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REPRESENTATION OF PROJECTORS INVOLVING MINKOWSKI INVERSE IN MINKOWSKI SPACE

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A certain class of results about the different representations of Oblique projectors is present in the literature. These results represent Oblique projectors as the functions of orthogonal projectors with given onto and along spaces. But these results are valid under the restriction that the functions of orthogonal projectors involved are invertible. In this paper we extend and generalize these results. The extension lies in making a transition from Euclidean space to Minkowski space $\mathscr{M}$ and the generalization is obtain by voiding the invertibility condition and use of the Minkowski inverse. Furthermore, the nobility lies in utilizing the *m*-projectors instead of the regular orthogonal projectors.

Key words : Minkowski inverse; *m*-symmetric; *m*-projectors; oblique projector.

1. Introduction

Let us denote by $M_{(m,n)}(\mathbb{C})$ the set of $m \times n$ matrices over the complex field and when $m = n$ we write $M_n(\mathbb{C})$ for $M_{(n,n)}(\mathbb{C})$. The symbols $\mathbf{A}^*$, $\mathbf{A}^\sim$, $\mathbf{A}^\oplus$, $\mathbf{A}^\dagger$, rk($\mathbf{A}$), $R(\mathbf{A})$, and $N(\mathbf{A})$ denote the conjugate transpose, Minkowski adjoint, Minkowski inverse, Moore-Penrose inverse, rank, range space, and null space of a matrix $\mathbf{A}$ respectively. $\mathbf{I_n}$ denotes the identity matrix of order $n \times n$. Further we denote by $\mathbb{C}_n^{mp}$ the set of all *m*-projections. i.e., $\mathbb{C}_n^{mp} = \{\mathbf{P} : \mathbf{P^2} = \mathbf{P} = \mathbf{P}^\sim\}$.

Indefinite inner product is a scalar product defined by

$$[\mathbf{u}, \mathbf{v}] = \langle \mathbf{u}, \mathbf{Hv} \rangle = \mathbf{u}^*\mathbf{Hv}, \tag{1}$$

where $\langle \ , \ \rangle$ denotes the conventional Hilbert Space inner product and $\mathbf{H}$ is a Hermitian matrix. This hermitian matrix $\mathbf{H}$ is referred to as metric matrix. Minkowski space $\mathscr{M}$ is an indefinite inner product space in which the metric matrix is denoted by $\mathbf{G}$ and is defined as

$$\mathbf{G} = \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I_{n-1}} \end{bmatrix}, \text{ satisfying } \mathbf{G}^2 = \mathbf{I}_n \text{ and } \mathbf{G}^* = \mathbf{G}.$$

$\mathbf{G}$ is called the Minkowski metric matrix. In case $\mathbf{u} = (\mathbf{u_0}, \mathbf{u_1}, ..., \mathbf{u_{n-1}}) \in \mathbb{C}^\mathbf{n}$, $\mathbf{G}$ is called the Minkowski metric tensor and is defined as $\mathbf{Gu} = (\mathbf{u_0}, -\mathbf{u_1}, ..., -\mathbf{u_{n-1}})$. For detailed study of indefinite linear algebra refer to [12].

The *m*-projectors, as a result of the impact of the Minkowski metric matrix on a projection, were identified by the author in [22]. Projections are widely scattered in the literature, see e.g., [3-6, 10, 15, 18, 19, 24, 25] and are used in pure as well as applied mathematics. Projections occupy a fundamental place in the literature of generalized inverses. There are many extensions and generalizations of the projections like generalized projections, k-generalized projections, hypergeneralized projections etc., for instance see [2, 11, 16, 23, 26]. Projections were also studied in indefinite inner product spaces see [27, and references therein].

An important projector known as oblique projector has been studied in different environmental settings, i.e., infinite and finite dimensional vector spaces over either complex or real Euclidean space resulting in establishment of various results of oblique projectors with given onto and along spaces, see e.g., [1, 13, 14, 21, 29]. But these results are valid only under the assumption of the nonsingularity of certain functions of projectors involved. In this paper we establish certain results from the literature in a more generalized and extended form, in a different environmental setting, obtained by making a transition from complex Euclidean space to Minkowski space $\mathscr{M}$ over the same field. This transition thus paves a way to go from a definite inner product space to an indefinite inner product space. Furthermore, the nobility lies in utilizing the *m*-projections instead of the orthogonal projections and the use of Minkowski inverse instead of ordinary inverse.

2. Preliminaries

In an indefinite inner product space the concept of orthogonality is analogous to that in the ordinary inner product space. Let $\mathbb{C}^n$ be an indefinite inner product space then a vector $\mathbf{x} \in \mathbb{C}^n$ is said to be orthogonal to a vector $\mathbf{y} \in \mathbb{C}^n$ if $[\mathbf{x}, \mathbf{y}] = \mathbf{0}$, where [,] denotes the indefinite inner product. Further, the orthogonal companion of a subspace say $\mathbf{S}$ of $\mathbb{C}^n$ is defined as $\mathbf{S}^{[\perp]} = \{\mathbf{x} \in \mathbb{C}^n : [\mathbf{x}, \mathbf{y}] = \mathbf{0} \; \forall \; \mathbf{y} \in \mathbf{S}\}$. We reserve the symbol $[\perp]$ to denote the orthogonality in the indefinite inner product space. Minkowski space $\mathscr{M}$ being an indefinite inner product space is nondegenerate in the sense that $[\mathbf{X}, \mathbf{Y}] = \mathbf{0} \; \forall \; \mathbf{Y} \in \mathscr{M}$ implies $\mathbf{X} = \mathbf{0}$. For every matrix $\mathbf{A} \in M_{(m,n)}(\mathbb{C})$, satisfying rk$(\mathbf{A}\mathbf{A}^\sim)$ = rk$(\mathbf{A}^\sim\mathbf{A})$ = rk$(\mathbf{A})$, there corresponds the *m*-projectors $\mathbf{A}\mathbf{A}^\oplus$ onto $R(\mathbf{A})$ along $N(\mathbf{A}^\sim)$, $\mathbf{I_m} - \mathbf{A}\mathbf{A}^\oplus$ onto $N(\mathbf{A}^\sim)$ along $R(\mathbf{A})$, $\mathbf{A}^\oplus\mathbf{A}$ onto $R(\mathbf{A}^\sim)$ along $N(\mathbf{A})$, and $\mathbf{I_n} - \mathbf{A}^\oplus\mathbf{A}$ onto $N(\mathbf{A})$

along $R(\mathbf{A}^{\sim})$. The rank equality rk$(\mathbf{A}\mathbf{A}^{\sim}) = $ rk$(\mathbf{A}^{\sim}\mathbf{A}) = $ rk$(\mathbf{A})$ is the necessary and sufficient condition for the existence of the Minkowski inverse $\mathbf{A}^{\oplus}$ of the matrix $\mathbf{A}$, which is the unique solution of equations

$$\mathbf{A}\mathbf{A}^{\oplus}\mathbf{A} = \mathbf{A}, \;\; \mathbf{A}^{\oplus}\mathbf{A}\mathbf{A}^{\oplus} = \mathbf{A}^{\oplus}, \;\; (\mathbf{A}\mathbf{A}^{\oplus})^{\sim} = \mathbf{A}\mathbf{A}^{\oplus}, \;\; (\mathbf{A}^{\oplus}\mathbf{A})^{\sim} = \mathbf{A}^{\oplus}\mathbf{A}. \tag{2}$$

For any subspace $\mathbf{S}$ of an indefinite inner product space $\mathbb{C}$, $\mathbf{S}^{[\perp]}$ is the direct complement to $\mathbf{S}$ if and only if $\mathbf{S}$ is nondegenerate, i.e., $\mathbf{S} \cap \mathbf{S}^{[\perp]} = \{\mathbf{0}\}$ or this can be equivalently expressed as $\mathbb{C} = \mathbf{S}\oplus^{[\perp]}\mathbf{S}^{[\perp]}$, where $\oplus^{[\perp]}$ indicates the orthogonality of the subspaces involved in the direct sum of an indefinite inner product space. Thus, for the *m*-projectors $\mathbf{A}\mathbf{A}^{\oplus}$ and $\mathbf{A}^{\oplus}\mathbf{A}$ corresponding to the matrix $\mathbf{A} \in M_{(m,n)}(\mathbb{C})$, we have

$$M_{(m,1)}(\mathbb{C}) = R(\mathbf{A})\oplus^{[\perp]}R(\mathbf{A})^{[\perp]} \text{ and } M_{(n,1)}(\mathbb{C}) = R(\mathbf{A}^{\sim})\oplus^{[\perp]}R(\mathbf{A}^{\sim})^{[\perp]}.$$

The following lemma gives a good insight of the relation between the subspaces of a matrix $\mathbf{A}$ in the Minkowski space $\mathscr{M}$.

Lemma 1 — Let $\mathbf{A} \in M_n(\mathbb{C})$ in the Minkowski space $\mathscr{M}$. Then, $N(\mathbf{A}) = R(\mathbf{A}^{\sim})^{[\perp]}$ and $N(\mathbf{A}^{\sim}) = R(\mathbf{A})^{[\perp]}$.

PROOF : For $\mathbf{A} \in M_n(\mathbb{C})$ and $\mathbf{x}, \mathbf{y} \in M_{(n,1)}(\mathbb{C})$, we have

$$[\mathbf{A}\mathbf{x}, \mathbf{y}] = \langle \mathbf{A}\mathbf{x}, \mathbf{G}\mathbf{y}\rangle = \langle \mathbf{x}, \mathbf{A}^{*}\mathbf{G}\mathbf{y}\rangle = [\mathbf{x}, \mathbf{A}^{\sim}\mathbf{y}]. \tag{3}$$

If $\mathbf{x} \in N(\mathbf{A})$ then, $[\mathbf{A}\mathbf{x}, \mathbf{y}] = \mathbf{0}\; \forall\; \mathbf{y} \in M_{(n,1)}(\mathbb{C})$. Thus, from (3), we get $[\mathbf{x}, \mathbf{A}^{\sim}\mathbf{y}] = \mathbf{0}$ which implies that $\mathbf{x}[\perp]\mathbf{R}(\mathbf{A}^{\sim})$ and hence, $N(\mathbf{A}) \subseteq R(\mathbf{A}^{\sim})^{[\perp]}$ and on the same lines we can prove the reverse inclusion. Therefore, $N(\mathbf{A}) = R(\mathbf{A}^{\sim})^{[\perp]}$ and its dual, i.e., $N(\mathbf{A}^{\sim}) = R(\mathbf{A})^{[\perp]}$ also holds.

