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SYMMETRIC INEQUALITIES WITH POWER-EXPONENTIAL FUNCTIONS

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We establish symmetric inequalities with power-exponential functions and propose a conjecture.

Key words : Inequalities; power-exponential functions; monotonically increasing functions; monotonically decreasing functions.

1. INTRODUCTION

Studying inequalities with power-exponential functions is the one of the most interesting fields of mathematical analysis. Coronel and Huancas [1] introduced the literature and history of this subject and mathematicians [1-13] studied inequalities with power-exponential functions and conjectured some open inequalities. Especially, the following symmetric inequality is the one of the simplest shaped form; if a and b are nonnegative real numbers with $a+b=1$, then the inequality $a^{2b}+b^{2a} \leq 1$ holds. The inequality is posed by Cîrtoaje [3] as Conjecture 4.8 and proved by himself in [4] and Matejíčka [6]. In [3, 4, 6] and [9], it is known that the following other symmetric inequalities hold; the inequalities $a^{2b}+b^{2a} \leq 2$ and $a^{3b}+b^{3a} \leq 2$ hold for nonnegative real numbers a and b with $a+b=2$ and moreover, the inequality with double power exponential functions $a^{(2b)^k}+b^{(2a)^k} \leq 1$ holds for nonnegative real numbers a and b with $a+b=1$ and $k \geq 1$. The above symmetric inequalities look like very simple forms, but these proofs are not immediate. In this paper, we establish symmetric inequalities as follows.

***Theorem* 1.1** — *If a and b are nonnegative real numbers with $a+b=1/2$, then the inequality $a^{2b}+b^{2a} \leq 1$ holds.*

***Theorem* 1.2** — *If a and b are nonnegative real numbers with $a+b=c$, then the inequality $a^{2b}+b^{2a} \leq 1$ holds for $1/2 \leq c \leq 1$.*

2. Proofs of Main Theorems

2.1 Proof of Theorem 1.1

Without loss of generality, we may assume that $0 \leq a \leq 1/4 \leq b \leq 1/2$. Here, we set

$$F(a) = 1 + (-4 + 8\ln 2)(a - 1/4) + \left(-4 - 16\ln 2 + 16(\ln 2)^2\right)(a - 1/4)^2$$

for $0 < a < 1/4$. We obtain lemmas related to $F(a)$.

Lemma 2.1 — For any $0 < a < 1/4$, we have

$$0 < F(a) < 1.$$

Proof : The derivatives of $F(a)$ are

$$F'(a) = -4 + 8\ln 2 + 2\left(-\frac{1}{4} + a\right)\left(-4 - 16\ln 2 + 16(\ln 2)^2\right)$$

and

$$F''(a) = 2\left(-4 - 16\ln 2 + 16(\ln 2)^2\right)$$
$$\cong 2 \times (-7.40311) < 0\,.$$

Thus, $F'(a)$ is strictly decreasing for $0 < a < 1/4$. Since $F'(a) > F'(1/4) = -4 + 8\ln 2 \cong 1.54518$, we have $F(a)$ is strictly increasing for $0 < a < 1/4$. From $F(0) = \left(7 - 12\ln 2 + 4(\ln 2)^2\right)/4 \cong 0.151011$ and $F(1/4) = 1$, we obtain $0 < F(a) < 1$. □

Lemma 2.2 — For any $0 < t < 1$, we have

$$\ln(1 + t) > (\ln 2)t.$$

Proof : We set $f(t) = \ln(1 + t) - (\ln 2)t$. The derivative of $f(t)$ is $f'(t) = 1/(1 + t) - \ln 2$. Therefore, $f'(t) > 0$ for $0 < t < (1 - \ln 2)/(\ln 2)$ and $f'(t) < 0$ for $(1 - \ln 2)/(\ln 2) < t < 1$. Since $f(t)$ is strictly increasing for $0 < t < (1 - \ln 2)/(\ln 2)$ and $f(t)$ is strictly decreasing for $(1 - \ln 2)/(\ln 2) < t < 1$, we can get $f(t) > \min\{f(0), f(1)\}$. From $f(0) = f(1) = 0$, we have $f(t) > 0$ for $0 < t < 1$. □

