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CONGRUENCES FOR $\ell-$REGULAR OVERPARTITION FOR $\ell \in \{5, 6, 8\}$

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Let $\bar{A}_\ell(n)$ denote the number of overpartitions of a non-negative integer n with no part divisible by ℓ, where ℓ is a positive integer. In this paper, we prove infinite family of congruences for $\bar{A}_5(n)$ modulo 4, $\bar{A}_6(n)$ modulo 3, and $\bar{A}_8(n)$ modulo 4. In the process, we also prove some other congruences.

Key words : ℓ-Regular overpartition; partition congruence; Ramanujan's theta-function.

1. INTRODUCTION

An overpartition of a positive integer n is a partition of n in which the first occurrence of each part can be over lined. If $\bar{p}(n)$ denotes the number of overpartitions of n, then the generating function of $\bar{p}(n)$ is given by

$$\sum_{n=0}^{\infty} \bar{p}(n) q^n = \frac{f_2}{f_1^2}, \tag{1.1}$$

where, here and throughout this paper, f_k is defined by

$$f_k := \prod_{n=1}^{\infty} (1 - q^{kn}); \qquad |q| < 1. \tag{1.2}$$

The arithmetic properties of overpartition function $\bar{p}(n)$ have been studied by many authors, for example see [9, 10, 12, 13, 16, 22] and references therein.

For any positive integer ℓ, ℓ-regular partition of a positive integer n is a partition of n such that none of its part is divisible by ℓ. If $b_\ell(n)$ denotes the number of ℓ-regular partitions of n, then generating function of $b_\ell(n)$ is given by

$$\sum_{n=0}^{\infty} b_\ell(n)q^n = \frac{f_\ell}{f_1}. \tag{1.3}$$

The arithmetic properties of ℓ-regular partitions have been studied by many authors, for example see [2, 3, 5, 7, 8, 11, 15, 17, 19-21] and references therein.

Lovejoy [14] investigated the partition function $\bar{A}_\ell(n)$ which counts the number of overpartitions of n with no parts divisible by ℓ. Shen [18] named the function $A_\ell(n)$ as $\ell-$regular overpartition and gave its generating function as

$$\sum_{n=0}^{\infty} \bar{A}_\ell(n)q^n = \frac{f_\ell^2 f_2}{f_1^2 f_{2\ell}}. \tag{1.4}$$

Shen [18] also derived some congruences for $\bar{A}_3(n)$ and $\bar{A}_4(n)$ modulo 3, 6 and 24 and gave their combinatorial interpretations. For example, Shen in [18, Corollary 2.4 & 2.8] proved that

$$\bar{A}_3(4n+1) \equiv 0 \pmod{2},$$

$$\bar{A}_3(4n+3) \equiv 0 \pmod{6},$$

$$\bar{A}_3(9n+3) \equiv 0 \pmod{6},$$

$$\bar{A}_3(9n+6) \equiv 0 \pmod{24}$$

and in [18, Theorem 2.9] he proved that

$$\bar{A}_4(12n+i) \equiv 0 \pmod{3}, \quad i = 4, 8$$

$$\bar{A}_4(12n+j) \equiv 0 \pmod{24}, \quad j = 7, 11.$$

Recently, Chern [6] proved some new infinite family of congruences modulo ℓ for $\bar{A}_\ell(n)$ where $\ell =$ 3, 5, 7. For example, Chern [6, Theorem 2.5] proved that: Let $p \neq 5$ be a prime and k and n are non-negative integers, then

$$\bar{A}_5(p^{4k+3}(pn+i)) \equiv 0 \pmod{5}, \quad i = 1, 2, 3, \ldots, p-1.$$

Motivated by above works, in this paper we prove some congruences for $\bar{A}_\ell(n)$ for $\ell = 5, 6, 8$. Specifically, in Section 3 we prove an infinite family of congruences for $\bar{A}_5(n)$ modulo 4. In Section 4, we prove an infinite family of congruences for $\bar{A}_6(n)$ modulo 3. In Section 5, we prove an infinite family of congruences for $\bar{A}_8(n)$ modulo 4. In this process, we also prove some other congruences.

