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BOUNDS FOR THE SIGNLESS LAPLACIAN ENERGY OF DIGRAPHS[1]

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Let G be a digraph with n vertices, a arcs, c_2 directed closed walks of length 2. Let $q_1, q_2, \ldots, q_n$ be the eigenvalues of the signless Laplacian matrix of G. The signless Laplacian energy of a digraph G is defined as $E_{SL}(G) = \sum_{i=1}^{n} |q_i - \frac{a}{n}|$. In this paper, some lower and upper bounds are derived for the signless Laplacian energy of digraphs.

Key words : Energy; signless Laplacian energy; digraph.

1. Introduction

In this paper we only consider finite digraphs without loops and multiple arcs. However, it allows to a pair of oppositely directed arcs join the same pair of vertices. Let $G = (V(G), E(G))$ be a digraph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and arc set $E(G)$. If two vertices are connected by an arc, then they are called adjacent. If there is an arc from v_i to v_j, we indicate this by writing (v_i, v_j), call v_j the head of (v_i, v_j), and v_i the tail of (v_i, v_j), respectively. For an arc (v_i, v_j), if (v_j, v_i) is not an arc in G, then (v_i, v_j) is simple and if (v_j, v_i) is also an arc in G, then (v_i, v_j) is symmetric. For any vertex v_i, let $N_i^+ = \{v_j \in V(G) \mid (v_i, v_j) \in E(G)\}$ denote the out-neighbors of v_i. Let $d_i^+ = |N_i^+|$ denote the outdegree of the vertex v_i in the digraph G. If every arc of a digraph is simple, then we call the digraph a simple digraph. Let $\overrightarrow{P_n}$ and $\overrightarrow{C_n}$ denote the directed path and the directed cycle on n vertices, respectively. Let $\overleftrightarrow{K}_n$ denote the complete digraph on n vertices obtained from complete undirected graph K_n

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by replacing each edge in K_n with a pair of symmetric arcs. If the outdegrees of all vertices are equal, then the digraph is called outdegree regular.

Let $A(G) = (a_{ij})$ denote the adjacency matrix of G, where $a_{ij} = 1$ if $(v_i, v_j) \in E(G)$ and $a_{ij} = 0$ otherwise. Let $D(G) = \text{diag}(d_1^+, d_2^+, \ldots, d_n^+)$ be the diagonal matrix with outdegrees of the vertices of G. Then the matrix $Q(G) = D(G) + A(G)$ is called the signless Laplacian matrix of G. Let $q_1, q_2, \ldots, q_n$ be the eigenvalues of the signless Laplacian matrix of G. Since $Q(G)$ is not symmetric and its eigenvalues can be complex numbers. If $Q(G)$ is a normal matrix, then G is called a signless Laplacian normal digraph.

The energy connected to undirected graphs has been well investigated in the literature, see [1, 6, 7, 8, 10, 17] and a book [12]. The energy of an undirected graph has close links to chemistry. However, in many cases we must use digraphs rather than undirected graphs. But there is not so much known results about the energy of digraphs.

Recently, Peña and Rada [14] introduced the concept of the energy for a digraph G:

$$E(G) = \sum_{i=1}^{n} |Re(z_i)|,$$

where $z_1, z_2, \ldots, z_n$ are the eigenvalues of $A(G)$ and $Re(z_i)$ denotes the real part of z_i, $1 \leq i \leq n$. For further results on the energy of digraphs see [5, 9, 15, 16].

Since the skew Laplacian matrix of a digraph contains the information of the degree of vertices and it is a skew symmetric matrix. It plays an important role in spectral theory of digraphs. Therefore, based on the definition of the Laplacian energy of undirected graphs, Adiga and Khoshbakht [2] gave the skew Laplacian energy of a simple digraph G as

$$SLE(G) = \sum_{i=1}^{n} |\mu_i - \frac{2m}{n}|,$$

where $\mu_1, \mu_2, \ldots, \mu_n$ are the eigenvalues of the skew Laplacian matrix of G and m is the number of arcs. They established some upper and lower bounds for the skew Laplacian energy of a simple digraph. There are also many other results on the skew Laplacian energy of a simple digraph, see [3, 4].