Lemma 2.3 — For any $0 < a < 1/4$, we have

$$(\ln 2)F(a) > 2a(\ln a).$$

PROOF : We set $f(a) = (\ln 2)F(a) - 2a \ln a$. The derivatives of $f(a)$ are

$$f'(a) = 2\left(-1 - \ln 2 - 4a\ln 2 + 8(\ln 2)^2 - 16a(\ln 2)^2 - 4(\ln 2)^3 + 16a(\ln 2)^3 - \ln a\right)$$

and

$$f''(a) = \frac{2\left(-1 - 4a\ln 2 + 16a(-1 + \ln 2)(\ln 2)^2\right)}{a}.$$

From $-1 + \ln 2 \cong -0.306853$, we have $f''(a) < 0$ and $f'(a)$ is strictly decreasing for $0 < a < 1/4$. Since

$$\begin{aligned} f'(a) >& f'\left(\frac{1}{4}\right) \\ =& 2\left(-1 - 2\ln 2 + 4(\ln 2)^2 + 2\ln 2\right) \\ \cong& 1.84362\,, \end{aligned}$$

we can get $f'(a) > 0$ and $f(a)$ is strictly increasing for $0 < a < 1/4$. From $f(0+) = (\ln 2)\left(7 - 12\ln 2 + 4(\ln 2)^2\right)/4$ and $-12\ln 2 + 4(\ln 2)^2 \cong -6.39595$, we can get $f(a) > 0$ for $0 < a < 1/4$. □

From Lemmas 2.1 and 2.2, the inequality $\ln(1 + F(a)) > (\ln 2)F(a)$ holds. Moreover, by Lemma 2.3, we have $(\ln 2)F(a) > 2a(\ln a)$. Hence, for $0 < a < 1/4$, we can get $1 + F(a) \geq a^{2a}$. Therefore, the inequality $a(1 + F(a)) \geq a^{1+2a}$ holds. Thus, it suffices to show that the inequality $1 - a\,(1 + F(a)) > (1/2 - a)^{2a}$ holds for $0 < a < 1/4$. We denote $t = 1/2 - a$. The inequality is equivalent to

$$1 - \left(\frac{1}{2} - t\right)\left\{2 + (-4 + 8\ln 2)\left(\frac{1}{4} - t\right) + \left(-4 - 16\ln 2 + 16(\ln 2)^2\right)\left(\frac{1}{4} - t\right)^2\right\} > t^{1-2t}$$

for $1/4 < t < 1/2$. We denote

$$G(t) = 1 - \left(\frac{1}{2} - t\right)\left\{2 + (-4 + 8\ln 2)\left(\frac{1}{4} - t\right) + \left(-4 - 16\ln 2 + 16(\ln 2)^2\right)\left(\frac{1}{4} - t\right)^2\right\} - t^{1-2t}.$$

The derivatives of $G(t)$ are

$$G'(t) = 2 + \frac{1}{4}(-17 + 64t - 48t^2 + 4\ln 2 + 64t\ln 2 - 192t^2\ln 2 + 20(\ln 2)^2 - 128t(\ln 2)^2 + 192t^2(\ln 2)^2) - t^{1-2t}\left(\frac{1 - 2t}{t} - 2\ln t\right),$$

$$G''(t) = \left(\frac{1+2t}{t^2}\right)t^{1-2t} + \frac{1}{4}(64 - 96t + 64\ln 2 - 384t\ln 2 - 128(\ln 2)^2 + 384t(\ln 2)^2) - t^{1-2t}\left(\frac{1-2t}{t} - 2\ln t\right)^2$$

and

$$G'''(t) = 24\left(-1 - 4\ln 2 + 4(\ln 2)^2\right) + 4t^{-1-2t}G_1(t)\,,$$

where $G_1(t) = 1 - 6t + 2t^2 - 9t\ln t + 6t^2\ln t - 3t(\ln t)^2 + 6t^2(\ln t)^2 + 2t^2(\ln t)^3$.

Lemma 2.4 — For any $1/4 < t < 1/2$, we have

$$G_1(t) < \frac{6}{5}\,.$$

PROOF : We set

$$f(t) = 1 - 6t + 2t^2 - 9t\ln t + 6t^2\ln t - 3t(\ln t)^2 + 5t^2(\ln t)^2 + 2t^2(\ln t)^3$$

and $g(t) = t^2(\ln t)^2$. Then the derivatives of $f(t)$ are

$$\begin{aligned} f'(t) &= -15 + 10t - 15\ln t + 22t\ln t - 3(\ln t)^2 + 16t(\ln t)^2 + 4t(\ln t)^3\,, \\ f''(t) &= 32 - \frac{15}{t} + 54\ln t - \frac{6\ln t}{t} + 28(\ln t)^2 + 4(\ln t)^3\,, \\ f'''(t) &= \frac{9 + 54t + 6\ln t + 56t\ln t + 12t(\ln t)^2}{t^2} \\ &= \frac{h(t)}{t^2}\,. \end{aligned}$$