2. PRELIMINARIES

Ramanujan's general theta-function $f(a,b)$ is defined by

$$f(a,b)=\sum_{n=-\infty}^{\infty}a^{n(n+1)/2}b^{n(n-1)/2},\ |ab|<1. \tag{2.1}$$

Three useful cases of $f(a,b)$ [4, p. 36, Entry 22(i)-(iii)] are

$$\phi(q):=f(q,q)=\sum_{n=0}^{\infty}q^{n^2}=\frac{f_2^5}{f_1^2f_4^2}, \tag{2.2}$$

$$f(-q):=f(-q,-q^2)=\sum_{n=-\infty}^{\infty}(-1)^nq^{n(3n+1)/2}=f_1, \tag{2.3}$$

and

$$\psi(q):=f(q,q^3)=\sum_{n=0}^{\infty}q^{n(n+1)/2}=\frac{f_2^2}{f_1}. \tag{2.4}$$

It is useful to note here that

$$\phi(-q)=\frac{f_1^2}{f_2}, \tag{2.5}$$

and from [4, P. 40, Entry 25(i) & (ii)] that

$$\phi(-q)=\phi(q^4)-2q\psi(q^8). \tag{2.6}$$

Lemma 2.1 — For any prime p, we have

$$f_p\equiv f_1^p \pmod{p}.$$

PROOF : Follows easily from binomial theorem. □

Lemma 2.2 — [1, Lemma 1.4] For any prime p, we have

$$f_{p^2}\equiv f_p^p \pmod{p^2}.$$

Lemma 2.3 — [7, Theorem 2.2] For any prime $p\geq 5$, we have

$$f_1=\sum_{\substack{k=-\frac{p-1}{2}\\ k\neq\frac{\pm p-1}{6}}}^{\frac{p-1}{2}}(-1)^kq^{(3k^2+k)/2}f\left(-q^{\frac{3p^2+(6k+1)p}{2}},-q^{\frac{3p^2-(6k+1)p}{2}}\right)+(-1)^{\frac{\pm p-1}{6}}q^{\frac{p^2-1}{24}}f_{p^2}, \tag{2.7}$$

where $\dfrac{\pm p-1}{6} := \begin{cases} \dfrac{p-1}{6}, & if\ p \equiv 1 \pmod 6, \\ \dfrac{-p-1}{6}, & if\ p \equiv -1 \pmod 6. \end{cases}$

Furthermore, if $-\dfrac{p-1}{2} \leq k \leq \dfrac{p-1}{2}$ and $k \neq \dfrac{\pm p-1}{2}$, then $\dfrac{3k^2+k}{2} \not\equiv \dfrac{p^2-1}{24} \pmod p$.

Lemma 2.4 — [7, Theorem 2.1]. For any odd prime p, we have

$$\psi(q) = \sum_{k=0}^{\frac{p-3}{2}} q^{\frac{k^2+k}{2}} f(q^{\frac{p^2+(2k+1)p}{2}}, q^{\frac{p^2-(2k+1)p}{2}}) + q^{\frac{p^2-1}{8}} \psi(q^{p^2}), \tag{2.8}$$

where

$$\frac{k^2+k}{2} \not\equiv \frac{p^2-1}{8} \pmod p \text{ for } 0 \leq k \leq \frac{p-3}{2}.$$

Lemma 2.5 — [11, Theorem 1]. We have

$$\frac{f_5}{f_1} = \frac{f_8 f_{20}^2}{f_2^2 f_{40}} + q\frac{f_4^3 f_{10} f_{40}}{f_2^3 f_8 f_{20}}.$$

Lemma 2.6 — [22, Lemma 2.1]. We have

$$\frac{1}{f_1^4} = \frac{f_4^{14}}{f_2^{14} f_8^4} + 4q\frac{f_4^2 f_8^4}{f_2^{10}}. \tag{2.9}$$

$$\frac{1}{f_1^2} = \frac{f_8^5}{f_2^5 f_{16}^2} + 2q\frac{f_4^2 f_{16}^2}{f_2^5 f_8}. \tag{2.10}$$

3. Congruences for 5-Regular Overpartition

In this section we prove an infinite family if congruences for $\bar{A}_5(n)$ modulo 4.