The skew Laplacian matrix of a digraph only allows simple arcs. Therefore the Laplacian energy of a digraph has attracted attention, where both simple and symmetric arcs are allowed. In 2010, Perera and Mizoguchi [13] presented another definition for the Laplacian energy of a digraph G, which was given as follow:

$$LE(G) = \sum_{i=1}^{n} \lambda_i^2,$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are all the eigenvalues of Laplacian matrix $L(G) = D(G) - A(G)$. They derived a formula for $LE(G)$ and presented lower and upper bounds for $LE(G)$ in term of the number of vertices.

In this paper, the signless Laplacian energy of a digraph G is defined. We obtain some lower and upper bounds about it and show that these bounds are sharp.

2. Bounds for the Signless Laplacian Energy of A Digraph

We give the definition of the signless Laplacian energy of a digraph.

Definition 2.1 — Let G be a digraph with n vertices and a arcs. Then the signless Laplacian energy of the digraph G is defined as

$$E_{SL}(G) = \sum_{i=1}^{n} |q_i - \frac{a}{n}|,$$

where $q_1, q_2, \ldots, q_n$ are the eigenvalues of the signless Laplacian matrix $Q(G) = D(G) + A(G)$ of G.

In order to illustrate the concept of the signless Laplacian energy of a digraph, we give the following two examples.

Example 2.2 : Let $\overleftrightarrow{K}_3$ be a complete digraph. Then

$$Q(\overleftrightarrow{K}_3) = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

The eigenvalues of the signless Laplacian matrix $Q(\overleftrightarrow{K}_3)$ are 1,1,4. Hence the signless Laplacian energy of $\overleftrightarrow{K}_3$ is 4.

Example 2.3 : Let $\overrightarrow{C_4}$ be a directed digraph with arc set {(1,2), (2,3), (3,4), (4,1)}. Then

$$Q(\overrightarrow{C_4}) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

The eigenvalues of the signless Laplacian matrix $Q(\overrightarrow{C_4})$ are 0, $1+i$, $1-i$, 2. Hence the signless Laplacian energy of $\overrightarrow{C_4}$ is 4.

Let G be a digraph with n vertices, a arcs, c_2 directed closed walks of length 2. Let $q_1, q_2, \ldots, q_n$ be the eigenvalues of the signless Laplacian matrix $Q(G)$. Then we have

$$\sum_{i=1}^{n} q_i = \sum_{i=1}^{n} d_i^+ = a. \tag{1}$$

From Vieta's rule, we can see that the sum of the determinants of all 2×2 principal submatrices of $Q(G)$ are equal to $\sum\limits_{i<j} q_i q_j$. That is

$$\begin{aligned}\sum_{i<j} q_i q_j &= \sum_{i<j} det \begin{pmatrix} d_i^+ & a_{ij} \\ a_{ji} & d_j^+ \end{pmatrix} \\ &= \sum_{i<j}(d_i^+ d_j^+ - a_{ij}a_{ji}) \\ &= \sum_{i<j} d_i^+ d_j^+ - \sum_{i<j} a_{ij}a_{ji}.\end{aligned}$$

Therefore

$$\sum_{i\neq j} q_i q_j = 2\sum_{i<j} q_i q_j = 2\sum_{i<j} d_i^+ d_j^+ - c_2. \tag{2}$$

Combining (1) and (2), we get

$$\begin{aligned}\sum_{i=1}^{n} {q_i}^2 &= (\sum_{i=1}^{n} q_i)^2 - 2\sum_{i<j} q_i q_j \\ &= (\sum_{i=1}^{n} d_i^+)^2 - 2\sum_{i<j} d_i^+ d_j^+ + c_2 \\ &= \sum_{i=1}^{n} {d_i^+}^2 + c_2.\end{aligned}$$