Here, the derivative of $h(t)$ is

$$\begin{aligned} h'(t) &= 110 + \frac{6}{t} + 80\ln t + 12(\ln t)^2 \\ &\geq 110 + 12 + 80\ln\left(\frac{1}{4}\right) \\ &\cong 11.0965. \end{aligned}$$

By $h'(t) > 0$ for $1/4 < t < 1/2$, $h(t)$ is strictly increasing for $1/4 < t < 1/2$. From

$$\begin{aligned} h\left(\frac{1}{4}\right) &= \frac{45}{2} - 40\ln 2 + 12(\ln 2)^2 \\ &\cong 0.539549 \end{aligned}$$

and $f'''(t) > 0$, $f''(t)$ is strictly increasing for $1/4 < t < 1/2$. By

$$\begin{aligned} f''\left(\frac{1}{2}\right) &= 2 - 42\ln 2 + 28(\ln 2)^2 - 4(\ln 2)^3 \\ &\cong -14.9916 \end{aligned}$$

and $f''(t) < 0$, $f'(t)$ is strictly decreasing for $1/4 < t < 1/2$. From

$$\begin{aligned} f'\left(\frac{1}{4}\right) &= -\frac{25}{2} + 19\ln 2 + 4(\ln 2)^2 - 8(\ln 2)^3 \\ &\cong -0.0725887 \end{aligned}$$

and $f'(t) < 0$, $f(t)$ is strictly decreasing for $1/4 < t < 1/2$. Therefore,

$$\begin{aligned} f(t) &\leq f\left(\frac{1}{4}\right) \\ &= -\frac{3}{8} + \frac{15\,(\ln 2)}{4} - \frac{7\,(\ln 2)^2}{4} - (\ln 2)^3 \end{aligned}$$

for $1/4 < t < 1/2$. On the other hand, $g(t) \leq g(1/e) = 1/e^2$ for $1/4 < t < 1/2$. Hence, we have

$$\begin{aligned} G_1(t) &= f(t) + g(t) \\ &\leq -\frac{3}{8} + \frac{15\,(\ln 2)}{4} - \frac{7\,(\ln 2)^2}{4} - (\ln 2)^3 + \frac{1}{e^2} \\ &\cong 1.18582 \\ &< \frac{6}{5}\,. \end{aligned}$$

□

Proof of Theorem 1.1 : Since t^{-1-2t} is strictly decreasing for $1/4 < t < 1/2$, by Lemma 2.4, we have

$$\begin{aligned} G'''(t) &< 24\left(-1 - 4\ln 2 + 4(\ln 2)^2\right) + 4t^{-1-2t}\left(\frac{6}{5}\right) \\ &< 24\left(-1 - 4\ln 2 + 4(\ln 2)^2\right) + 4\left(\frac{1}{4}\right)^{-1-2\left(\frac{1}{4}\right)}\left(\frac{6}{5}\right) \\ &\cong -6.01864\,. \end{aligned}$$

Thus, $G''(t)$ is strictly decreasing for $1/4 < t < 1/2$. Since we have

$$\begin{aligned} G''\left(\frac{1}{4}\right) &= -2(-10 + 8(\ln 2)^2 + 8\ln 2) \\ &\cong 1.2224 \end{aligned}$$

and

$$G''\left(\frac{1}{2}\right) = 4(3 - 8\ln 2 + 3(\ln 2)^2)$$
$$\cong -4.41527\,,$$

there exists a unique real number t_1 with $1/4 < t_1 < 1/2$ such that $G''(t) > 0$ for $1/4 < t < t_1$ and $G''(t) < 0$ for $t_1 < t < 1/2$. Hence, $G'(t)$ is strictly increasing for $1/4 < t < t_1$ and $G'(t)$ is strictly decreasing for $t_1 < t < 1/2$. From $G'(1/4) = 0$ and

$$G'\left(\frac{1}{2}\right) = \frac{11}{4} - 5\ln 2 + (\ln 2)^2$$
$$\cong -0.235283\,,$$

there exists a unique real number t_2 with $1/4 < t_2 < 1/2$ such that $G'(t) > 0$ for $1/4 < t < t_2$ and $G'(t) < 0$ for $t_2 < t < 1/2$. Thus, $G(t)$ is strictly increasing for $1/4 < t < t_2$ and $G(t)$ is strictly decreasing for $t_2 < t < 1/2$. By $G(1/4) = G(1/2) = 0$, we can obtain $G(t) > 0$ for $1/4 < t < 1/2$. From $a(1 + F(a)) \geq a^{1+2a}$, $1 - a\,(1 + F(a)) > (1/2 - a)^{2a}$ holds for $0 < a < 1/4$, so the proof of Theorem 1.1 is complete. □

