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BOUNDARY VALUE PROBLEM FOR ONE EVOLUTION EQUATION

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The aim of the paper is to investigate the boundary value problem of the evolution equation

$$Lu = K(x,t)\,u_t - \Delta u + a(x,t)\,u = f(x,t)\,.$$

The characteristic property of this type of equations is the failure of the Petrovski's "A" condition when coefficients are constant [1]. In this case, Cauchy problem is incorrect in the sense of Hadamard. Hence in this paper, the space, guaranteeing the correctness of the boundary value problem in the sense of Hadamard, is selected by adding some additional conditions to the coefficients of the equation.

Key words : Evolution equations; boundary value problem; parabolic equations changing with direction of time; Sobolev type differential equations.

1. Introduction and Main Results

Let's consider the equation

$$Lu = K(x,t)\,u_t - \Delta u + a(x,t)\,u = f(x,t) \tag{1}$$

on the cylindrical area $Q = (0,T) \times D$ where D is bounded in the space R^n and $\partial D \in C^2$.

In this equation $K(x,0) \leq 0, \;\; K(x,T) \geq 0$ and there is no restriction on the sign of $K(x,t)$ inside the area Q.

Equation (1) expresses turbulent processes coming across. In the theory of partial differential equations, this kind of equations are called parabolic equations changing with the direction of time. Boundary value problems for such equations are the main research topic of many mathematicians. As an example, we can provide references [2-14]. In mentioned studies, the solvability of boundary value problems was shown in appropriate Sobolev spaces.

Boundary value problem : Finding the solution of equation (1) satisfying the condition

$$uI_{\Gamma} = 0 \tag{2}$$

in the area *Q*, where $\Gamma = (0,T) \times \partial D$.

Method : As the method, we used regularizing operators technique. If the following conditions are satisfied, operator family depending on the regularizing parameter ε is called regularizing operators family for the given problem:

1. For any $\varepsilon > 0$, the corresponding problem for the family is correct in the sense of Hadamard.
2. Under existing conditions, the solution of the given problem is the limit of solutions of problems for the families when $\varepsilon \to 0$

***Theorem* 1** — *Assume that $K(x,t)$, $a(x,t) \in C^1(Q)$ and $2a - |K_t(x,t)| \geq \delta > 0$. Then, for each function $f(x,t)$, $f_t(x,t) \in L^2(Q)$, there is a unique solution of the problem (1), (2) on the space $W^{(1,2)}_{2(t,x)}(Q)$.*

PROOF : Consider the following problem on the area *Q*:

Finding the solution of the equation

$$L_\varepsilon u_\varepsilon \equiv -\varepsilon u_{\varepsilon tt} - \Delta u_\varepsilon + K u_{\varepsilon t} + a u_\varepsilon = f, \tag{3}$$

which satisfies conditions

$$u_\varepsilon|_\Gamma = 0, \quad u_{\varepsilon t}|_{t=0} = 0, \; u_{\varepsilon t}|_{t=T} = 0. \tag{4}$$

Solvability of the problem (3), (4) for any fixed $\varepsilon > 0$ is well known from the general elliptical type equations theory [15]. Thus, if $f, \; f_t \in L^2(Q)$ then $u_{\varepsilon t} \in W_2^2(Q)$.

Now let's get the necessary inequalities.

By multiplying both sides of equation (3) by u_ε and integrating on the area *Q* ,we get the equality

$$\int_Q L_\varepsilon u_\varepsilon u_\varepsilon dQ = \int_Q f u_\varepsilon dQ.$$

Integrating by parts and using Holder inequality give us

$$\varepsilon \int_Q u_{et}^2 dQ + \int_Q (|\nabla u_\varepsilon|^2 + u_\varepsilon^2) dQ \leq c_1 \int_Q f^2 dQ, \tag{5}$$

where c_1 is a constant independent of ε.