Remark 1 : The Minkowski inverse of a matrix $\mathbf{A} \in M_{(m,n)}(\mathbb{C})$ exists if and only if rk$(\mathbf{A}\mathbf{A}^{\sim}) =$ rk$(\mathbf{A}^{\sim}\mathbf{A}) =$ rk$(\mathbf{A})$. This condition is trivially satisfied by the *m*-projections. For a *m*-projection $\mathbf{P}$, we have $\mathbf{P}\mathbf{P}^{\sim} = \mathbf{P^2} = \mathbf{P} = \mathbf{P^2} = \mathbf{P}^{\sim}\mathbf{P}$. Using this equality, we have rk$(\mathbf{P}\mathbf{P}^{\sim}) =$ rk$(\mathbf{P}^{\sim}\mathbf{P}) =$ rk $(\mathbf{P})$.

The representation obtained in the Corollary (2.5) in [22] facilitates to formulate the Minkowski inverse of the range symmetric matrices and in particular for the *m*-projections in the Minkowski space $\mathscr{M}$. A matrix $\mathbf{A} \in M_n(\mathbb{C})$ is said to be G-unitary if and only if $\mathbf{A}\mathbf{A}^{\sim} = \mathbf{A}^{\sim}\mathbf{A} = \mathbf{I}$. If $\mathbf{U} \in M_n(\mathbb{C})$ is unitary, then

$$\mathbf{U}\mathbf{G} = \mathbf{G}\mathbf{U} \Leftrightarrow \mathbf{U}\mathbf{G}\mathbf{U}^{*} = \mathbf{G} \Leftrightarrow \mathbf{U}\mathbf{G}\mathbf{U}^{*}\mathbf{G} = \mathbf{I_n} \Leftrightarrow \mathbf{U}\mathbf{U}^{\sim} = \mathbf{I_n}.$$

Similarly, $\mathbf{U}^{\sim}\mathbf{U} = \mathbf{I_n}$ also holds. We will use this assumption in formulating the Minkowski inverse of the *m*-projections in the forthcoming results.

Let $\mathbf{L}$ be an *m*-projector in the Minkowski space $\mathscr{M}$. Taking into account Remark 1 and the observations made from the Corollaries of the Theorem 2.4 in [22], there exists a G-unitary matrix $\mathbf{U}$ such that

$$\mathbf{L} = \mathbf{U}\begin{bmatrix}\mathbf{I_r} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\mathbf{U}^{\sim}. \tag{4}$$

The representation (4) of the *m*-projector can be used to determine the partitioning of any other *m*- projector say $\mathbf{M}$ with the use of the same G-unitary matrix $\mathbf{U}$ such that

$$\mathbf{M} = \mathbf{U}\begin{bmatrix}\mathbf{W} & \mathbf{X}\\ \mathbf{Y} & \mathbf{Z}\end{bmatrix}\mathbf{U}^{\sim}. \tag{5}$$

Using the later part of the definition of a *m*-projection, i.e., $\mathbf{M} = \mathbf{M}^{\sim}$, the representation (5) becomes

$$\mathbf{M} = \mathbf{U}\begin{bmatrix}\mathbf{W} & \mathbf{X}\\ -\mathbf{G_1}\mathbf{X}^{\sim} & \mathbf{Z}\end{bmatrix}\mathbf{U}^{\sim}, \tag{6}$$

where $\mathbf{W} \in M_r(\mathbb{C})$ is *m*-symmetric, $\mathbf{Z} \in M_{n-r}(\mathbb{C})$ is hermitian and $\mathbf{G_{n-r}}$ is the metric matrix of order $\mathbf{n-r\times n-r}$. We will use $\mathbf{G_1}$ in place of $\mathbf{G_{n-r}}$. The following results from [22] give the relation between the submatrices $\mathbf{W}$, $\mathbf{X}$ and $\mathbf{Z}$ of the matrix $\mathbf{M}$ given in (6) and will be used extensively in the main results. We use the notation $\bar{\mathbf{R}} = \mathbf{I_k} - \mathbf{R}$, where $\mathbf{I_k}$ is the identity matrix of order $\mathbf{k}$ and $\mathbf{R}$ is any matrix or a block of the given matrix of order $\mathbf{k}$. Also we use the convection according to which $\mathbf{P_R} = \mathbf{R}\mathbf{R}^{\oplus}$ and $\tilde{\mathbf{P}}_\mathbf{R} = \mathbf{I_k} - \mathbf{R}\mathbf{R}^{\oplus}$, where $\mathbf{R} \in \{\mathbf{W}, \bar{\mathbf{W}}, \mathbf{Z}, \bar{\mathbf{Z}}\}$.

Lemma 2 — Let $\mathbf{M} \in \mathbb{C}_n^{mp}$ be partitioned as in (6). Then:

(i) $\mathbf{W} = \mathbf{W^2} - \mathbf{X}\mathbf{G_1}\mathbf{X}^{\sim}$ or equivalently $\mathbf{W}\bar{\mathbf{W}} = -\mathbf{X}\mathbf{G_1}\mathbf{X}^{\sim}$,

(ii) $\mathbf{X} = \mathbf{W}\mathbf{X} + \mathbf{X}\mathbf{Z}$ or equivalently $\mathbf{X}^{\sim} = \mathbf{X}^{\sim}\mathbf{W} + \mathbf{G_1}\mathbf{Z}\mathbf{G_1}\mathbf{X}^{\sim}$,

(iii) $\mathbf{G_1}\mathbf{X}^{\sim} = \mathbf{G_1}\mathbf{X}^{\sim}\mathbf{W} + \mathbf{Z}\mathbf{G_1}\mathbf{X}^{\sim}$ or equivalently $\mathbf{G_1}\mathbf{X}^{\sim}\bar{\mathbf{W}} = \mathbf{Z}\mathbf{G_1}\mathbf{X}^{\sim}$,

(iv) $\mathbf{Z} = \mathbf{Z^2} - \mathbf{G_1}\mathbf{X}^{\sim}\mathbf{X}$ or equivalently $\mathbf{Z}\bar{\mathbf{Z}} = -\mathbf{G_1}\mathbf{X}^{\sim}\mathbf{X}$.

Lemma 3 — Let $\mathbf{M} \in \mathbb{C}_n^{mp}$ be partitioned as in (6). Then:

(i) $\bar{\mathbf{W}} = \bar{\mathbf{W}}^\mathbf{2} - \mathbf{X}\mathbf{G_1}\mathbf{X}^{\sim}$,

(ii) $\mathbf{W}\mathbf{X} = \mathbf{X}\bar{\mathbf{Z}}$,

(iii) $\mathbf{X}\mathbf{Z} = \bar{\mathbf{W}}\mathbf{X}$,

(iv) $\mathbf{G_1X^{\sim}} = \mathbf{G_1X^{\sim}\bar{W}} + \mathbf{\bar{Z}G_1X^{\sim}}$,

(v) $\mathbf{\bar{Z}} = \mathbf{\bar{Z}^2} - \mathbf{G_1X^{\sim}X}$.

***Theorem* 1** — *Let* $\mathbf{M} \in \mathbb{C}_n^{mp}$ *be partitioned as in (6). Then:*

(i) $\mathbf{WW^{\oplus}X} = \mathbf{X}$,

(ii) $\mathbf{\bar{W}\bar{W}^{\oplus}X} = \mathbf{X}$,

(iii) $\mathbf{ZZ^{\oplus}G_1X^{\sim}} = \mathbf{G_1X^{\sim}}$,

(iv) $\mathbf{\bar{Z}\bar{Z}^{\oplus}G_1X^{\sim}} = \mathbf{G_1X^{\sim}}$,

(v) $\mathbf{W^{\oplus}X} = \mathbf{X\bar{Z}^{\oplus}}$,

(vi) $\mathbf{XZ^{\oplus}} = \mathbf{\bar{W}^{\oplus}X}$.

***Theorem* 2** — *Let* $\mathbf{M} \in \mathbb{C}_n^{mp}$ *be partitioned as in (6). Then:*

(i) $\mathbf{P_w} = \mathbf{W} - \mathbf{X\bar{Z}^{\oplus}G_1X^{\sim}}$ *and* $\mathbf{\tilde{P}_w} = \mathbf{\bar{W}} + \mathbf{X\bar{Z}^{\oplus}G_1X^{\sim}}$,

(ii) $\mathbf{P_{\bar{w}}} = \mathbf{\bar{W}} - \mathbf{XZ^{\oplus}G_1X^{\sim}}$ *and* $\mathbf{\tilde{P}_{\bar{w}}} = \mathbf{W} + \mathbf{XZ^{\oplus}G_1X^{\sim}}$,

(iii) $\mathbf{P_z} = \mathbf{Z} - \mathbf{G_1X^{\sim}\bar{W}^{\oplus}X}$ *and* $\mathbf{\tilde{P}_z} = \mathbf{\bar{Z}} + \mathbf{G_1X^{\sim}\bar{W}^{\oplus}X}$,

(iv) $\mathbf{P_{\bar{z}}} = \mathbf{\bar{Z}} - \mathbf{G_1X^{\sim}W^{\oplus}X}$ *and* $\mathbf{\tilde{P}_{\bar{z}}} = \mathbf{Z} + \mathbf{G_1X^{\sim}W^{\oplus}X}$.

Lemma 4 — Let $\mathbf{M} \in \mathbb{C}_n^{mp}$ be partitioned as in (6). Then:

(i) $rk(\mathbf{\bar{W}}) = r - rk(\mathbf{W}) + rk(\mathbf{X})$,

(ii) $rk(\mathbf{\bar{Z}}) = n - r + rk(\mathbf{X}) - rk(\mathbf{Z})$.

PROOF : From [28, 2.12] it follows that $\text{rk}(\mathbf{W\bar{W}}) = \text{rk}(\mathbf{W}) + \text{rk}(\mathbf{\bar{W}}) - \mathbf{r}$. Using (i) of Lemma 2, we get $\text{rk}(\mathbf{\bar{W}}) = \mathbf{r} + \text{rk}(\mathbf{XG_1X^{\sim}}) - \text{rk}(\mathbf{W})$. Which on simplifying and using the fact that $\mathbf{G_1}$ is nonsingular is equivalent to the statement (i). The statement (ii) follows analogously.