2.2 Proof of Theorem 1.2

Without loss of generality, we may assume that $0 \leq a \leq c/2 \leq b \leq c$. Here, we set $H(c) = a^{2(-a+c)} + (-a + c)^{2a} - 1$. The derivative of $H(c)$ is

$$\begin{aligned} H'(c) &= 2a(-a+c)^{-1+2a} + 2a^{2(-a+c)}\ln a \\ &= 2a^{2(-a+c)}\left(a^{1-2(-a+c)}(-a+c)^{-1+2a} + \ln a\right) \\ &= 2a^{2(-a+c)}I(c) \end{aligned}$$

and the derivative of $I(c)$ is

$$I'(c) = \frac{a^{1+2a-2c}(-a+c)^{2a}(-1+2a+2a\ln a - 2c\ln a)}{(a-c)^2}\,.$$

Lemma 2.5 — For any $1/2 \leq c \leq 1$ and $0 < a < c/2$, we have

$$-1 + 2a + 2a\ln a - 2c\ln a > 0.$$

Proof : We set $f(a) = -1 + 2a + 2a\ln a - 2c\ln a$. The derivative of $f(a)$ is

$$f'(a) = \frac{2(2a - c + a\ln a)}{a} < 0\,.$$

Since $f(a)$ is strictly decreasing for $0 < a < c/2$, $f(a) > f(c/2) = -1 + c - c\ln\,(c/2)$. We denote $g(c) = -1 + c - c\ln\,(c/2)$. The derivative of $g(c)$ is $g'(c) = -\ln\,(c/2) > 0$. Since $g(c)$ is strictly increasing for $1/2 < c < 1$ and $g(c) > g(1/2) = -1/2 + \ln 2 \cong 0.193147$, we can get $f(a) > 0$. □

From Lemma 2.5, we have $I'(c) > 0$ and $I(c)$ is strictly increasing for $1/2 < c < 1$. If $c = 1/2$, then by Theorem 1.1, we have $(1/2 - a)^{2a} < 1 - a^{1-2a}$ for $0 < a < 1/4$. Thus the following inequality holds.

$$
\begin{aligned}
I\left(\frac{1}{2}\right) &= \left(\frac{1}{2} - a\right)^{-1+2a} a^{1-2\left(\frac{1}{2}-a\right)} + \ln a \\
&= \frac{\left(\frac{1}{2} - a\right)^{2a} a^{2a} + (\frac{1}{2} - a)\ln a}{\frac{1}{2} - a} \\
&\leq \frac{a^{2a} - a + \left(\frac{1}{2} - a\right)\ln a}{\frac{1}{2} - a}\,.
\end{aligned}
$$

Lemma 2.6 — For any $0 < a < 1/4$, we have

$$
a^{2a} - a + \left(\frac{1}{2} - a\right)\ln a < 0.
$$

PROOF : We set

$$
f(a) = (\ln a)2a - \ln\,\left(a - \left(\frac{1}{2} - a\right)\ln a\right)\,.
$$

The derivative of $f(a)$ is

$$
f'(a) = \frac{(-1 + 2a)\left(-1 + 2a + 4a\ln a + 2a(\ln a)^2\right)}{a(2a + (-1 + 2a)\ln a)}\,.
$$