***Theorem* 3.1** — *Let $p \geq 5$ be a prime with $\left(\dfrac{-5}{p}\right) = -1$. Then for non-negative integers α and n, we have*

$$\sum_{n=0}^{\infty} \bar{A}_5\left(4p^{2\alpha}n + p^{2\alpha}\right) q^n \equiv 2 f_1 f_5 \pmod 4, \tag{3.1}$$

where, here and throughout the paper $\left(\dfrac{\cdot}{\cdot}\right)$ denotes the Legendre symbol.

PROOF : Setting $\ell = 5$ in (1.4), we obtain

$$\sum_{n=0}^{\infty} \bar{A}_5(n) q^n = \frac{f_5^2 f_2}{f_1^2 f_{10}}. \tag{3.2}$$

Employing Lemma 2.5 in (3.2), we obtain

$$\sum_{n=0}^{\infty} \bar{A}_5\,(n)\,q^n = \frac{f_2}{f_{10}}\left[\frac{f_8^2 f_{20}^4}{f_2^4 f_{40}^2} + q^2\frac{f_4^6 f_{10}^2 f_{40}^2}{f_2^6 f_8^2 f_{20}^2} + 2q\frac{f_{20} f_{10} f_4^3}{f_2^5}\right]. \tag{3.3}$$

Extracting the terms involving q^{2n+1} from (3.3), dividing by q and replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty} \bar{A}_5\,(2n+1)\,q^n = 2\frac{f_{10} f_2^3}{f_1^4}. \tag{3.4}$$

Employing (2.9) in (3.4), we obtain

$$\sum_{n=0}^{\infty} \bar{A}_5\,(2n+1)\,q^n = 2\frac{f_{10} f_4^{14}}{f_8^4 f_2^{11}} + 8q\frac{f_{10} f_4^2 f_8^4}{f_2^7}. \tag{3.5}$$

Extracting the terms involving q^{2n} from (3.5) and replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty} \bar{A}_5\,(4n+1)\,q^n = 2\frac{f_5 f_2^{14}}{f_4^4 f_1^{11}}. \tag{3.6}$$

Simplifying (3.6) by using Lemma 2.2 with $p=2$, we obtain

$$\sum_{n=0}^{\infty} \bar{A}_5\,(4n+1)\,q^n \equiv 2 f_1 f_5 \pmod{4}. \tag{3.7}$$

So (3.1) holds for $\alpha = 0$ case.

Assume (3.1) holds for α. Employing Lemma 2.3 in (3.1), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} \bar{A}_5\left(4p^{2\alpha}n + p^{2\alpha}\right)q^n \equiv 2\Bigg[&\sum_{\substack{k=-\frac{p-1}{2}\\ k\neq\frac{\pm p-1}{6}}}^{\frac{p-1}{2}} (-1)^k q^{(3k^2+k)/2} f\left(-q^{\frac{3p^2+(6k+1)p}{2}}, -q^{\frac{3p^2-(6k+1)p}{2}}\right)\\ &+(-1)^{\frac{\pm p-1}{6}} q^{\frac{p^2-1}{24}} f_{p^2}\Bigg]\\ \times\Bigg[&\sum_{\substack{m=-\frac{p-1}{2}\\ m\neq\frac{\pm p-1}{6}}}^{\frac{p-1}{2}} (-1)^m q^{5(3m^2+m)/2} f\left(-q^{5\frac{3p^2+(6m+1)p}{2}}, -q^{5\frac{3p^2-(6m+1)p}{2}}\right) + (-1)^{\frac{\pm p-1}{6}} q^{5\frac{p^2-1}{24}} f_{5p^2}\Bigg] \pmod{4}.\end{aligned} \tag{3.8}$$

Consider the congruence

$$\frac{3k^2+k}{2}+5\left(\frac{3m^2+m}{2}\right)\equiv 6\left(\frac{p^2-1}{24}\right) \pmod{p}. \tag{3.9}$$

The congruence (3.9) is equivalent to

$$(6k+1)^2+5(6m+1)^2\equiv 0 \pmod{p}. \tag{3.10}$$

For $\left(\frac{-5}{p}\right)=-1$ the congruence (3.10) has a unique solution $k=m=\frac{\pm p-1}{6}$.