3. m-PROJECTORS ONTO CERTAIN SUBSPACES

In this section we develop the representations of *m*-projectors onto certain subspace including their sum and intersection. The first lemma gives a powerful tool for constructing the *m*-projectors onto given spaces which is obtained as an analogous result from orthogonal projectors.

Lemma 5 — Let $\mathbf{L},\ \mathbf{M} \in \mathbb{C}_n^{mp}$. Then:

(i) $\mathbf{L} + \mathbf{\bar{L}}(\mathbf{\bar{L}M})^{\oplus}$ is the m-projector onto $R(\mathbf{L}) + R(\mathbf{M})$,

(ii) $\mathbf{L} - \mathbf{L}(\mathbf{L}\bar{\mathbf{M}})^{\oplus}$ is the m-projector onto $R(\mathbf{L}) \cap R(\mathbf{M})$.

PROOF : The proof follows analogously from the equivalent conditions of the Theorems 3 and 4 in [24].

Using Lemma 5 we obtain the following representations of the *m*-projectors onto the sums and intersection of certain subspaces including their dimensions.

Lemma 6 — Let $\mathbf{L},\ \mathbf{M} \in \mathbb{C}_n^{mp}$ and let $\mathbf{M}$ be partitioned as in (6). Then:

(i) $\mathbf{P}_{R(\mathbf{L})+R(\mathbf{M})} = \mathbf{U}\begin{bmatrix} \mathbf{I_r} & \mathbf{0} \\ \mathbf{0} & \mathbf{P_z} \end{bmatrix}\mathbf{U}^{\sim}$, where $\dim[R(\mathbf{L}) + R(\mathbf{M})] = \mathbf{r} + rk(\mathbf{Z})$,

(ii) $\mathbf{P}_{R(\mathbf{L})+N(\mathbf{M})} = \mathbf{U}\begin{bmatrix} \mathbf{I_r} & \mathbf{0} \\ \mathbf{0} & \mathbf{P_{\bar{z}}} \end{bmatrix}\mathbf{U}^{\sim}$, where $\dim[R(\mathbf{L}) + N(\mathbf{M})] = \mathbf{r} + rk(\mathbf{X}) + rk(\mathbf{Z})$,

(iii) $\mathbf{P}_{N(\mathbf{L})+R(\mathbf{M})} = \mathbf{U}\begin{bmatrix} \mathbf{P_w} & \mathbf{0} \\ \mathbf{0} & \mathbf{I_{n-r}} \end{bmatrix}\mathbf{U}^{\sim}$, where $\dim[N(\mathbf{L}) + R(\mathbf{M})] = \mathbf{n} - \mathbf{r} + rk(\mathbf{W})$,

(iv) $\mathbf{P}_{N(\mathbf{L})+N(\mathbf{M})} = \mathbf{U}\begin{bmatrix} \mathbf{P_{\bar{w}}} & \mathbf{0} \\ \mathbf{0} & \mathbf{I_{n-r}} \end{bmatrix}\mathbf{U}^{\sim}$, where $\dim[N(\mathbf{L}) + N(\mathbf{M})] = \mathbf{n} - rk(\mathbf{W}) + rk(\mathbf{X})$.

PROOF : For given $\mathbf{L}$ and $\mathbf{M}$, we have $(\bar{\mathbf{L}}\mathbf{M}) = \mathbf{U}\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ -\mathbf{G_1}\mathbf{X}^{\sim} & \mathbf{Z} \end{bmatrix}\mathbf{U}^{\sim}$. Utilising the conditions (iv) of Lemma 2 and (iii) of Theorem 1, can be easily verified that the Minkowski inverse of $\bar{\mathbf{L}}\mathbf{M}$ is

$$(\bar{\mathbf{L}}\mathbf{M})^{\oplus} = \mathbf{U}\begin{bmatrix} \mathbf{0} & \mathbf{X}\mathbf{Z}^{\oplus} \\ \mathbf{0} & \mathbf{P_z} \end{bmatrix}\mathbf{U}^{\sim}. \tag{7}$$

Hence, using the statement (i) of Lemma 5 and substituting (7), we obtain the *m*-projector as claimed in point (i). The remaining part is obvious from the representation of the projector. The remaining points can be obtained in a similar fashion.

Lemma 7 — Let $\mathbf{L},\ \mathbf{M} \in \mathbb{C}_n^{mp}$ and let $\mathbf{M}$ be partitioned as in (6). Then:

(i) $\mathbf{P}_{R(\mathbf{L})\cap R(\mathbf{M})} = \mathbf{U}\begin{bmatrix} \tilde{\mathbf{P}}_{\bar{\mathbf{w}}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}\mathbf{U}^{\sim}$, where $\dim[R(\mathbf{L}) \cap R(\mathbf{M})] = rk(\mathbf{W}) - rk(\mathbf{X})$,

(ii) $\mathbf{P}_{R(\mathbf{L})\cap N(\mathbf{M})} = \mathbf{U}\begin{bmatrix} \tilde{\mathbf{P}}_{\mathbf{w}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}\mathbf{U}^{\sim}$, where $\dim[R(\mathbf{L}) \cap N(\mathbf{M})] = \mathbf{r} - rk(\mathbf{W})$,

(iii) $\mathbf{P}_{N(\mathbf{L})\cap R(\mathbf{M})} = \mathbf{U}\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{P}}_{\bar{\mathbf{z}}} \end{bmatrix}\mathbf{U}^{\sim}$, where $\dim[R(\mathbf{L}) \cap R(\mathbf{M})] = rk(\mathbf{Z}) - rk(\mathbf{X})$,

(iv) $\mathbf{P}_{N(\mathbf{L})\cap R(\mathbf{M})} = \mathbf{U}\begin{bmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \tilde{\mathbf{P}}_{\mathbf{z}}\end{bmatrix}\mathbf{U}^{\sim}$, where $\dim[R(\mathbf{L}) \cap R(\mathbf{M})] = \mathbf{n} - \mathbf{r} - rk(\mathbf{Z})$.

PROOF : With given $\mathbf{L}$ and $\mathbf{M}$, we have $\mathbf{L}\bar{\mathbf{M}} = \mathbf{U}\begin{bmatrix}\bar{\mathbf{W}} & -\mathbf{X}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\mathbf{U}^{\sim}$. Using the conditions (i) and (ii) of Lemma 2 and Theorem 1 respectively, direct verification shows that the Minkowski inverse $(\mathbf{L}\bar{\mathbf{M}})^{\oplus}$ of $\mathbf{L}\bar{\mathbf{M}}$ is

$$(\mathbf{L}\bar{\mathbf{M}})^{\oplus} = \mathbf{U}\begin{bmatrix}\bar{\mathbf{W}}\bar{\mathbf{W}}^{\oplus} & \mathbf{0}\\ \mathbf{G_1}\mathbf{X}^{\sim}\bar{\mathbf{W}}^{\oplus} & \mathbf{0}\end{bmatrix}\mathbf{U}^{\sim}. \tag{8}$$

Now using the statement (i) of the Lemma 5 and substituting (8) we obtain the representation claimed in point (i) of the lemma. The dimensions of the onto space involved can be obtained easily by using the point (i) of Lemma 4. The remaining statements follow on the same lines.

***Theorem* 3** — *Let* $\mathbf{L},\ \mathbf{M} \in \mathbb{C}_n^{mp}$ *and let* $\mathbf{M}$ *be partitioned as in (6). Then:*

(i) $R(\mathbf{L}) \cap R(\mathbf{M}) = 0 \Leftrightarrow rk(\mathbf{W}) = rk(\mathbf{X})$,

(ii) $R(\mathbf{L}) + R(\mathbf{M}) = M_{(n,1)}(\mathbb{C}) \Leftrightarrow rk(\mathbf{Z}) = \mathbf{n} - \mathbf{r}$,

(iii) $R(\mathbf{L})[\perp]R(\mathbf{M}) \Leftrightarrow rk(\mathbf{W}) = 0$,

(iii) $R(\mathbf{L}) \oplus R(\mathbf{M}) = M_{(n,1)}(\mathbb{C}) \Leftrightarrow rk(\mathbf{W}) = rk(\mathbf{X})\ and\ rk(\mathbf{Z}) = \mathbf{n} - \mathbf{r}$,

(iv) $R(\mathbf{L}) \oplus^{[\perp]} R(\mathbf{M}) = M_{(n,1)}(\mathbb{C}) \Leftrightarrow rk(\mathbf{W}) = 0\ and\ rk(\mathbf{Z}) = \mathbf{n} - \mathbf{r}$.

PROOF : The points (i) and (ii) follow directly from the point (i) of Lemmas 6 and 7. The statement (iii) is obtained by noticing the fact $R(\mathbf{L})[\perp]R(\mathbf{M}) \Leftrightarrow \mathbf{LM} = 0$, which holds if and only if rk$(\mathbf{W}) = \mathbf{0}$. Finally, the points (iv) and (v) follow at once by combining the implications (i) with (ii) and (ii) with (iii) respectively.

