \in R$. So $f(R) \subseteq Z(R)$ by Lemma 2.5(iii) and R is a commutative ring by Lemma 2.13. □

***Theorem* 3.4** — *Let R be a near-ring with a non-zero generalized derivation f and a non-zero right R-subgroup U of R. Suppose $f(uv) = f(vu)$ for all $u, v \in U$ and there exist $u_o, v_o \in U$ such that $d(u_o), v_o$ are not zero divisors, where d is the associated derivation with f. Then R is a commutative ring.*

PROOF : Replace v by uv in $f(uv) = f(vu)$ to get

$$0 = f(u(uv - vu)) = f((uv - vu)u) = (uv - vu)d(u)$$

for all $u, v \in U$. Since $d(u_o)$ is not a zero divisor, we have $u_o v = vu_o$ for all $v \in U$. Thus, $f(u_o uv) = f(vu_o u) = f(u_o vu)$ for all $u, v \in U$. So

$$0 = f(u_o(uv - vu)) = f((uv - vu)u_o) = (uv - vu)d(u_o)$$

and hence $uv = vu$ for all $u, v \in U$. Now, for all $x, y \in R$, we have $v_o v_o xy = (v_o x) v_o y = v_o y(v_o x) = v_o(v_o y)x$ and hence $xy = yx$ which means R is a commutative near-ring. Now, for all $x, y \in R$, we get that

$$\begin{aligned}(v_o + v_o)(x + y) &= (v_o + v_o)x + (v_o + v_o)y \\ &= v_o x + v_o x + v_o y + v_o y\end{aligned}$$

and on the other hand

$$\begin{aligned}(v_o + v_o)(x + y) &= v_o(x + y) + v_o(x + y) \\ &= v_o x + v_o y + v_o x + v_o y.\end{aligned}$$

Comparing the last two equations, we have that $v_o(x + y - x - y) = 0$ and hence $x + y = y + x$ for all $x, y \in R$. Therefor R is a commutative ring. □

Corollary 3.5 — Let R be a near-ring with a non-zero generalized derivation f and a non-zero semigroup right ideal U of R. Suppose $f(uv) = f(vu)$ for all $u, v \in U$ and there exist $u_o, v_o \in U$ such that $d(v_o u_o), v_o$ are not zero divisors, where d is the associated derivation with f. Then R is a commutative ring.

PROOF : From Lemma 2.16, we get that $V = v_o R$ is a right R-subgroup of R. Observe that $V \subseteq U$ and $f(uv) = f(vu)$ for all $u, v \in V$. If $V = \{0\}$, then $v_o v_o = 0$ and hence $v_o = 0$, a contradiction. So $V \neq \{0\}$. Also, $v_o v_o, v_o u_o \in V$ and $v_o v_o, d(v_o u_o)$ are not zero divisors. So R is a commutative ring by Theorem 3.4. □

***Theorem* 3.6** — *Let R be a 3-prime near-ring with a non-zero generalized derivation f associated with the zero derivation. If $f(uv) = f(vu)$ for all $u, v \in U$, where U is a non-zero semigroup right ideal of R, then either $f(U) = \{0\}$ or R is a commutative ring.*

PROOF : Replace u by vu in $f(uv) = f(vu)$ to get $f(vuv) = f(vvu)$ for all $u, v \in U$. It follows that $f(v)uv = f(v)vu$. Replacing u by uw where $w \in U$, we have $f(v)uwv = f(v)vuw = f(v)uvw$ and hence $f(v)u(wv - vw) = 0$. So $f(v)ur(wv - vw) = 0$ for all $u, v, w \in U, r \in R$. If $f(U)U = \{0\}$, then we have $0 = f(ur)v = f(u)rv$ for all $u, v \in U, r \in R$, and hence $f(U) = \{0\}$. Otherwise, there exists $0 \neq a \in U$ such that $aw = wa$ for all $w \in U$ which implies $a \in Z(R)$ by Lemma 2.10. Now, for all $u, v \in U$ we have $f((au)v) = f(v(au)) = f(avu)$ which means $f(a)uv = f(a)vu$. Replacing u by uw and by the same way above, we get $f(a)u(vw - wv) = 0$ for all $u, v, w \in U$. Since we are in the case $f(U)U \neq \{0\}$, we have $vw = wv$ for all $v, w \in U$. So $U \subseteq Z(R)$ by Lemma 2.10 and R is a commutative ring by Lemma 2.11. □

The following result generalizes Theorem 3.7 of [11].