That is

$$\sum_{i=1}^{n} {q_i}^2 = \sum_{i=1}^{n} {d_i^+}^2 + c_2. \tag{3}$$

Define $\alpha_i = q_i - \dfrac{a}{n}$ for $i = 1, 2, \ldots, n$. Using (1) and (3), we see that

$$\sum_{i=1}^{n} \alpha_i = \sum_{i=1}^{n} (q_i - \frac{a}{n}) = \sum_{i=1}^{n} q_i - \sum_{i=1}^{n} \frac{a}{n} = 0. \tag{4}$$

and

$$\sum_{i=1}^{n} {\alpha_i}^2 = \sum_{i=1}^{n} (q_i - \frac{a}{n})^2 = \sum_{i=1}^{n} {q_i}^2 - 2\frac{a}{n} \sum_{i=1}^{n} q_i + \sum_{i=1}^{n} (\frac{a}{n})^2$$
$$= \sum_{i=1}^{n} {d_i^+}^2 + c_2 - 2\frac{a}{n} \sum_{i=1}^{n} d_i^+ + \sum_{i=1}^{n} (\frac{a}{n})^2$$
$$= c_2 + \sum_{i=1}^{n} (d_i^+ - \frac{a}{n})^2.$$

Let $M = c_2 + \sum_{i=1}^{n} (d_i^+ - \frac{a}{n})^2$. Then we have

$$\sum_{i=1}^{n} {\alpha_i}^2 = M \tag{5}$$

Since $\frac{a}{n}$ is the average vertex outdegree, we obtain that $M = c_2 \geq 0$ if and ony if G is the outdegree regular, and $M > c_2 \geq 0$ otherwise.

Theorem 2.4 — *Let G be a digraph with n vertices, a arcs, c_2 directed closed walks of length 2, and d_i^+ be the outdegree of the i^{th} vertex of G, $i = 1, 2, \ldots, n$. If $q_1, q_2, \ldots, q_n$ are the eigenvalues of the signless Laplacian matrix $Q(G) = D(G) + A(G) = (b_{ij})$ of G, where $D(G) = diag(d_1^+, d_2^+, \ldots, d_n^+)$ and $A(G) = (a_{ij})$ is the adjacency matrix of G. Then*

$$E_{SL}(G) \leq \sqrt{M_1 n},$$

where $M_1 = a + \sum_{i=1}^{n} (d_i^+ - \frac{a}{n})^2$. Moreover, this upper bound is sharp.

PROOF : From (1), we have

$$\sum_{i=1}^{n} Re(q_i) = \sum_{i=1}^{n} d_i^+ = a. \tag{6}$$

By Schur's unitary triangularization theorem [11], there is a unitary matrix U such that $U^*Q(G)U = T = (t_{ij})$, where $T = (t_{ij})$ is an upper triangular matrix with diagonal entries $t_{ii} = q_i$, $i = 1, 2, \ldots, n$. Therefore

$$T^*T = U^*Q^*(G)UU^*Q(G)U = U^*Q^*(G)Q(G)U.$$

Then we have $tr(T^*T) = tr(Q^*(G)Q(G))$. So

$$\sum_{i=1}^{n} |q_i|^2 = \sum_{i=1}^{n} |t_{ii}|^2 \leq \sum_{i,j=1}^{n} |t_{ij}|^2 = tr(T^*T) = tr(Q^*(G)Q(G)) = \sum_{i,j=1}^{n} |b_{ij}|^2.$$

Thus

$$\sum_{i=1}^{n}|q_i|^2 \leq \sum_{i,j=1}^{n}|b_{ij}|^2 = \sum_{i=1}^{n} d_i^{+2} + a. \tag{7}$$

Let $\alpha_i = q_i - \dfrac{a}{n}$ for $i = 1, 2, \ldots, n$. Applying the Cauchy-Schwarz inequality to the vector

$$(|\alpha_1|, |\alpha_2|, \ldots, |\alpha_n|) \text{ and } (1, 1, \ldots, 1).$$

We have

$$E_{SL}(G) = \sum_{i=1}^{n}|q_i - \frac{a}{n}| = \sum_{i=1}^{n}|\alpha_i| \leq \sqrt{n\sum_{i}|\alpha_i|^2}. \tag{8}$$

However, by (6) and (7), we have

$$\begin{aligned}\sum_{i=1}^{n}|\alpha_1|^2 = \sum_{i=1}^{n}|q_i - \frac{a}{n}|^2 &= \sum_{i=1}^{n}(q_i - \frac{a}{n})(\overline{q_i} - \frac{a}{n}) \\ &= \sum_{i=1}^{n}|q_i|^2 - 2\frac{a}{n}\sum_{i=1}^{n} Re(q_i) + \sum_{i=1}^{n}(\frac{a}{n})^2 \\ &\leq \sum_{i=1}^{n} d_i^{+2} + a - 2\frac{a}{n}\sum_{i=1}^{n} d_i^{+} + \sum_{i=1}^{n}(\frac{a}{n})^2 \\ &= a + \sum_{i=1}^{n}(d_i^{+} - \frac{a}{n})^2.\end{aligned}$$