4. MAIN RESULTS

Taking Minkowski adjoint of the (8) and using the point (vi) of Theorem 1, we get

$$(\bar{\mathbf{M}}\mathbf{L})^{\oplus} = \mathbf{U}\begin{bmatrix}\mathbf{P}_{\bar{\mathbf{w}}} & -\mathbf{X}\mathbf{Z}^{\oplus}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\mathbf{U}^{\sim}. \tag{9}$$

It can be easily verified that representation (9) is idempotent. We now determine the onto and along spaces for this projector. Using the first half of the condition (ii) of the Theorem 2, it can be easily verified that the *m*- projector onto $R((\bar{\mathbf{M}}\mathbf{L})^{\oplus})$, i.e., $\mathbf{P}_{R[(\bar{\mathbf{M}}\mathbf{L})^{\oplus}]}$ is given by

$$\mathbf{P}_{R[(\bar{\mathbf{M}}\mathbf{L})^{\oplus}]} = \mathbf{U}\begin{bmatrix} \mathbf{P}_{\bar{\mathbf{w}}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}\mathbf{U}^{\sim}. \tag{10}$$

Also the *m*-projector onto $N[(\bar{\mathbf{M}}\mathbf{L})^{\oplus}]$, i.e., $\mathbf{P}_{N[\bar{\mathbf{M}}\mathbf{L})^{\oplus}]} = \mathbf{I_n} - (\bar{\mathbf{M}}\mathbf{L})(\bar{\mathbf{M}}\mathbf{L})^{\oplus}$, obtained by using the conditions (ii) and (iv) of the Theorem 1 and the condition (iii) of the Theorem 2, is given by

$$\mathbf{P}_{N[(\bar{\mathbf{M}}\mathbf{L})^{\oplus}]} = \mathbf{U}\begin{bmatrix} \bar{\mathbf{W}} & \mathbf{X} \\ -\mathbf{G_1}\mathbf{X}^{\sim} & \mathbf{Z} + \tilde{\mathbf{P}_{\mathbf{z}}} \end{bmatrix}\mathbf{U}^{\sim}. \tag{11}$$

On applying the statement (ii) of Lemma 5 to the projectors given in (4) and point (iv) of the Lemma 6, we have

$$\mathbf{P}_{R(\mathbf{L})\cap N(\mathbf{L})+N(\mathbf{M})} = \mathbf{U}\begin{bmatrix} \mathbf{P}_{\bar{\mathbf{w}}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}\mathbf{U}^{\sim}. \tag{12}$$

Further, on applying the statement (i) of the Lemma 5 to the projectors given by (8) and the point (iv) of the Lemma 7 and utilizing the condition (iii) of Theorem 1, we get

$$\mathbf{P}_{R(\mathbf{M})\oplus^{[\perp]}N(\mathbf{L})\cap N(\mathbf{M})} = \mathbf{U}\begin{bmatrix} \mathbf{W} & \mathbf{X} \\ -\mathbf{G_1}\mathbf{X}^{\sim} & \mathbf{Z} + \tilde{\mathbf{P}_{\mathbf{z}}} \end{bmatrix}\mathbf{U}^{\sim}. \tag{13}$$

Observing (10) and (12) leads to

$$R((\bar{\mathbf{M}}\mathbf{L})^{\oplus}) = R(\mathbf{L}) \cap N(\mathbf{L}) + N(\mathbf{M}). \tag{14}$$

Also from (11) and (13), we have

$$N((\bar{\mathbf{M}}\mathbf{L})^{\oplus}) = R(\mathbf{M}) \oplus^{[\perp]} N(\mathbf{L}) \cap N(\mathbf{M}). \tag{15}$$

From the above two equations we can observe that if the range spaces of $\mathbf{L} \text{ and } \mathbf{M}$ are complementary then $(\bar{\mathbf{M}}\mathbf{L})^{\oplus}$ is an oblique projector onto the $R(\mathbf{L})$ along $R(\mathbf{M})$. This observation was also pointed out by Greville in [14] with the restriction that $R(\mathbf{L})$ and $R(\mathbf{M})$ are complementary. Afriat in [1] produced one more representation of the oblique projector in terms of orthogonal projectors with a restrictive condition that $R(\mathbf{L}) \cap R(\mathbf{M}) = \{0\}$ establishes the invertibility of $\mathbf{I_n} - \mathbf{L}\mathbf{M}$. The next theorem generalizes this result by weakening the disjointness condition $R(\mathbf{L}) \cap R(\mathbf{M}) = \{0\}$.

***Theorem* 4** — *Let* $\mathbf{L},\ \mathbf{M} \in \mathbb{C}_n^{mp}$. *Then,*

$$(\bar{\mathbf{M}}\mathbf{L})^{\oplus} = (\mathbf{I_n} - \mathbf{L}\mathbf{M})^{\oplus}\mathbf{L}\bar{\mathbf{M}} \tag{16}$$

is the oblique projector onto $R(\mathbf{L}) \cap [N(\mathbf{L}) + N(\mathbf{M})]$ *along* $R(\mathbf{M}) \oplus^{[\perp]} [N(\mathbf{L}) \cap N(\mathbf{M})]$.

PROOF : Using the point (ii) of Theorem 1, the Minkowski inverse of

$$(\mathbf{I_n} - \mathbf{LM}) = \mathbf{U} \begin{bmatrix} \bar{\mathbf{W}} & -\mathbf{X} \\ \mathbf{0} & \mathbf{I_{n-r}} \end{bmatrix} \mathbf{U}^{\sim}, \tag{17}$$

i.e., $(\mathbf{I_n} - \mathbf{LM})^{\oplus}$, as a result of direct verification is given by

$$(\mathbf{I_n} - \mathbf{LM})^{\oplus} = \mathbf{U} \begin{bmatrix} \bar{\mathbf{W}}^{\oplus} & \bar{\mathbf{W}}^{\oplus}\mathbf{X} \\ \mathbf{0} & \mathbf{I_{n-r}} \end{bmatrix} \mathbf{U}^{\sim}. \tag{18}$$

Post multiplying (18) by $\mathbf{L}\bar{\mathbf{M}}$, we get

$$(\mathbf{I_n} - \mathbf{LM})^{\oplus}\mathbf{L}\bar{\mathbf{M}} = \mathbf{U} \begin{bmatrix} \mathbf{P}_{\bar{\mathbf{w}}} & -\bar{\mathbf{W}}^{\oplus}\mathbf{X} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{U}^{\sim}. \tag{19}$$

Which is the same representation as obtained in (9). The column space and the null space of $(\bar{\mathbf{M}}\mathbf{L})^{\oplus}$ are already characterized in (14) and (15).

The representation given in the Theorem 4 can be written as

$$(\bar{\mathbf{M}}\mathbf{L})^{\oplus} = (\mathbf{I_n} - \mathbf{LM})^{\oplus}\mathbf{L}(\mathbf{I_n} - \mathbf{LM}),$$

which resembles a similarity transformation carrying $\mathbf{L}$ into the projector $(\mathbf{LM})^{\oplus}$. Premultiplying (18) by $\mathbf{L}\bar{\mathbf{M}}$ and utilizing the points (ii) and (vi) of the Theorem 1, we have

$$\mathbf{L}\bar{\mathbf{M}}(\mathbf{I_n} - \mathbf{LM})^{\oplus} = \mathbf{U} \begin{bmatrix} \mathbf{P}_{\bar{\mathbf{w}}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{U}^{\sim},$$

which is same as the *m*-projector onto $R[(\bar{\mathbf{M}}\mathbf{L})]$ given in (10).

Corollary 1 — Let $\mathbf{L},\ \mathbf{M} \in \mathbb{C}_n^{mp}$ and $R(\mathbf{L}) \cap R(\mathbf{M}) = \{\mathbf{0}\}$. Then, $(\mathbf{I_n} - \mathbf{LM})$ is nonsingular and $(\bar{\mathbf{M}}\mathbf{L})^{\oplus} = (\mathbf{I_n} - \mathbf{LM})^{-1}\mathbf{L}\bar{\mathbf{M}}$ is an oblique projector onto $R(\mathbf{L})$ along $R(\mathbf{M}) \oplus^{[\perp]} [N(\mathbf{L}) \cap N(\mathbf{M})]$.

Utilizing representations (17) and (18), it can be easily verified that the *m*-projector onto the $R(\mathbf{I_n} - \mathbf{LM})$ is given by

$$\mathbf{P}_{R(\mathbf{I_n} - \mathbf{LM})} = \mathbf{U} \begin{bmatrix} \mathbf{P}_{\bar{\mathbf{w}}} & \mathbf{0} \\ \mathbf{0} & \mathbf{I_{n-r}} \end{bmatrix} \mathbf{U}^{\sim}. \tag{20}$$

Further, it can observed from (20) that $(\mathbf{I_n} - \mathbf{LM})$ is nonsingular if and only if rk$(\bar{\mathbf{W}}) = \mathbf{r}$. Combining this statement with the point (i) of the Lemma 4 and statement (i) of the Theorem 3 leads to $R(\mathbf{L}) \cap R(\mathbf{M}) = \{\mathbf{0}\}$. Thus, the invertibility of $(\mathbf{I_n} - \mathbf{LM})$ is equivalent to $R(\mathbf{L}) \cap R(\mathbf{M}) = \{\mathbf{0}\}$.

***Theorem* 5** — *Let* $\mathbf{L},\ \mathbf{M} \in \mathbb{C}_n^{mp}$. *Then:*

(i) $(\mathbf{I_n} - \mathbf{LM})^{\oplus}\mathbf{L\bar{M}}$ *is the oblique projector onto* $R(\mathbf{L}) \cap [N(\mathbf{L}) + N(\mathbf{M})] \oplus^{[\perp]} [N(\mathbf{L}) \cap N(\mathbf{M})]$ *along* $R(\mathbf{M})$,

(ii) $\mathbf{L}(\mathbf{L} + \mathbf{M} - \mathbf{ML})^{\oplus}$ *is the oblique projector onto* $R(\mathbf{L})$ *along* $R(\mathbf{M}) \cap [N(\mathbf{L}) + N(\mathbf{M})] \oplus^{[\perp]} [N(\mathbf{L}) \cap N(\mathbf{M})]$.