So extracting the terms involving $q^{pn+(p^2-1)/4}$ from (3.8), dividing by $q^{(p^2-1)/4}$ and replacing q^p by q, we deduce that

$$\sum_{n=0}^{\infty}\bar{A}_5\left(4p^{2\alpha+1}n+p^{2\alpha+2}\right)q^n\equiv 2f_pf_{5p} \pmod{4}. \tag{3.11}$$

Extracting the terms involving q^{pn} from (3.11) and replacing q^p by q, we obtain

$$\sum_{n=0}^{\infty}\bar{A}_5\left(4p^{2(\alpha+1)}n+p^{2(\alpha+1)}\right)q^n\equiv 2f_1f_5 \pmod{4}, \tag{3.12}$$

which is the case $\alpha+1$ of (3.1). Hence, the proof is complete. □

Corollary 3.2 — Let $p\geq 5$ be an odd prime with $\left(\frac{-5}{p}\right)=-1$. Then for non-negative integers α and n, we have

$$\bar{A}_5\left(4p^{2\alpha+2}n+4p^{2\alpha+1}j+p^{2\alpha+2}\right)\equiv 0 \pmod{4}, \tag{3.13}$$

where $j=1,2,3,\ldots,p-1$.

PROOF : Extracting the terms involving q^{pn+j} for $j=1,2,3,\ldots,p-1$ from (3.11), we arrive at the desired result. □

Corollary 3.3 — We have

$$\bar{A}_5(4n+3)\equiv 0 \pmod{8}.$$

PROOF : This follows easily from (3.5) by extracting the terms involving q^{2n+1}. □

4. CONGRUENCES FOR 6-REGULAR OVERPARTITION

In this section we prove an infinite family if congruences for $\bar{A}_6(n)$ modulo 3.

***Theorem* 4.1** — *Let p be an odd prime such that $\left(\frac{-2}{p}\right)=-1$. Then for non-negative integers α and n, we have*

$$\sum_{n=0}^{\infty}\bar{A}_6\left(8p^{2\alpha}n+3p^{2\alpha}\right)q^n\equiv 2\psi(q)\psi(q^2)\pmod 3. \tag{4.1}$$

PROOF : Setting $\ell=6$ in (1.4), we obtain

$$\sum_{n=0}^{\infty}\bar{A}_6\left(n\right)q^n=\frac{f_6^2f_2}{f_1^2f_{12}}. \tag{4.2}$$

Employing (2.10) in (4.2), we obtain

$$\sum_{n=0}^{\infty}\bar{A}_6\left(n\right)q^n=\frac{f_6^2f_2}{f_{12}}\left[\frac{f_8^5}{f_2^5f_{16}^2}+2q\frac{f_4^2f_{16}^2}{f_2^5f_8}\right]. \tag{4.3}$$

Extracting the terms involving q^{2n+1}, dividing by q and replacing q^2 by q, we deduce that

$$\sum_{n=0}^{\infty}\bar{A}_6\left(2n+1\right)q^n=2\frac{f_2^2f_3^2f_8^2}{f_6f_4f_1^4}. \tag{4.4}$$

Simplifying (4.4) using Lemma (2.1) with $p=3$ and employing (2.4) and (2.5), we obtain

$$\sum_{n=0}^{\infty}\bar{A}_6\left(2n+1\right)q^n\equiv 2\frac{f_1^2f_8^2}{f_2f_4}=2\phi(-q)\psi(q^4)\pmod 3. \tag{4.5}$$

Employing (2.6) in (4.5), we obtain

$$\sum_{n=0}^{\infty}\bar{A}_6\left(2n+1\right)q^n\equiv 2\phi(q^4)\psi(q^4)-4q\psi(q^4)\psi(q^8)\pmod 3. \tag{4.6}$$

Extracting the terms involving q^{4n+1} in (4.6), dividing by q and replacing q^4 by q, we obtain

$$\sum_{n=0}^{\infty}\bar{A}_6\left(8n+3\right)q^n\equiv 2\psi(q)\psi(q^2)\pmod 3, \tag{4.7}$$

which is the case $\alpha=0$ of (4.1). Assume that (4.1) holds for α. Employing Lemma 2.4 in (4.1), we obtain

$$\sum_{n=0}^{\infty}\bar{A}_6\left(8p^{2\alpha}n+3p^{2\alpha}\right)q^n\equiv 2\left[\sum_{k=0}^{\frac{p-3}{2}}q^{\frac{k^2+k}{2}}f(q^{\frac{p^2+(2k+1)p}{2}},q^{\frac{p^2-(2k+1)p}{2}})+q^{\frac{p^2-1}{8}}\psi(q^{p^2})\right]$$

$$\times\left[\sum_{m=0}^{\frac{p-3}{2}} q^{2\frac{m^2+m}{2}} f(q^{2\frac{p^2+(2m+1)p}{2}}, q^{2\frac{p^2-(2m+1)p}{2}}) + q^{2\frac{p^2-1}{8}}\psi(q^{2p^2})\right] \pmod 3. \tag{4.8}$$