Since $M_1 = a + \sum_{i=1}^{n}(d_i^{+} - \frac{a}{n})^2 \geq 0$. Then we have

$$\sum_{i=1}^{n}|\alpha_1|^2 \leq M_1. \tag{9}$$

Therefore, from (8) and (9), we conclude that

$$E_{SL}(G) \leq \sqrt{M_1 n}.$$

Evidently, from (7) and (8), we know that the equality holds if and only if $T = (t_{ij})$ is a diagonal matrix and $|\alpha_1| = |\alpha_2| = \ldots = |\alpha_n|$.

From Schur's unitary triangularization theorem [11], we know that $T = (t_{ij})$ is a diagonal matrix if and only if $Q(G)$ is a normal matrix. That is

$$Q^*(G)Q(G) = Q(G)Q^*(G).$$

This equality holds if and only if G is a signless Laplacian normal digraph and $|q_1 - \frac{a}{n}| = |q_2 - \frac{a}{n}| = \ldots = |q_n - \frac{a}{n}|$.

It is natural to ask: do such digraphs exist? The answer is yes. Let $\overrightarrow{C_n}$ denote the directed cycle on n vertices. Note that $d_i^+ = 1$ for all v_i in G. And $\overrightarrow{C_n}$ has n arcs, so $\frac{a}{n} = 1 = d_i^+$, $M_1 = a + \sum_{i=1}^{n}(d_i^+ - \frac{a}{n})^2 = n$, $\sqrt{M_1 n} = n$. The signless Laplacian matrix of $\overrightarrow{C_n}$ is

$$Q(\overrightarrow{C_n}) = \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix}.$$

We know that $Q(\overrightarrow{C_n})$ is a normal matrix, hence $\overrightarrow{C_n}$ is a signless Laplacian normal digraph. The eigenvalues of the signless Laplacian matrix $Q(\overrightarrow{C_n})$ are $1+\cos(\frac{2k\pi}{n})+i\sin(\frac{2k\pi}{n})$, $k = 1, 2, \ldots, n$. Hence$|q_1-\frac{a}{n}| = |q_2-\frac{a}{n}| = \ldots = |q_n-\frac{a}{n}|=1$, the signless Laplacian energy of $\overrightarrow{C_n}$ is $n = \sqrt{M_1 n}$, which implies that the upper bound is sharp. □

Corollary 2.5 — If G has no isolated vertices, then

$$E_{SL}(G) \le \sqrt{2}M_1.$$

PROOF : Since G has no isolated vertices, we have $n \le \sum_{i=1}^{n} d_i^+ + \sum_{i=1}^{n} d_i^- = 2a$. But $M_1 = a + \sum_{i=1}^{n}(d_i^+ - \frac{a}{n})^2 \ge a$, so

$$E_{SL}(G) \le \sqrt{M_1 n} \le \sqrt{2M_1 a} \le \sqrt{2}M_1.$$

This completes the proof. □

Let G be a digraph with n vertices, a arcs, c_2 directed closed walks of length 2. If $E(G)$ is the energy of a digraph G, then we have known [9] that

$$E(G) \le \frac{c_2}{n} + \sqrt{(n-1)\left[a - \left(\frac{c_2}{n}\right)^2\right]}. \tag{10}$$

We prove an inequality similar to (10) for the signless Laplacian energy of a digraph.

***Theorem* 2.6** — *Let G be a digraph with n vertices, a arcs, c_2 directed closed walks of length 2. If $q_1, q_2, \ldots, q_n$ are the eigenvalues of the signless Laplacian matrix $Q(G)$, with $|\alpha_1| \leq |\alpha_2| \leq \ldots \leq |\alpha_n| = k$, where $\alpha_i = q_i - \frac{a}{n}$ for $i = 1, 2, \ldots, n$. Then*

$$E_{SL}(G) \leq k + \sqrt{(n-1)(M_1 - k^2)},$$

where $M_1 = a + \sum\limits_{i=1}^{n}(d_i^+ - \frac{a}{n})^2$.