PROOF : Direct verification shows that the Minkowski inverse of

$$\mathbf{I_n} - \mathbf{ML} = \mathbf{U}\begin{bmatrix} \mathbf{\bar{W}} & \mathbf{0} \\ \mathbf{G_1X^{\sim}} & \mathbf{I_{n-r}} \end{bmatrix}\mathbf{U}^{\sim}$$

is given by

$$(\mathbf{I_n} - \mathbf{ML})^{\oplus} = \mathbf{U}\begin{bmatrix} \mathbf{\bar{W}}^{\oplus} & \mathbf{0} \\ -\mathbf{G_1X^{\sim}\bar{W}}^{\oplus} & \mathbf{I_{n-r}} \end{bmatrix}\mathbf{U}^{\sim}.$$

Direct verification, on utilizing the conditions (iii) of Lemma 2, (i) of Lemma 3, (ii) and (iii) of Theorem 1, and the later part of point (iii) of Theorem 2, shows that the Minkowski inverse of

$$(\mathbf{I_n} - \mathbf{ML})^{\oplus}\mathbf{\bar{M}} = \mathbf{U}\begin{bmatrix} \mathbf{P_{\bar{w}}} & -\mathbf{\bar{W}}^{\oplus}\mathbf{X} \\ -\mathbf{0} & \mathbf{\tilde{P_z}} \end{bmatrix}\mathbf{U}^{\sim},$$

i.e., $[(\mathbf{I_n} - \mathbf{ML})^{\oplus}\mathbf{\bar{M}}]^{\oplus}$ is given by

$$[(\mathbf{I_n} - \mathbf{ML})^{\oplus}\mathbf{\bar{M}}]^{\oplus} = \mathbf{U}\begin{bmatrix} \mathbf{\bar{W}} & \mathbf{0} \\ \mathbf{G_1X^{\sim}} & \mathbf{\tilde{P_z}} \end{bmatrix}\mathbf{U}^{\sim}.$$

Therefore, using the conditions (iii) and (ii) of the Theorems 1 and 2 respectively leads to the fact that

$$\mathbf{P}_{R(\mathbf{I_n}-\mathbf{ML})^{\oplus}\mathbf{\bar{M}}} = \mathbf{U}\begin{bmatrix} \mathbf{P_{\bar{w}}} & \mathbf{0} \\ \mathbf{0} & \mathbf{\tilde{P_z}} \end{bmatrix}\mathbf{U}^{\sim}. \tag{21}$$

Thus, the point (i) of the result will be proved if we establish that the matrix on the LHS of (21) represents the *m*-projector with given onto and along spaces. This can be easily established by applying Lemma 5 to the equation (4) and the point (iv) of the Lemmas 6 and 7 respectively. For proving point (ii) of the results we observe that the Minkowski inverse of

$$\mathbf{L} + \mathbf{M} - \mathbf{ML} = \mathbf{U}\begin{bmatrix} \mathbf{I_r} & \mathbf{X} \\ \mathbf{0} & \mathbf{Z} \end{bmatrix}\mathbf{U}^{\sim}, \tag{22}$$

i.e., $(\mathbf{L}+\mathbf{M}+\mathbf{ML})^{\oplus}$ is given by

$$(\mathbf{L}+\mathbf{M}-\mathbf{ML})^{\oplus}=\mathbf{U}\begin{bmatrix}\mathbf{I_r} & -\bar{\mathbf{W}}^{\oplus}\mathbf{X}\\ \mathbf{0} & \mathbf{Z}^{\oplus}\end{bmatrix}\mathbf{U}^{\sim}. \tag{23}$$

Premultiplying (23) by $\mathbf{L}$ and using condition (vi) of Theorem 1, we get

$$\mathbf{L}(\mathbf{L}+\mathbf{M}-\mathbf{ML})^{\oplus}=\mathbf{U}\begin{bmatrix}\mathbf{I_r} & -\bar{\mathbf{W}}^{\oplus}\mathbf{X}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\mathbf{U}^{\sim}. \tag{24}$$

Now utilizing the conditions (ii) and (iii) of Theorem 2, (ii) of Theorem 1, (i) of Lemma 3, and (iii) of Lemma 2, direct verification shows that the Minkowski inverse of $\mathbf{L}(\mathbf{L}+\mathbf{M}-\mathbf{ML})^{\oplus}$ is of the form

$$[\mathbf{L}(\mathbf{L}+\mathbf{M}-\mathbf{ML})^{\oplus}]^{\oplus}=\mathbf{U}\begin{bmatrix}\bar{\mathbf{W}}+\tilde{\mathbf{P}}_{\bar{\mathbf{w}}} & \mathbf{0}\\ -\mathbf{G_1}\mathbf{X}^{\sim} & \mathbf{0}\end{bmatrix}\mathbf{U}^{\sim}.$$

As a consequence, we have

$$\mathbf{P}_{R[\mathbf{L}(\mathbf{L}+\mathbf{M}-\mathbf{ML})^{\oplus}]}=\mathbf{L} \text{ and } \mathbf{P}_{N[\mathbf{L}(\mathbf{L}+\mathbf{M}-\mathbf{ML})^{\oplus}]}=\mathbf{U}\begin{bmatrix}\mathbf{P}_{\bar{\mathbf{w}}}-\mathbf{W} & \mathbf{X}\\ \mathbf{G_1}\mathbf{X}^{\sim} & \tilde{\mathbf{P}}_{\mathbf{z}}+\mathbf{Z}\end{bmatrix}\mathbf{U}^{\sim}. \tag{25}$$

Finally, the onto and along spaces can be easily obtained by applying the analogous reasoning as used for the statement (i).

From Theorem 5 it can be easily observed that $(\bar{\mathbf{M}}\mathbf{L})^{\oplus}=(\mathbf{I_n}-\mathbf{ML})^{-\mathbf{1}}\bar{\mathbf{M}}$ when $R(\mathbf{L})\cap R(\mathbf{M})=\{\mathbf{0}\}$ and the invertibility of $(\mathbf{L}+\mathbf{M}-\mathbf{ML})$, as a part of the projector in the point (ii) of the Theorem 5, is due if and only if $R(\mathbf{L})+R(\mathbf{M})=M_{(n,1)}(\mathbb{C})$, which on using the statement (ii) of Theorem 3 is equivalent to rk$(\mathbf{Z})=\mathbf{n}-\mathbf{r}$. This observation can be alternatively seen from the *m*-projector

$$\mathbf{P}_{R(\mathbf{L}+\mathbf{M}-\mathbf{ML})}=\mathbf{U}\begin{bmatrix}\mathbf{I_r} & \mathbf{0}\\ \mathbf{0} & \mathbf{P_Z}\end{bmatrix}\mathbf{U}^{\sim},$$

obtained by utilizing the representation obtained in (22) and (23). Taking into account the Theorems 4 and 5, the following result establishes the conditions to ensure the equality of the projectors involved therein.

***Theorem* 6** — *Let* $\mathbf{L},\ \mathbf{M}\in\mathbb{C}_n^{mp}$. *Then:*

(i) $(\bar{\mathbf{M}}\mathbf{L})^{\oplus}=(\mathbf{I_n}-\mathbf{ML})^{\oplus}\bar{\mathbf{M}}\Leftrightarrow R(\mathbf{L})+R(\mathbf{M})=M_{(n,1)}(\mathbb{C})$,

(ii) $(\bar{\mathbf{M}}\mathbf{L})^{\oplus}=\mathbf{L}(\mathbf{L}+\mathbf{M}+\mathbf{ML})^{\oplus}\Leftrightarrow R(\mathbf{L})\cap R(\mathbf{M})=\{\mathbf{0}\}$,

(iii) $(\mathbf{I_n} - \mathbf{ML})^{\oplus}\bar{\mathbf{M}} = \mathbf{L}(\mathbf{L} + \mathbf{M} + \mathbf{ML})^{\oplus} \Leftrightarrow R(\mathbf{L}) \oplus R(\mathbf{M}) = M_{(n,1)}(\mathbb{C})$.

PROOF : On observing the representation given in (9) and the representation of $(\mathbf{I_n} - \mathbf{ML})^{\oplus}\bar{\mathbf{M}}$ given in Theorem 5, the condition (i) follows if and only if rk$(\mathbf{Z}) = \mathbf{n} - \mathbf{r}$. This follows directly from the equivalence (ii) of Theorem (3). Also on account of the representations given in (9) and (24), the statement (ii) follows if and only if rk$(\bar{\mathbf{W}}) = \mathbf{r}$. Combining this condition with the point (i) of Lemma 4 and Theorem 3 leads to the statement (ii). The statement (iii) follows if and only if both $\bar{\mathbf{W}}$ and $\mathbf{Z}$ are invertible, which in turn, on using the statement (iv) of 3, implies that $R(\mathbf{L}) \oplus R(\mathbf{M}) = M_{(n,1)}(\mathbb{C})$.

The next result gives a representations of the projector $(\bar{\mathbf{M}}\mathbf{L})^{\oplus}$.

***Theorem* 7** — *Let* $\mathbf{L},\ \mathbf{M} \in \mathbb{C}_n^{mp}$. *Then,* $(\bar{\mathbf{M}}\mathbf{L})^{\oplus} = (\mathbf{I_n} - \mathbf{LML})^{\oplus}\mathbf{L}\bar{\mathbf{M}}$.

PROOF : Using the representations of $\mathbf{L}$ and $\mathbf{M}$ given in (4) and (6), it can be easily verified that the Minkowski inverse of

$$(\mathbf{I_n} - \mathbf{LML}) = \mathbf{U}\begin{bmatrix} \bar{\mathbf{W}} & \mathbf{0} \\ \mathbf{0} & \mathbf{I_{n-r}} \end{bmatrix}\mathbf{U}^{\sim} \tag{26}$$

is given by

$$(\mathbf{I_n} - \mathbf{LML})^{\oplus} = \mathbf{U}\begin{bmatrix} \bar{\mathbf{W}}^{\oplus} & \mathbf{0} \\ \mathbf{0} & \mathbf{I_{n-r}} \end{bmatrix}\mathbf{U}^{\sim}. \tag{27}$$

Postmultiplying (26) by $(\mathbf{L}\bar{\mathbf{M}})$, we get

$$(\mathbf{I_n} - \mathbf{LML})^{\oplus}(\mathbf{L}\bar{\mathbf{M}}) = \mathbf{U}\begin{bmatrix} \mathbf{P_{\bar{w}}} & -\mathbf{XZ}^{\oplus} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}\mathbf{U}^{\sim},$$

which is same as the representation (9). The onto and along spaces of the projector $(\bar{\mathbf{M}}\mathbf{L})^{\oplus}$ = $(\mathbf{I_n} - \mathbf{LML})^{\oplus}(\mathbf{L}\bar{\mathbf{M}})$ are already characterized in Theorem 4. It is obvious from (27) that $(\mathbf{I_n} - \mathbf{LML})$ is nonsingular if and only if rk$(\bar{\mathbf{W}}) = \mathbf{r}$, which is equivalent to $R(\mathbf{L}) \cap R(\mathbf{M}) = \{\mathbf{0}\}$ as shown in Theorem 6.

The next theorem gives one more representation of the projector $(\bar{\mathbf{M}}\mathbf{L})^{\oplus}$ as a product of two functions of $\mathbf{L}$ and $\mathbf{M}$.

***Theorem* 8** — *Let* $\mathbf{L},\ \mathbf{M} \in \mathbb{C}_n^{mp}$. *Then,* $(\bar{\mathbf{M}}\mathbf{L})^{\oplus} = \mathbf{L}(\mathbf{L} - \mathbf{M})^{\oplus}$.