Consider the congruence

$$\frac{k^2+k}{2} + 2\left(\frac{m^2+m}{2}\right) \equiv 3\left(\frac{p^2-1}{8}\right) \pmod p. \tag{4.9}$$

The congruence (4.9) is equivalent to

$$(2k+1)^2 + 2(2m+1)^2 \equiv 0 \pmod p. \tag{4.10}$$

For $\left(\frac{-2}{p}\right) = -1$ the congruence (4.10) has a unique solution $k = m = \frac{p-1}{2}$.

So extracting the terms involving $q^{pn+3(p^2-1)/8}$ from (4.8), dividing by $q^{3(p^2-1)/8}$ and replacing q^{pn} by q, we deduce that

$$\sum_{n=0}^{\infty} \bar{A}_6\left(8p^{2\alpha+1}n + 3p^{2\alpha+2}\right) q^n \equiv 2\psi(q^p)\psi(q^{2p}) \pmod 3. \tag{4.11}$$

Extracting the terms involving q^{pn} from (4.11) and replacing q^{pn} by q, we obtain

$$\sum_{n=0}^{\infty} \bar{A}_6\left(8p^{2(\alpha+1)}n + 3p^{2(\alpha+1)}\right) q^n \equiv 2\psi(q)\psi(q^2) \pmod 3, \tag{4.12}$$

which is the case $\alpha+1$ of (4.1). So the proof is complete.

Corollary 4.2 — Let p be an odd prime such that $\left(\frac{-2}{p}\right) = -1$. Then for non-negative integers α and n, we have

$$\bar{A}_6\left(8p^{2\alpha+2}n + 8p^{2\alpha+1}j + 3p^{2\alpha+2}\right) \equiv 0 \pmod 3, \tag{4.13}$$

where $j = 1, 2, 3, \ldots, p-1$.

PROOF: Extracting the terms involving q^{pn+j} for $j = 1, 2, 3, \ldots, p-1$ from (4.11), we arrive at the desired result. □

Corollary 4.3 — We have

(i) $\bar{A}_6(8n+5) \equiv 0 \pmod 3$,

(ii) $\bar{A}_6(8n+7) \equiv 0 \pmod 3$.

PROOF : Extracting the terms involving q^{4n+2} and q^{4n+3} in (4.6) we easily arrive at (i) and (ii), respectively. □

5. CONGRUENCES FOR 8-REGULAR OVERPARTITION

***Theorem* 5.1** — *We have*

(i) $\bar{A}_8(12n+4i+1) \equiv 0 \pmod 3, \quad where \quad i = 1, 2.$

(ii) $\bar{A}_8(8n+k) \equiv 0 \pmod 4, \quad where \quad k = 3, 5.$

(iii) $\bar{A}_8(28n+4j) \equiv 0 \pmod 7, \quad where \quad j = 1, 2, 3, 4, 5, 6.$

PROOF : Setting $\ell = 8$ in (1.4), we obtain

$$\sum_{n=0}^{\infty} \bar{A}_8\,(n)\,q^n = \frac{f_8^2 f_2}{f_1^2 f_{16}}. \tag{5.1}$$

Employing (2.10) in (5.1), we obtain

$$\sum_{n=0}^{\infty} \bar{A}_8\,(n)\,q^n = \frac{f_8^2 f_2}{f_{16}} \left[\frac{f_8^5}{f_2^5 f_{16}^2} + 2q \frac{f_4^2 f_{16}^2}{f_2^5 f_8} \right]. \tag{5.2}$$

Extracting the terms involving q^{2n+1} from (5.2), dividing by q and replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty} \bar{A}_8\,(2n+1)\,q^n = 2 \frac{f_2^2 f_4 f_8}{f_1^4}. \tag{5.3}$$