PROOF : Let $\mathbf{X} = (|\alpha_1|, |\alpha_2|, \ldots, |\alpha_{n-1}|)$ and $\mathbf{Y} = (1, 1, \ldots, 1)$. By the Cauchy-Schwarz inequality we have

$$(\sum_{i=1}^{n-1} |\alpha_i|)^2 \leq (n-1) \sum_{i=1}^{n-1} |\alpha_i|^2.$$

That is

$$(E_{SL}(G) - |\alpha_n|)^2 \leq (n-1)(\sum_{i=1}^{n} |\alpha_i|^2 - |\alpha_n|^2).$$

By (9), we have

$$(E_{SL}(G) - k)^2 \leq (n-1)(M_1 - k^2).$$

Therefore

$$E_{SL}(G) \leq k + \sqrt{(n-1)(M_1 - k^2)},$$

This completes the proof. □

***Theorem* 2.7** — *Let G be a digraph with n vertices, a arcs, c_2 directed closed walks of length 2. Then*

$$E_{SL}(G) \geq \sqrt{2M},$$

where $M = c_2 + \sum\limits_{i=1}^{n}(d_i^+ - \frac{a}{n})^2$. Moreover, this lower bound is sharp.

PROOF : Since $\sum\limits_{i=1}^{n} \alpha_i = 0$, we have

$$(\sum_{i=1}^{n} \alpha_i)^2 = \sum_{i=1}^{n} {\alpha_i}^2 + 2\sum_{i<j} \alpha_i \alpha_j = 0.$$

By (5), we have

$$2\sum_{i<j} \alpha_i \alpha_j = -M.$$

This implies

$$M = 2|\sum_{i<j} \alpha_i \alpha_j| \leq 2\sum_{i<j} |\alpha_i||\alpha_j|. \tag{11}$$

Using (5) again,

$$M = |\sum_{i=1}^{n} {\alpha_i}^2| \leq \sum_{i=1}^{n} |\alpha_i|^2, \tag{12}$$

therefore

$$\begin{aligned}(E_{SL}(G))^2 = (\sum_{i=1}^{n} |\alpha_i|)^2 &= \sum_{i=1}^{n} |\alpha_i|^2 + 2\sum_{i<j} |\alpha_i||\alpha_j| \\ &\geq M + 2\sum_{i<j} |\alpha_i||\alpha_j|.\end{aligned}$$

Combining with (11), we have $(E_{SL}(G))^2 \geq 2M$. Thus

$$E_{SL}(G) \geq \sqrt{2M}.$$

It follows from (11) and (12) that the equality holds if and only if for each pair of $\alpha_{i_1}\alpha_{j_1}$ and $\alpha_{i_2}\alpha_{j_2}$, $(i_1 \neq j_1, i_2 \neq j_2)$, there exists a nonnegative real number h such that $\alpha_{i_1}\alpha_{j_1} = h\alpha_{i_2}\alpha_{j_2}$, for each pair of ${\alpha_{i_1}}^2$ and ${\alpha_{i_2}}^2$, there exists a nonnegative real number l such that ${\alpha_{i_1}}^2 = l{\alpha_{i_2}}^2$.

Next we give an example to illustrate the existence of such digraphs. Let G_1 be the complete bipartite digraph $\overleftrightarrow{K}_{n,n}$ on $2n$ vertices obtained from complete bipartite undirected graph $K_{n,n}$ by replacing each edge in $K_{n,n}$ with a pair of symmetric arcs, as shown in Figure 1. Assume that $\{X_1, X_2\}$ is the bipartition of G_1 and $|X_1| = |X_2| = n$.

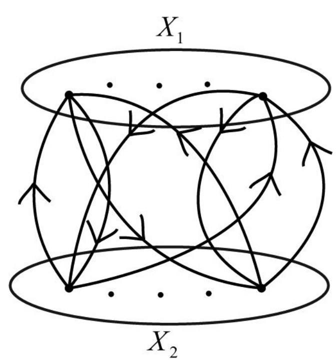

Figure 1 : The digraph $G_1 = \overleftrightarrow{K}_{n,n}$.