PROOF : The Minkowski inverse of the difference of the *m*-projectors $\mathbf{L}$ and $\mathbf{M}$, i.e.,

$$\mathbf{L} - \mathbf{M} = \mathbf{U}\begin{bmatrix} \bar{\mathbf{W}} & -\mathbf{X} \\ \mathbf{G_1}\mathbf{X}^{\sim} & -\mathbf{Z} \end{bmatrix}\mathbf{U}^{\sim}, \tag{28}$$

as a result of direct verifications, on using the points (iii) of Lemma 2, (ii) and (iii) of Theorem 1, and (ii) and (iii) of Theorem 2, is given by

$$(\mathbf{L}-\mathbf{M})^{\oplus}=\mathbf{U}\begin{bmatrix}\mathbf{P}_{\bar{\mathbf{w}}} & -\mathbf{X}\mathbf{Z}^{\oplus}\\ \mathbf{Z}^{\oplus}\mathbf{G_1}\mathbf{X}^{\sim} & -\mathbf{P}_{\mathbf{z}}\end{bmatrix}\mathbf{U}^{\sim}. \tag{29}$$

The result follows at once by premultiplying (29) by $\mathbf{L}$.

From (28) and (29) doing simple algebra we can easily verify that

$$\mathbf{P}_{R(\mathbf{L}-\mathbf{M})}=\mathbf{U}\begin{bmatrix}\mathbf{P}_{\bar{\mathbf{w}}} & \mathbf{0}\\ \mathbf{0} & \mathbf{P}_{\mathbf{z}}\end{bmatrix}\mathbf{U}^{\sim}. \tag{30}$$

As a simple observation from (30), the invertibility of $(\mathbf{L}-\mathbf{M})$ directly depends on the invertibility of $\bar{\mathbf{W}}$ and $\mathbf{Z}$, which is equivalent to the statement, given in the proof of the Theorem 6, that the range space of $\mathbf{L}$ and $\mathbf{M}$ are complementary. Another representation for $(\mathbf{L}-\mathbf{M})^{\oplus}$ can be formulated by using the representations of $(\bar{\mathbf{M}}\mathbf{L})^{\oplus}$ and $(\mathbf{M}\bar{\mathbf{L}})^{\oplus}$ obtained in (7) and (9), i.e.,

$$(\mathbf{L}-\mathbf{M})^{\oplus}=(\bar{\mathbf{M}}\mathbf{L})^{\oplus}-(\mathbf{M}\bar{\mathbf{L}})^{\oplus}. \tag{31}$$

In case $\bar{\mathbf{W}}$ and $\mathbf{Z}$ are nonsingular, the representation reduces to

$$(\mathbf{L}-\mathbf{M})^{-\mathbf{1}}=(\bar{\mathbf{M}}\mathbf{L})^{\oplus}-(\mathbf{M}\bar{\mathbf{L}})^{\oplus}.$$

In [7], the invertibility of difference of two projectors $\mathbf{P}$ and $\mathbf{Q}$ is established as the existence of the oblique projector $\mathbf{M}$ with $R(\mathbf{M})=R(\mathbf{P})$ and $N(\mathbf{M})=N(\mathbf{Q})$ and if rk$(\mathbf{P}-\mathbf{Q})=\mathbf{n}$, then

$$(\mathbf{P}-\mathbf{Q})^{-\mathbf{1}}=\mathbf{M}+\mathbf{M}^{*}-\mathbf{I_n}. \tag{32}$$

Same result was given in [9] with an extension that the projector involved in the above equation is unique and $\mathbf{M}=(\bar{\mathbf{Q}}\mathbf{P})^{\dagger}$ and whence

$$(\mathbf{P}-\mathbf{Q})^{-\mathbf{1}}=(\bar{\mathbf{Q}}\mathbf{P})^{\dagger}+(\mathbf{P}\bar{\mathbf{Q}})^{\dagger}-\mathbf{I_n}. \tag{33}$$

This remains valid in Minkowski space by replacing the conjugate transpose by the Minkowski adjoint and the Moore-Penrose inverse by Minkowski inverse, and the expression becomes

$$(\mathbf{L}-\mathbf{M})^{-\mathbf{1}}=(\bar{\mathbf{M}}\mathbf{L})^{\oplus}+(\mathbf{L}\bar{\mathbf{M}})^{\oplus}-\mathbf{I_n}. \tag{34}$$

Replacing the ordinary inverse by the Minkowski inverse and using the corresponding representations, we get

$$(\mathbf{L}-\mathbf{M})^{\oplus}-(\bar{\mathbf{M}}\mathbf{L})^{\oplus}-(\mathbf{L}\bar{\mathbf{M}})^{\oplus}+\mathbf{I_n}=\mathbf{U}\begin{bmatrix}\tilde{\mathbf{P}}_{\bar{\mathbf{w}}} & \mathbf{0}\\ \mathbf{0} & \tilde{\mathbf{P}}_{\mathbf{z}}\end{bmatrix}\mathbf{U}^{\sim}. \tag{35}$$

The representation obtained in (35) can be alternatively expressed as the following result.

***Theorem* 9** — *Let* $\mathbf{L},\ \mathbf{M}\in\mathbb{C}_n^{mp}$. *Then,*

$$(\mathbf{L}-\mathbf{M})^{\oplus}=\mathbf{P}_{[R(\mathbf{L})\cap R(\mathbf{M})]\oplus^{[\perp]}[N(\mathbf{L})\cap N(\mathbf{M})]}+(\bar{\mathbf{M}}\mathbf{L})^{\oplus}+(\mathbf{L}\bar{\mathbf{M}})^{\oplus}-\mathbf{I_n}.$$

PROOF : The result follows by applying point (i) of the Lemma 5 to the projectors in the points (i) and (iv) of the Lemma 7.

Combining the expressions given in Theorem 9 and equation (31), results in the following corollary.

Corollary 2 — Let $\mathbf{L},\ \mathbf{M}\in\mathbb{C}_n^{mp}$. Then,

$$\mathbf{P}_{[R(\mathbf{L})\cap R(\mathbf{M})]\oplus^{[\perp]}[N(\mathbf{L})\cap N(\mathbf{M})]}+\mathbf{P}_{[R(\mathbf{L})\cap R(\mathbf{M})]\cap[N(\mathbf{L})\cap N(\mathbf{M})]}=\mathbf{I_n}.$$

PROOF : The proof will follow at once if we show that $(\bar{\mathbf{M}}\mathbf{L})^{\oplus}+(\mathbf{L}\bar{\mathbf{M}})^{\oplus}$ is a *m*-projector onto $[R(\mathbf{L})\cap R(\mathbf{M})]\cap[N(\mathbf{L})\cap N(\mathbf{M})]$, which in fact can be easily obtained by applying point (ii) of the Lemma 5 to the projectors in the points (i) and (iv) of the Lemma 6.

In the forthcoming results we will utilize the sum of two projectors and the invertibility of their sum. As a consequence of this consideration, we have

$$\mathbf{L}+\mathbf{M}=\mathbf{U}\begin{bmatrix}\mathbf{I_r}+\mathbf{W} & \mathbf{X}\\ -\mathbf{G_1}\mathbf{X}^{\sim} & \mathbf{Z}\end{bmatrix}\mathbf{U}^{\sim}. \tag{36}$$

The Minkowski inverse of (36), as a result of direct verification on using the points (ii) and (iii) of Theorem 2, (ii) of Lemma 2, and (iii) of Theorem 1, is given by

$$(\mathbf{L}+\mathbf{M})^{\oplus}=\mathbf{U}\begin{bmatrix}\mathbf{I_r}-\frac{\mathbf{1}}{\mathbf{2}}\tilde{\mathbf{P}}_{\bar{\mathbf{w}}} & -\mathbf{X}\mathbf{Z}^{\oplus}\\ \mathbf{Z}^{\oplus}\mathbf{G_1}\mathbf{X}^{\sim} & \mathbf{2}\mathbf{Z}^{\oplus}-\mathbf{P_z}\end{bmatrix}\mathbf{U}^{\sim}. \tag{37}$$

From (36) and (37) it is easy to observe that

$$\mathbf{P}_{R(\mathbf{L}+\mathbf{M})}=\mathbf{U}\begin{bmatrix}\mathbf{I_r} & \mathbf{0}\\ \mathbf{0} & \mathbf{P_z}\end{bmatrix}\mathbf{U}^{\sim}. \tag{38}$$

Whence, it is obvious that $(\mathbf{L}+\mathbf{M})$ is nonsingular if and only if $\text{rk}(\mathbf{Z}) = \mathbf{n}-\mathbf{r}$. In [17] Groß and Trenkler established that the invertibility of $(\mathbf{L}-\mathbf{M})$ implies the invertibility of $(\mathbf{L}+\mathbf{M})$. The next result establishes the relation between the sum and difference of the *m*-projectors involving their Minkowski inverse.

***Theorem* 10** — *Let* $\mathbf{L},\ \mathbf{M} \in \mathbb{C}_n^{mp}$. *Then,*

$$(\mathbf{L}-\mathbf{M})^{\oplus} = (\mathbf{L}+\mathbf{M})^{\oplus}(\mathbf{L}-\mathbf{M})(\mathbf{L}+\mathbf{M})^{\oplus}.$$

PROOF : The proof follows by substituting the representations from (28), (29), and (37) and utilizing the conditions (ii), (iii), (v), and (vi) of Theorem 1 and (ii) and (iii) of Theorem 2.

***Theorem* 11** — *Let* $\mathbf{L},\ \mathbf{M} \in \mathbb{C}_n^{mp}$. *Then,*

$$(\mathbf{L}+\mathbf{M})^{\oplus} - (\mathbf{L}-\mathbf{M})^{\oplus}(\mathbf{L}+\mathbf{M})(\mathbf{L}-\mathbf{M})^{\oplus} = \frac{\mathbf{1}}{\mathbf{2}}\mathbf{P}_{R(\mathbf{L})\cap R(\mathbf{M})}.$$

PROOF : The result follows by using the point (i) of Lemma 7, the representations (29), (36), (37) and utilizing the points (ii), (iii), (v), and (vi) of Theorem 1 and (ii) and (iii) of Theorem 2.