Denoting $u_{\varepsilon t} = \vartheta$, we can see that the function ν satisfies the following conditions

$$\varepsilon\vartheta_{tt} - \Delta\vartheta + (a + K_t)\,\vartheta + K\vartheta_t = f_t - a_t u_\varepsilon = f, \qquad f \in L^2(Q) \tag{6}$$

$$\vartheta|_\Gamma = 0, \;\; \vartheta|_{t=0} = 0, \;\; \vartheta|_{t=T} = 0. \tag{7}$$

By multiplying both sides of equation (6) by ν and integrating on the area Q, we get the equality

$$\int_Q \left[\varepsilon\vartheta_{tt} - \Delta\vartheta + (a + K_t)\,\vartheta + K\vartheta_t\right]\vartheta dQ = \int_Q f\vartheta dQ.$$

Taking into account the conditions (7) in this equality and using partial integrating formula, equation (5) and Holder inequality we obtain the inequality

$$\varepsilon\int_Q \vartheta_t^2 dQ + \int_Q (|\nabla\vartheta|^2 + \vartheta^2)dQ \leq c_2\int_Q (f^2 + f_t^2)dQ, \tag{8}$$

where c_2 is a constant independent of ε.

Now taking into account the problem (3), (4), and inequalities (5), (8),we get

$$\int_Q \sum_{i=1}^{n} u_{\varepsilon x_i x_i}^2 dQ \leq c_3\int_Q (f^2 + f_t^2)dQ\ , \tag{9}$$

where c_3 is a constant independent of ε.

As inequalities (5), (8), (9) are uniformly limited by ε, it is possible to extract converging subsequence $\{u_{\varepsilon_n}\}$ from the sequence $\{u_\varepsilon\}$.

When $\varepsilon_n \to 0$

$u_{\varepsilon_n} \to u$ in $L^2(Q)$ norm,
$u_{\varepsilon_n t} \to u_t$ in $L^2(Q)$ weakly,
$u_{\varepsilon_n x_i x_i} \to u_{x_i x_i}$ in $L^2(Q)$ weakly.

Hence, the limit function $u(x,t)$ is one solution of the problem (1), (2).

Proof of uniqueness :
Assume that there are two different solutions u_1 and u_2 of the problem (1), (2). Then the function $u = u_1 - u_2$ is also the solution of the problem

$$Lu = 0 \tag{10}$$

$$u|_\Gamma = 0. \tag{11}$$

Multiplying both sides of equation (6) by u and integrating on the area Q ,we get the equality

$$\int_Q Lu\, udQ = \int_Q \left(K\left(x,t\right)u_t - \Delta u + a\left(x,t\right)u\right)udQ = 0$$

Integrating this equality by parts, we get

$$\int_Q \left(K\left(x,t\right)u_t - \Delta u + a\left(x,t\right)u\right)udQ = \int_Q \left(\left|\nabla \mathrm{u}\right|^2 + \frac{2a - K_t}{2}u^2\right)dQ -$$

$$- \int_D K\left(x,t\right)u^2|_{t=0}dD + \int_D K\left(x,t\right)u^2|_{t=T}dD = 0.$$

Taking into account the boundary conditions and the conditions of Theorem 1,we obtain $u \equiv 0$. The theorem is proven.

Let's consider an example showing the importance of the conditions of Theorem 1.

Example : Define the following problem for the equation

$$Lu = tu_t - \Delta u - \left(1 + \lambda^2\right)u = 0 \tag{12}$$

on the area $Q_1 = (0,1) \times D$.

Find the solution of equation (12) on the area Q_1 satisfying the condition

$$u|_\Gamma = 0, \quad \Gamma = (0,1) \times dD. \tag{13}$$

It can be easily seen that the function $u = t\vartheta\left(x\right)$ is an unknown solution of the problem (12), (13). Here the function $\vartheta\left(x\right)$ is the solution of the problem

$$-\Delta\vartheta - \lambda^2\vartheta = 0 \tag{14}$$

$$v|_{dD} = 0. \tag{15}$$

This example shows that if conditions of Theorem 1 are not satisfied, then the solution of the problem (1), (2) is not unique.

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