Note that $d_i^+ = n$ for all v_i in G_1, $a = 2n^2$, $\frac{a}{2n} = n = d_i^+$, the number of directed closed walks of length 2 are a. So $M = c_2 + \sum_{i=1}^{2n}(d_i^+ - \frac{a}{2n})^2 = a = 2n^2$, $\sqrt{2M} = 2n$. The signless

Laplacian matrix of G_1 is

$$Q(G_1) = \begin{pmatrix} nI_n & J_n \\ J_n & nI_n \end{pmatrix},$$

where J_n is the $n \times n$ matrix in which each entry is 1. The eigenvalues of the signless Laplacian matrix $Q(G_1)$ are 0, $2n$, n with multiplicity 1, 1, $2n-2$, respectively. Hence the signless Laplacian energy of G_1 is $2n = \sqrt{2M}$, which implies that the lower bound is sharp. This completes the proof. □

Remark 2.8 : If we define the Laplacian energy of a digraph G as $E_L(G) = \sum_{i=1}^{n} |\lambda_i - \frac{a}{n}|$, where the Laplacian matrix of G is $L(G) = D(G) - A(G)$ and $\lambda_1, \lambda_2, \ldots, \lambda_n$ are all the eigenvalues of $L(G)$. Then we can find that the above upper and lower bounds are also valid for the Laplacian energy of G.

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References

1. N. Abreu, D. M. Cardoso, I. Gutman, E. A. Martins and M. Robbiano, Bounds for the signless Laplacian energy, *Linear Algebra Appl.*, **435** (2011), 2365-2374.
2. C. Adiga and Z. Khoshbakht, On some inequalities for the skew Laplacian energy of digraphs, *JIPAM. J. Inequal. Pure and Appl. Math.*, **10**(3) (2009), Art. 80, 6 pp.
3. C. Adiga and M. Smitha, On the skew Laplacian energy of a digraph, *Int. Math. Forum*, **4(3)** (2009), 1907-1914.
4. Q. Q. Cai, X. L. Li and J. L. Song, New skew Laplacian enery of simple digraphs, *Transactions on Combinatorics*, **2**(1) (2013), 27-37.
5. R. Cruz, H. Giraldo and J. Rada, An upper bound for the energy of radial digraphs, *Linear Algebra Appl.*, **442** (2014), 75-81.
6. K. C. Das and S. A. Mojallal, Upper bounds for the energy of graphs, *MATCH Commun. Math. Comput. Chem.*, **70**(2) (2013), 657-662.
7. K. C. Das, I. Gutman, A. S. Çevik and B. Zhou, On Laplacian energy, *MATCH Commun. Math. Comput. Chem.*, **70**(2) (2013), 689-696.
8. I. Gutman, *The energy of a graph: old and new results*, In: A Betten, A. Kohnert, R. Laue, A. Wassermann (Eds), Algebraic Combinatorics and Applications, Springer-Verlag, Berlin, 2001, 196-211.

9. E. Gudiño and J. Rada, A lower bound for the spectral radius of a digraph, *Linear Algebra Appl.*, **433** (2010), 233-240.

10. I. Gutman and B. Zhou, Laplacian energy of a graph, *Linear Algebra Appl.*, **414** (2006), 29-37.

11. R. A. Horn and C. R. Johnson, *Matrix analysis*, Cambridge Univ. Press, 1990.

12. X. L. Li, Y. C. Shi and I. Gutman, *Graph Energy*, Springer, New York, 2012.

13. K. Perera and Y. Mizoguchi, Laplacian energy of directed graphs and minimizing maximum outdegree algorithms, MI Preprint Series (2010) 2010-35.

14. I. Peña and J. Rada, Energy of digraphs, *Linear and Multilinear Algebra*, **56**(5) (2008), 565-579.

15. S. Pirzada, M. A Bhat, I. Gutman and J. Rada, On the energy of digraphs, *Bull. Int. Math. Virtual Inst.*, **3** (2013), 69-76.

16. J. Rada, The McClelland inequality for the energy of digraphs, *Linear Algebra Appl.*, **430** (2009), 800-804.

17. B. Zhou and I. Gutman, On Laplacian energy of graphs, *MATCH Commun. Math. Comput. Chem.*, **57** (2007), 211-220.