From the above result it can be easily observed that $R(\mathbf{L}) \cap R(\mathbf{M}) = \{\mathbf{0}\}$ is equivalent to

$$(\mathbf{L}+\mathbf{M})^{\oplus} = (\mathbf{L}-\mathbf{M})^{\oplus}(\mathbf{L}+\mathbf{M})(\mathbf{L}-\mathbf{M})^{\oplus},$$

which is a generalization of statement (2.6) in [20], under the restriction of the invertibility of the terms $(\mathbf{L}+\mathbf{M})$ and $(\mathbf{L}-\mathbf{M})$.

Further using the proof of the Theorems 10 and 11, it can be observed that

$$(\mathbf{L}+\mathbf{M})^{\oplus}(\mathbf{L}-\mathbf{M}) = (\mathbf{L}-\mathbf{M})^{\oplus}(\mathbf{L}+\mathbf{M}) = (\mathbf{L}\bar{\mathbf{M}})^{\oplus} - (\mathbf{M}\bar{\mathbf{L}})^{\oplus}.$$

Turning back to the main interest of the paper which involves the matrix

$$(\mathbf{L}+\mathbf{M}-\mathbf{I_n}) = \mathbf{U}\begin{bmatrix} \mathbf{W} & \mathbf{X} \\ -\mathbf{G_1}\mathbf{X}^{\sim} & -\bar{\mathbf{Z}} \end{bmatrix}\mathbf{U}^{\sim}. \tag{39}$$

Using the points (i) and (iv) of Theorem 1 and (i) of Theorem 2, direct verification shows that the Minkowski inverse of $(\mathbf{L}+\mathbf{M}-\mathbf{I_n})$ is

$$(\mathbf{L}+\mathbf{M}-\mathbf{I_n})^{\oplus} = \mathbf{U}\begin{bmatrix} \mathbf{P_w} & \mathbf{W}^{\oplus}\mathbf{X} \\ -\mathbf{G_1}\mathbf{X}^{\sim}\mathbf{W}^{\oplus} & -\mathbf{P}_{\bar{\mathbf{z}}} \end{bmatrix}\mathbf{U}^{\sim}. \tag{40}$$

While verifying the conditions of the Minkowski inverse of $(\mathbf{L}+\mathbf{M}-\mathbf{I_n})$, it can be easily observed that

$$\mathbf{P}_{R(\mathbf{L}+\mathbf{M}-\mathbf{I_n})} = \mathbf{U}\begin{bmatrix}\mathbf{P_w} & 0\\ \mathbf{0} & \mathbf{P_{\bar{z}}}\end{bmatrix}\mathbf{U}^{\sim}. \tag{41}$$

Hence, the invertibility of $(\mathbf{L}+\mathbf{M}-\mathbf{I_n})$ is equivalent to the invertibility of both $\mathbf{W}$ and $\bar{\mathbf{Z}}$. Another equivalent condition for the invertibility of $(\mathbf{L}+\mathbf{M}-\mathbf{I_n})$ is invertibility of the *m*-projectors given in points (ii) and (iii) of Lemma 6 or equivalently the two projectors reduce to the Identity matrix $\mathbf{I_n}$.

In [8] the author formulates results involving projections in the Banach space settings, under the assumption of the invertibility of the functions of the projectors involved. We extend and generalize the statements (ii) and (iv) of the Lemma 2.4 in [8], in Minkowski space setting, by relaxing the invertibility condition and use Minkowski inverse of the functions of *m*- projectors.

***Theorem* 12** — *Let* $\mathbf{L},\ \mathbf{M}\in\mathbb{C}_n^{mp}$. *Then:*

(i) $(\mathbf{L}+\mathbf{M}-\mathbf{I_n})^{\oplus}\mathbf{L}=(\mathbf{LM})^{\oplus}$,

(ii) $\mathbf{M}(\mathbf{L}+\mathbf{M}-\mathbf{I_n})^{\oplus}=(\mathbf{LM})^{\oplus}$,

(iii) $\mathbf{L}(\mathbf{L}+\mathbf{M}-\mathbf{I_n})^{\oplus}=(\mathbf{ML})^{\oplus}$,

(iv) $(\mathbf{L}+\mathbf{M}-\mathbf{I_n})^{\oplus}\mathbf{M}=(\mathbf{ML})^{\oplus}$.

PROOF : Using the point (i) of the Theorems 1 and 2, it can be easily verified that the Minkowski inverse of $(\mathbf{LM})$ is

$$(\mathbf{LM})^{\oplus} = \mathbf{U}\begin{bmatrix}\mathbf{P_w} & 0\\ -\bar{\mathbf{Z}}\mathbf{G_1}\mathbf{X}^{\sim} & \mathbf{0}\end{bmatrix}\mathbf{U}^{\sim}. \tag{42}$$

Now using the matrix representation given in (4), (40), and (42), we obtain the points (i) and (ii) of the Theorem. The remaining two points follow on the same lines by using the respective representations.

***Theorem* 13** — *Let* $\mathbf{L},\ \mathbf{M}\in\mathbb{C}_n^{mp}$. *Then:*

(i) $[(\mathbf{L}+\mathbf{M}-\mathbf{I_n})^{\oplus}]^{\mathbf{2}}\mathbf{L}=(\mathbf{LML})^{\oplus}=\mathbf{L}[(\mathbf{L}+\mathbf{M}-\mathbf{I_n})^{\oplus}]^{\mathbf{2}}$,

(ii) $[(\mathbf{L}+\mathbf{M}-\mathbf{I_n})^{\oplus}]^{\mathbf{2}}\mathbf{M}=(\mathbf{MLM})^{\oplus}=\mathbf{M}[(\mathbf{L}+\mathbf{M}-\mathbf{I_n})^{\oplus}]^{\mathbf{2}}$.

PROOF : Using the matrix representations of $\mathbf{L}$ and $(\mathbf{L}+\mathbf{M}-\mathbf{I_n})^{\oplus}$ and the point (i) of Theorem

1 and the points (i) and (iv) of Theorem 2, we obtain

$$[(\mathbf{L}+\mathbf{M}-\mathbf{I_n})^{\oplus}]^{\mathbf{2}} = \mathbf{U}\begin{bmatrix}\mathbf{W}^{\oplus} & 0\\ \mathbf{0} & 0\end{bmatrix}\mathbf{U}^{\sim}.$$

Also

$$(\mathbf{LML})^{\oplus} = [(\mathbf{L}+\mathbf{M}-\mathbf{I_n})^{\oplus}]^{\mathbf{2}} = \mathbf{U}\begin{bmatrix}\mathbf{W}^{\oplus} & 0\\ \mathbf{0} & 0\end{bmatrix}\mathbf{U}^{\sim}.$$

Thus, the LHS of the equality (i) follows. Taking the Minkowski conjugate of the first half of the equality, the RHS of the equality (i) follows. The proof of the point (ii) follows on the same lines.

In [8] the authors showed that under the conditions of the invertibility of $(\mathbf{L}+\mathbf{M}-\mathbf{I_n})$, the function given by

$$\mathbf{K}(\mathbf{L},\mathbf{M}) = \mathbf{L}[(\mathbf{L}+\mathbf{M}-\mathbf{I_n})]^{-\mathbf{2}}\mathbf{M}.$$

is a unique projector onto the $R(\mathbf{L})$ along $N(\mathbf{M})$. The function involved is actually know as the Kovarik formula defined by Kovarik in [21]. Redefining the function, by relaxing the invertibility condition, as

$$\mathbf{K_r}(\mathbf{L},\mathbf{M}) = \mathbf{L}[(\mathbf{L}+\mathbf{M}-\mathbf{I_n})^{\oplus}]^{\mathbf{r}}\mathbf{M};\ \mathbf{r}\in\mathbb{N}, (\mathbb{N}\text{ is the set of natural numbers}). \tag{43}$$

For $\mathbf{r}=\mathbf{1}$, we have

$$\mathbf{K_1}(\mathbf{L},\mathbf{M}) = \mathbf{L}[(\mathbf{L}+\mathbf{M}-\mathbf{I_n})^{\oplus}]\mathbf{M},$$

which upon using the point (iii) of Theorem 12 gives

$$\mathbf{K_1}(\mathbf{L},\mathbf{M}) = \mathbf{L}[(\mathbf{L}+\mathbf{M}-\mathbf{I_n})^{\oplus}]\mathbf{M} = (\mathbf{ML})^{\oplus}. \tag{44}$$

The next result shows that the statement (i) of the Theorem (2.7) in [8] remains valid by relaxing the assumption of invertibility of $(\mathbf{L}+\mathbf{M}-\mathbf{I_n})$.

***Theorem* 14** — *Let* $\mathbf{L},\ \mathbf{M}\in\mathbb{C}_n^{mp}$. *Let* $\mathbf{K_r}(\mathbf{L},\mathbf{M})$ *be defined as in* (43). *Then,*

$$\mathbf{K_{2r+1}}(\mathbf{L},\mathbf{M}) = \mathbf{K_{2(r+1)}}(\mathbf{L},\mathbf{M}).$$

PROOF : From the definition given in (43), we have

$$\mathbf{K_{2r+1}}(\mathbf{L},\mathbf{M}) = \mathbf{L}[(\mathbf{L}+\mathbf{M}-\mathbf{I_n})^{\oplus}]^{\mathbf{2r+1}}\mathbf{M} = \mathbf{L}[(\mathbf{L}+\mathbf{M}-\mathbf{I_n})^{\oplus}]^{\mathbf{2r}}(\mathbf{L}+\mathbf{M}-\mathbf{I_n})^{\oplus}\mathbf{M}.$$

Mathematical induction, on using the point (i) of Theorem 13, shows that $\mathbf{L}$ commutes with $[(\mathbf{L}+\mathbf{M}-\mathbf{I_n})^{\oplus}]^{\mathbf{2r}}$. Therefore, we have

$$\mathbf{K_{2r+1}}(\mathbf{L},\mathbf{M}) = [(\mathbf{L}+\mathbf{M}-\mathbf{I_n})^{\oplus}]^{\mathbf{2r}}\mathbf{L}(\mathbf{L}+\mathbf{M}-\mathbf{I_n})^{\oplus}\mathbf{M} = [(\mathbf{L}+\mathbf{M}-\mathbf{I_n})^{\oplus}]^{\mathbf{2r}}\mathbf{K_1}(\mathbf{L},\mathbf{M}).$$

But the point (i) of Theorem 13 gives

$$\mathbf{K_{2(r+1)}}(\mathbf{L},\mathbf{M}) = [(\mathbf{L}+\mathbf{M}-\mathbf{I_n})^{\oplus}]^{\mathbf{2(r+1)}}\mathbf{L}\mathbf{M}. \tag{45}$$

Since $\mathbf{LM} = (\mathbf{L}+\mathbf{M}-\mathbf{I_n})\mathbf{M}$, we have

$$\mathbf{K_{2(r+1)}}(\mathbf{L},\mathbf{M}) = [(\mathbf{L}+\mathbf{M}-\mathbf{I_n})^{\oplus}]^{\mathbf{2r}}[(\mathbf{L}+\mathbf{M}-\mathbf{I_n})^{\oplus}]^{\mathbf{2}}(\mathbf{L}+\mathbf{M}-\mathbf{I_n})\mathbf{M}.$$

Now using the fact that $(\mathbf{L}+\mathbf{M}-\mathbf{I_n})$ commutes with its Minkowski inverse and the condition second in the definition of Minkowski inverse of a matrix, we get

$$\mathbf{K_{2(r+1)}}(\mathbf{L},\mathbf{M}) = [(\mathbf{L}+\mathbf{M}-\mathbf{I_n})^{\oplus}]^{\mathbf{2r}}[(\mathbf{L}+\mathbf{M}-\mathbf{I_n})^{\oplus}]\mathbf{M}. \tag{46}$$

Finally, using the point (iv) of Theorem 12, we get the result.