Employing (2.9) in (5.3), we obtain

$$\sum_{n=0}^{\infty} \bar{A}_8\,(2n+1)\,q^n = 2 f_2^2 f_4 f_8 \left[\frac{f_4^{14}}{f_2^{14} f_8^4} + 4q \frac{f_4^2 f_8^4}{f_2^{10}} \right]. \tag{5.4}$$

Extracting the terms involving q^{2n} in (5.4) and replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty} \bar{A}_8\,(4n+1)\,q^n = 2 \frac{f_2^{15}}{f_1^{12} f_4^3}. \tag{5.5}$$

Simplifying (5.5) using Lemma 2.1 with $p = 3$, we obtain

$$\sum_{n=0}^{\infty} \bar{A}_8\,(4n+1)\,q^n \equiv 2 \frac{f_6^5}{f_3^4 f_{12}} \pmod 3. \tag{5.6}$$

Extracting the terms involving q^{3n+1} and q^{3n+2}, we complete the proof of (i).

Again, simplifying (5.3) using Lemma 2.2 with $p=2$, we obtain

$$\sum_{n=0}^{\infty} \bar{A}_8(2n+1)\,q^n \equiv 2f_4f_8 \pmod 4. \tag{5.7}$$

Extracting the terms involving q^{4n+k} for $k=1,2$, we complete the proof of (ii).

Next, extracting the terms involving q^{2n} from (5.2) and replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty} \bar{A}_8(2n)\,q^n = \frac{f_4^7}{f_8^3 f_1^4}. \tag{5.8}$$

Employing (2.9) in (5.8), we obtain

$$\sum_{n=0}^{\infty} \bar{A}_8(2n)\,q^n = \frac{f_4^{21}}{f_8^7 f_2^{14}} + 4q\frac{f_4^9 f_8}{f_2^{10}}. \tag{5.9}$$

Extracting the terms involving q^{2n} from (5.9) and replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty} \bar{A}_8(4n)\,q^n = \frac{f_2^{21}}{f_4^7 f_1^{14}}. \tag{5.10}$$

Simplifying (5.10) using Lemma 2.1 with $p=7$, we obtain

$$\sum_{n=0}^{\infty} \bar{A}_8(4n)\,q^n \equiv \frac{f_{14}^3}{f_{28} f_7^2} \pmod 7. \tag{5.11}$$

Extracting the terms involving q^{7n+j} for $j=1,2,3,4,5,6$, we complete the proof of (iii).

***Theorem* 5.2** — *Let $p \geq 5$ be a prime such that $\left(\frac{-2}{p}\right) = -1$. Then for non-negative integers α and n*

$$\sum_{n=0}^{\infty} \bar{A}_8\left(8p^{2\alpha}n + p^{2\alpha}\right) q^n \equiv 2f_1f_2 \pmod 4. \tag{5.12}$$

PROOF : Extracting the terms involving q^{4n} in (5.7) and replacing q^4 by q, we obtain

$$\sum_{n=0}^{\infty} \bar{A}_8(8n+1)\,q^n \equiv 2f_1f_2 \pmod 4 \tag{5.13}$$

which is the $\alpha=0$ case of (5.12). Assume (5.12) holds for α. Employing Lemma 2.3 in (5.13), we obtain

$$\sum_{n=0}^{\infty} \bar{A}_8\left(8p^{2\alpha}n + p^{2\alpha}\right) q^n \equiv 2[\sum_{\substack{k=\frac{-p-1}{2}\\ k\neq \frac{\pm p-1}{6}}}^{\frac{p-1}{2}} (-1)^k q^{(3k^2+k)/2} f\left(-q^{\frac{3p^2+(6k+1)p}{2}}, -q^{\frac{3p^2-(6k+1)p}{2}}\right)$$

$$+(-1)^{\frac{\pm p-1}{6}} q^{\frac{p^2-1}{24}} f_{p^2}]$$

$$\times \left[\sum_{\substack{m=\frac{-p-1}{2} \\ m\neq \frac{\pm p-1}{6}}}^{\frac{p-1}{2}} (-1)^m q^{2(3m^2+m)/2} f\left(-q^{2\frac{3p^2+(6m+1)p}{2}}, -q^{2\frac{3p^2-(6m+1)p}{2}}\right) + (-1)^{\frac{\pm p-1}{6}} q^{2\frac{p^2-1}{24}} f_{2p^2} \right] \pmod 4. \tag{5.14}$$

Consider the congruence

$$\frac{3k^2+k}{2} + 2\left(\frac{3m^2+m}{2}\right) \equiv 3\left(\frac{p^2-1}{24}\right) \pmod p. \tag{5.15}$$

The congruence (5.15) is equivalent to

$$(6k+1)^2 + 2(6m+1)^2 \equiv 0 \pmod p. \tag{5.16}$$

For $\left(\frac{-2}{p}\right) = -1$ the congruence (5.16) has a unique solution $k = m = \frac{\pm p - 1}{6}$.