Corollary 3.7 — Let R be an integral near-ring with a non-zero generalized derivation f and a non-zero semigroup right ideal U of R. If $f(uv) = f(vu)$ for all $u, v \in U$, then either $f(U) = \{0\}$ or R is a commutative ring.

PROOF : Let d be the associated derivation with f. If $d = 0$, then either $f(U) = \{0\}$ or R is a commutative ring by using Lemma 2.12 and Theorem 3.6. Now, suppose $d \neq 0$. So $d(U) \neq \{0\}$ by Lemma 2.8. Thus, there exists $u_o \in U$ such that $d(u_o) \neq 0$. If $d(u_o v) = 0$ for all $v \in U$, then $0 = d(u_o u_o u_o) = u_o u_o d(u_o)$, a contradiction. So there exists $v_o \in U$ such that $d(u_o v_o) \neq 0$. Therefore, R is a commutative ring by Corollary 3.5. □

Let R be a ring. For all $x \in R$, define a map $f : R \to R$ by $f(x) = cx$, where $c \in R$. Then f is called a left multiplication. It is well-known that any left multiplication is a generalized derivation associated with the zero derivation. We will use it in the following example which shows that some conditions in some previous results are not redundant.

Example 3.1 : Let $R = M_2(F)$, where F is any field. Then R is a non-commutative prime ring. Define a left multiplication f on R by

$$f(x) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix}.$$

Observe that $f \neq 0$. Take the right ideal $U = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} R = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} |a, b \in F \right\}$. So $U \neq \{0\}$, $f(U) = \{0\} \subseteq Z(R)$ and $f(uv) = f(vu)$ for all $u, v \in U$. This shows that the condition "$f(U) = \{0\}$" in Theorem 3.2 and Theorem 3.6 is not redundant. Also, observe that $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} U = \{0\}$ and $d = 0$ which shows that the conditions "there exist $u_o, v_o \in U$ such that $d(u_o), v_o$ are not zero divisors" in Theorem 3.4 and "there exist $u_o, v_o \in U$ such that $d(v_o u_o), v_o$ are not zero divisors" in Corollary 3.5 are not redundant.

Now, let R be a ring with a derivation d and c a fixed element of R. From [5], the map $f(x) = cx + d(x)$ defined on R, is a generalized derivation on R. We will use this fact to give an example to show that the condition "$f(U) = \{0\}$" in Theorem 3.3 is not redundant.

Example 3.2 : Let $R = M_2(\mathbb{Z}_2)$. Then R is a non-commutative prime ring. Take d to be the inner derivation on R induced by $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then $d \neq 0$ and

$$d\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} c & a+d \\ 0 & c \end{bmatrix}.$$

Now, take the non-zero left ideal $V = R\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \left\{\begin{bmatrix} 0 & b \\ 0 & d \end{bmatrix} | b, d \in \mathbb{Z}_2\right\}$.

Observe that $d(V) = \left\{\begin{bmatrix} 0 & d \\ 0 & 0 \end{bmatrix} | d \in \mathbb{Z}_2\right\}$. Now, define a generalized derivation f on R by

$$f(x) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + d(x) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} c & a+d \\ 0 & c \end{bmatrix} = \begin{bmatrix} 0 & a \\ 0 & c \end{bmatrix}.$$

Notice that $f(V) = \{0\} \subseteq Z(R)$.

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