We denote $g(a) = -1/2 + 4a\ln a + 2a(\ln a)^2$. The derivative of $g(a)$ is $g'(a) = 2\left(2 + 4\ln a + (\ln a)^2\right)$. Therefore, we obtain $g'(a) > 0$ for $0 < a < e^{-2-\sqrt{2}}$ and $g'(a) < 0$ for $e^{-2-\sqrt{2}} < a < 1/4$. Hence, we have

$$
\begin{aligned}
g(a) &\leq g\left(e^{-2-\sqrt{2}}\right) \\
&= \frac{e^{-2-\sqrt{2}}}{2}\left(8 + 8\sqrt{2} - e^{2+\sqrt{2}}\right) \\
&\cong -0.182268
\end{aligned}
$$

for $0 < a < 1/4$. Since $-1 + 2a + 4a\ln a + 2a(\ln a)^2 < g(a) < 0$ and $(-1 + 2a)\ln a > 0$, we have $f'(a) > 0$ and $f(a)$ is strictly increasing for $0 < a < 1/4$. From

$$f\left(\frac{1}{4}\right) = -\ln 2 - \ln\left(\frac{1}{4} + \frac{\ln 2}{2}\right) \cong -0.176595\,,$$

we have $f(a) < 0$ for $0 < a < 1/4$. □

By Lemma 2.6, we have $I(1/2) < 0$. If $c = 1$, then $I\,(1) = (1-a)^{-1+2a}a^{-1+2a} + \ln a$. We may show that $I(1) > 0$.

Lemma 2.7 — For any $0 < a < 1/2$, we have

$$(1-a)^{-1+2a} > 1.$$

PROOF : We set $f(a) = (-1 + 2a)\ln(1 - a)$. The derivatives of $f(a)$ are

$$f'(a) = -\frac{-1+2a}{1-a} + 2\ln(1-a)$$

and

$$f''(a) = \frac{-3+2a}{(-1+a)^2}\,.$$

By $f''(a) < 0$ for $0 < a < 1/2$, $f'(a)$ is strictly decreasing for $0 < a < 1/2$. From $f'(0) = 1$ and $f'(1/2) = -2\ln 2 \cong -1.38629$, there exists a unique real number a_1 with $0 < a_1 < 1/2$ such that $f'(a) > 0$ for $0 < a < a_1$ and $f'(a) < 0$ for $a_1 < a < 1/2$. Therefore, $f(a)$ is strictly increasing for $0 < a < a_1$ and $f(a)$ is strictly decreasing for $a_1 < a < 1/2$. By $f(0) = f(1/2) = 0$, we can obtain $f(a) > 0$ for $0 < a < 1/2$. □

Lemma 2.8 — For any $0 < a < 1/2$, we have

$$a\ln a > -\frac{2}{5}\,.$$

PROOF : We set $f(a) = a\ln a + 2/5$. The derivative of $f(a)$ is $f'(a) = 1 + \ln a$. Since $f'(a)$ is strictly increasing for $0 < a < 1/2$ and we have $f'(a) < 0$ for $0 < a < 1/e$ and $f'(a) > 0$ for $1/e < a < 1/2$, $f(a) > f(1/e) = 2/5 - 1/e \cong 0.0321206$. Thus, we can get $f(a) > 0$ for $0 < a < 1/2$. □

Lemma 2.9 — For any $0 < a < 1/2$, we have

$$a^{2a} > \frac{2}{5}\,.$$

PROOF : We set $f(a) = 2a\ln a - \ln\,(2/5)$. The derivative of $f(a)$ is $f'(a) = 2(1 + \ln a)$. Since $f'(a) < 0$ for $0 < a < 1/e$ and $f'(a) > 0$ for $1/e < a < 1/2$, $f(a) > f(1/e) = -2/e + \ln\,(5/2) \cong 0.180532$. Thus, we can get $f(a) > 0$ for $0 < a < 1/2$. □

PROOF OF THEOREM 1.2 : By Lemmas 2.7, 2.8 and 2.9, we have $I(1) > 0$. Since $I(c)$ is strictly increasing for $1/2 < c < 1$ and $I(1/2) < 0$ and $I(1) > 0$, there exists a unique function $c = J(a)$ such that $I(J(a)) < 0$ for $1/2 < c < J(a)$ and $I(J(a)) > 0$ for $J(a) < c < 1$. Thus, $H(c)$ is strictly decreasing for $1/2 < c < J(a)$ and $H(c)$ is strictly increasing for $J(a) < c < 1$. Since Theorem 1.1 and the inequality $a^{2b} + b^{2a} \leq 1$ holds for $a + b = 1$, we have $H(1/2) \leq 0$ and $H(1) \leq 0$. Hence, we can obtain $H(c) \leq 0$ and the proof of Theorem 1.2 is complete. □

We propose the following conjecture.

Conjecture 2.10 — If a and b are nonnegative real numbers with $a + b = 1/2$, then the inequality

$$\frac{1}{2} \leq a^{(2b)^k} + b^{(2a)^k} \leq 1$$

holds for $0 \leq k \leq 1$.

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