Corollary 3 — Let $\mathbf{L},\ \mathbf{M} \in \mathbb{C}_n^{mp}$. Let $\mathbf{K_1}(\mathbf{L},\mathbf{M})$ and $\mathbf{K_2}(\mathbf{L},\mathbf{M})$ be defined as in (43). Then, $\mathbf{K_1}(\mathbf{L},\mathbf{M}) = \mathbf{K_2}(\mathbf{L},\mathbf{M}) = (\mathbf{ML})^{\oplus}$

The next Theorem generalizes and extends the result (2.6) from [13] to the Minkowski space $\mathscr{M}$.

***Theorem* 15** — *Let* $\mathbf{L},\ \mathbf{M} \in \mathbb{C}_n^{mp}$. *Let* $\mathbf{K_2}(\mathbf{L},\mathbf{M})$ *be defined as in* (43). *Then:*

(i) $\mathbf{K_2}(\mathbf{L},\mathbf{M}) + \mathbf{K_2}(\bar{\mathbf{M}},\bar{\mathbf{L}}) = \mathbf{P}_{R(\mathbf{L}+\mathbf{M}-\mathbf{I_n})}$,

(ii) $\mathbf{K_2}[\mathbf{K_2}(\mathbf{M},\mathbf{L}),\mathbf{K_2}(\mathbf{L},\mathbf{M})] = \mathbf{P}_{R(\mathbf{ML})}$.

PROOF : The Minkowski inverse of $(\bar{\mathbf{L}}\bar{\mathbf{M}}) = \mathbf{U}\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{G_1}\mathbf{X}^{\sim} & \bar{\mathbf{Z}} \end{bmatrix}\mathbf{U}^{\sim}$, as a result of direct verification on using the points (i) and (iv) of Theorem 1 and (iv) of Theorem 2, is given by

$$(\bar{\mathbf{L}}\bar{\mathbf{M}})^{\oplus} = \mathbf{U}\begin{bmatrix} \mathbf{0} & -\mathbf{X}\bar{\mathbf{Z}}^{\oplus} \\ \mathbf{0} & \mathbf{P}_{\bar{\mathbf{z}}} \end{bmatrix}\mathbf{U}^{\sim}. \tag{47}$$

Hence, from the matrix representations given in (41), (42), and (47), we have $(\mathbf{ML})^{\oplus} + (\bar{\mathbf{L}}\bar{\mathbf{M}})^{\oplus} = \mathbf{P}_{R(\mathbf{L}+\mathbf{M}-\mathbf{I_n})}$. Using the fact that $\mathbf{K_2}(\bar{\mathbf{M}},\bar{\mathbf{L}}) = (\bar{\mathbf{L}}\bar{\mathbf{M}})^{\oplus}$ and Corollary 3, the statement (i) of the Theorem is established. Again from the definition given in (43) and Corollary 3, we get

$$\mathbf{K_2}[\mathbf{K_2}(\mathbf{M},\mathbf{L}),\mathbf{K_2}(\mathbf{L},\mathbf{M})] = (\mathbf{LM})^{\oplus}\{[(\mathbf{LM})^{\oplus} + (\mathbf{ML})^{\oplus} - \mathbf{I_n}]^{\oplus}\}^{\mathbf{2}}(\mathbf{ML})^{\oplus}.$$

Now

$$(\mathbf{LM})^{\oplus}+(\mathbf{ML})^{\oplus}-\mathbf{I_n}=\mathbf{U}\left\{\begin{bmatrix}\mathbf{P_w} & \mathbf{W}^{\oplus}\mathbf{X}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}+\begin{bmatrix}\mathbf{P_w} & \mathbf{0}\\ -\bar{\mathbf{Z}}^{\oplus}\mathbf{G_1}\mathbf{X}^{\sim} & \mathbf{0}\end{bmatrix}-\begin{bmatrix}\mathbf{I_r} & \mathbf{0}\\ \mathbf{0} & \mathbf{I_{n-r}}\end{bmatrix}\right\}\mathbf{U}^{\sim}$$
$$=\mathbf{U}\begin{bmatrix}\mathbf{2P_w}-\mathbf{I_r} & \mathbf{W}^{\oplus}\mathbf{X}\\ -\bar{\mathbf{Z}}^{\oplus}\mathbf{G_1}\mathbf{X}^{\sim} & -\mathbf{I_{n-r}}\end{bmatrix}\mathbf{U}^{\sim}.$$

Utilizing the points (i) and (iv) of Theorem 1 and the points (i) and (iv) of Theorem 2, it can be verified that the Minkowski inverse of $[(\mathbf{LM})^{\oplus}+(\mathbf{ML})^{\oplus}-\mathbf{I_n}]$ is

$$[(\mathbf{LM})^{\oplus}+(\mathbf{ML})^{\oplus}-\mathbf{I_n}]^{\oplus}=\mathbf{U}\begin{bmatrix}\mathbf{W}-\tilde{\mathbf{P}}_{\mathbf{w}} & \mathbf{X}\\ -\mathbf{G_1}\mathbf{X}^{\sim} & -(\bar{\mathbf{Z}}+\tilde{\mathbf{P}}_{\bar{\mathbf{z}}})\end{bmatrix}\mathbf{U}^{\sim}.$$

Furthermore, using the points (i) of Lemma 2, (i), (iii), and (iv) of Theorem 2 and (v) of Lemma 3, we obtain

$$\{[(\mathbf{LM})^{\oplus}+(\mathbf{ML})^{\oplus}-\mathbf{I_n}]^{\oplus}\}^{\mathbf{2}}=\mathbf{U}\begin{bmatrix}\mathbf{W}+\tilde{\mathbf{P}}_{\mathbf{w}} & \mathbf{0}\\ \mathbf{0} & \bar{\mathbf{Z}}+\tilde{\mathbf{P}}_{\bar{\mathbf{z}}}\end{bmatrix}\mathbf{U}^{\sim}.$$

Using the respective matrix representations of $(\mathbf{LM})^{\oplus}$,$\{[(\mathbf{LM})^{\oplus}+(\mathbf{ML})^{\oplus}-\mathbf{I_n}]^{\oplus}\}^{\mathbf{2}}$, and $(\mathbf{ML})^{\oplus}$ and utilizing the points (i) of Theorem 1 and (iv) of Theorem 2, we get

$$\mathbf{K_2}[\mathbf{K_2}(\mathbf{M},\mathbf{L}),\mathbf{K_2}(\mathbf{L},\mathbf{M})]=\mathbf{U}\begin{bmatrix}\mathbf{W} & \mathbf{X}\\ -\mathbf{G_1}\mathbf{X}^{\sim} & \mathbf{P}_{\bar{\mathbf{z}}}+\bar{\mathbf{Z}}\end{bmatrix}\mathbf{U}^{\sim}. \tag{48}$$

Finally, $\mathbf{P}_{R(\mathbf{ML})} = (\mathbf{ML})(\mathbf{ML})^{\oplus}$, which is equal to the representation obtained in (48), the result follows.

Corollary 4 — Let $\mathbf{L},\ \mathbf{M}\in\mathbb{C}_n^{mp}$. Let $\mathbf{K_1}(\mathbf{L},\mathbf{M})$ be defined as in (43). Then,

$$\mathbf{K_1}[\mathbf{K_1}(\mathbf{M},\mathbf{L}),\mathbf{K_1}(\mathbf{L},\mathbf{M})]=\mathbf{P}_{R(\mathbf{ML})}.$$

From the representation (48), it can be observed that $\mathbf{K_1}[\mathbf{K_1}(\mathbf{M},\mathbf{L}),\mathbf{K_1}(\mathbf{L},\mathbf{M})] = \mathbf{M}$ if and only if $\mathbf{P}_{\bar{\mathbf{z}}}-\bar{\mathbf{Z}} = \mathbf{Z}$, or equivalently rk$(\bar{\mathbf{Z}}) = \mathbf{n}-\mathbf{r}$. But we have already mentioned that the invertibility of $(\mathbf{L}+\mathbf{M}-\mathbf{I_n})$ implies the invertibility of $\bar{\mathbf{Z}}$. This is also established in the result (2.6) in [13].

***Theorem* 16** — *Let* $\mathbf{L},\ \mathbf{M}\in\mathbb{C}_n^{mp}$. *Let* $\mathbf{K_2}(\mathbf{L},\mathbf{M})$ *be defined as in* (43). *Then,*

$$\mathbf{K_2}(\mathbf{L},\mathbf{M})+\mathbf{K_2}(\mathbf{M},\mathbf{L})=(\mathbf{L}+\mathbf{M}-\mathbf{I_n})^{\oplus}+\mathbf{P}_{R(\mathbf{L}+\mathbf{M}-\mathbf{I_n})}.$$

PROOF : The proof follows at once by using the representations obtained in (39), (40), (41) and (42).

Using the results obtained in this paper, a large number of formulas regarding *m*-projections can be further developed.

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