So extracting the terms involving $q^{pn+(p^2-1)/8}$ from (5.14), dividing by $q^{(p^2-1)/8}$ and replacing q^p by q, we deduce that

$$\sum_{n=0}^{\infty} \bar{A}_8\left(8p^{2\alpha+1}n + p^{2\alpha+2}\right) q^n \equiv 2f_p f_{2p} \pmod 4. \tag{5.17}$$

Extracting the terms involving q^{pn} from (5.17) and replacing q^p by q, we obtain

$$\sum_{n=0}^{\infty} \bar{A}_8\left(8p^{2(\alpha+1)}n + p^{2(\alpha+1)}\right) q^n \equiv 2f_1 f_2 \pmod 4, \tag{5.18}$$

which is the case $\alpha + 1$ of (5.12). Hence the proof is complete. □

Corollary 5.3 — Let $p \geq 5$ be a prime with $\left(\frac{-2}{p}\right) = -1$. Then for non-negative integers α and n, we have

$$\bar{A}_8\left(8p^{2(\alpha+1)}n + 8p^{2\alpha+1}j + p^{2\alpha+2}\right) \equiv 0 \pmod 4, \tag{5.19}$$

where $j = 1, 2, 3, \ldots, p-1$.

PROOF : Extracting the terms involving q^{pn+j} for $j = 1, 2, 3, \ldots, p-1$ from (5.17), we arrive at the desired result. □

***Theorem* 5.4** — *We have*

(i) $\bar{A}_8(4n+3) \equiv 0 \pmod 8$.

$(ii)\ \bar{A}_8(8n+7) \equiv 0 \pmod{64}.$

$(iii)\ \bar{A}_8(24n+8i+7) \equiv 0 \pmod{3}, \quad where \quad i=1,2.$

$(iv)\ \bar{A}_8(56n+8j+7) \equiv 0 \pmod{7}, \quad where \quad j=1,2,3,4,5,6.$

PROOF : Extracting the terms involving q^{2n+1} in (5.4), we obtain

$$\sum_{n=0}^{\infty} \bar{A}_8\,(4n+3)\,q^n = 8\frac{f_2^3 f_4^5}{f_1^8}. \tag{5.20}$$

Now (i) follows easily from (5.20).

Employing (2.9) in (5.20), we obtain

$$\sum_{n=0}^{\infty} \bar{A}_8\,(4n+3)\,q^n = 8\frac{f_4^{33}}{f_2^{25} f_8^8} + 128q^2\frac{f_4^9 f_8^8}{f_2^{17}} + 64q\frac{f_4^{21}}{f_2^{21}}. \tag{5.21}$$

Extracting the terms involving q^{2n+1} from (5.21), dividing by q, and replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty} \bar{A}_8\,(8n+7)\,q^n = 64\frac{f_2^{21}}{f_1^{21}}. \tag{5.22}$$

Now (ii) follows easily from (5.22).

Simplifying (5.22) using Lemma 2.1 with $p=3$, we obtain

$$\sum_{n=0}^{\infty} \bar{A}_8\,(8n+7)\,q^n \equiv \frac{f_6^7}{f_3^7} \pmod{3}. \tag{5.23}$$

Extracting the terms involving q^{3n+1} and q^{3n+2} in (5.23), we complete the proof of (iii).

Again, simplifying (5.22) using Lemma 2.1 with $p=7$, we obtain

$$\sum_{n=0}^{\infty} \bar{A}_8\,(8n+7)\,q^n \equiv \frac{f_{14}^3}{f_7^3} \pmod{7}. \tag{5.24}$$

Extracting the terms involving q^{7n+j} for $j=1,2,3,4,5,6$ in (5.24), we complete the proof of (iv). □

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