Indian J. Pure Appl. Math., **48**(3): 449-465, September 2017
 DOI: 10.1007/s13226-017-0237-4

RAMANUJAN-TYPE CONGRUENCES MODULO POWERS OF 5 AND 7

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(*Received* 5 *January* 2017; *accepted* 28 *March* 2017)

Let $b_\ell(n)$ denote the number of ℓ-regular partitions of n. In 2012, using the theory of modular forms, Furcy and Penniston presented several infinite families of congruences modulo 3 for some values of ℓ. In particular, they showed that for $\alpha, n \geq 0$, $b_{25}\Big(3^{2\alpha+3}n + 2 \cdot 3^{2\alpha+2} - 1\Big) \equiv 0 \pmod{3}$. Most recently, congruences modulo powers of 5 for $c_5(n)$ was proved by Wang, where $c_N(n)$ counts the number of bipartitions (λ_1, λ_2) of n such that each part of λ_2 is divisible by N. In this paper, we prove some interesting Ramanujan-type congruences modulo powers of 5 for $b_{25}(n)$, $B_{25}(n)$, $c_{25}(n)$ and modulo powers of 7 for $c_{49}(n)$. For example, we prove that for $j \geq 1$,

$$c_{25}\left(5^{2j}n + \frac{11 \cdot 5^{2j} + 13}{12}\right) \equiv 0 \pmod{5^{j+1}},$$
$$c_{49}\left(7^{2j}n + \frac{11 \cdot 7^{2j} + 25}{12}\right) \equiv 0 \pmod{7^{j+1}}$$

and

$$b_{25}\Big(3^{2\alpha+3} \cdot 5^{2j-1} \cdot n + 2 \cdot 3^{2\alpha+2} \cdot 5^{2j-1} - 1\Big) \equiv 0 \pmod{3 \cdot 5^{2j-1}}.$$

Key words : Congruences; bipartitions; ℓ-regular partitions.

1. Introduction

Let $p(n)$ denote the number of unrestricted partitions of the positive integer n. In [7], Ramanujan presented the following generating functions for $p(5n+4)$ and $p(7n+5)$.

$$\sum_{n=0}^{\infty} p(5n+4)q^n = 5\frac{(q^5;q^5)_\infty^5}{(q;q)_\infty^6} \tag{1.1}$$

and

$$\sum_{n=0}^{\infty} p(7n+5)q^n = 7\frac{(q^7;q^7)_\infty^3}{(q;q)_\infty^4} + 49q\frac{(q^7;q^7)_\infty^7}{(q;q)_\infty^8}, \tag{1.2}$$

where

$$(a;q)_\infty := \prod_{k=0}^{\infty}(1-aq^k), \qquad |q|<1.$$

It follow from (1.1) and (1.2) that

$$p(5n+4) \equiv 0 \pmod 5 \quad \text{and} \quad p(7n+5) \equiv 0 \pmod 7.$$

Ramanujan [7] further conjectured that for any integer $j \geq 1$,

$$p\Big(5^j n + \delta_5(j)\Big) \equiv 0 \pmod{5^j}, \tag{1.3}$$

$$p\Big(7^j n + \delta_7(j)\Big) \equiv 0 \pmod{7^j}, \tag{1.4}$$

where $0 < \delta_\ell(j) < \ell^j$ satisfies the congruence $24\delta_\ell(j) \equiv 1 \pmod{\ell^j}$. Ramanujan gave a brief sketch of a proof of (1.3) in an unpublished manuscript [8]. Ramanujan's conjecture (1.4) was incorrect and a corrected version (1.4) was proved by Watson [10] on utilizing modular equation of seventh order. In fact he proved that if $j \geq 1$ and $n \geq 0$,

$$p\Big(7^{2j-1} n + \delta_7(2j-1)\Big) \equiv 0 \pmod{7^j} \tag{1.5}$$

and

$$p\Big(7^{2j} n + \delta_7(2j)\Big) \equiv 0 \pmod{7^{j+1}}. \tag{1.6}$$

By utilizing the classical identities of Euler and Jacobi, Hirschhorn and Hunt [5] and Garvan [4] gave a simple proof of (1.3) and (1.5)-(1.6), respectively.

Very Recently, Ranganatha [9] proved an analogue of (1.3) for the sequence $A_5(n)$:

$$A_5\Big(5^\alpha n + 5^\alpha - 2\Big) \equiv 0 \pmod{5^\alpha}, \qquad \alpha \geqq 1,$$

where

$$\sum_{n=0}^{\infty} A_5(n)q^n = \frac{(q^5;q^5)_\infty^{10}}{(q;q)_\infty^2}.$$

An ℓ-regular partition is a partition where none of its part is divisible by ℓ. The generating function for $b_\ell(n)$, which counts the number of ℓ-regular partitions of n is given by

$$\sum_{n=0}^{\infty} b_\ell(n)q^n = \frac{(q^\ell;q^\ell)_\infty}{(q;q)_\infty}. \tag{1.7}$$

Using the theory of modular forms, Furcy and Penniston [3] discovered several infinite families of congruences modulo 3 for $b_\ell(n)$ where $\ell \in \{4, 7, 13, 19, 25, 34, 37, 43, 49\}$. For example, they proved that for all $\alpha \geq 0$ and $n \geq 0$,

$$b_{25}\left(3^{2\alpha+3}n + 2 \cdot 3^{2\alpha+2} - 1\right) \equiv 0 \pmod 3. \tag{1.8}$$

A bipartition of n is an ordered pair of partitions (λ_1, λ_2) such that the sum of all the parts of λ_1 and λ_2 equals n. A bipartition (λ_1, λ_2) of n is said to be ℓ-regular bipartition of n if none of the parts of λ_1 and λ_2 are divisible by ℓ. The generating function for $B_\ell(n)$, the number of ℓ-regular bipartitions of n is given by

$$\sum_{n=0}^{\infty} B_\ell(n)q^n = \frac{(q^\ell;q^\ell)_\infty^2}{(q;q)_\infty^2}. \tag{1.9}$$

Recently, by following the strategy of Hirschhorn and Hunt [5], Wang [11, 12] established several infinite families of congruences modulo powers of 5 for $b_5(n)$ and $B_5(n)$. In recent days, a number of mathematicians studied the congruence properties for $b_\ell(n)$ and $B_\ell(n)$. For example, see the references listed in [11, 12].

Let $c_N(n)$ denote the number of bipartitions (λ_1, λ_2) of n such that each part of λ_2 is divisible by N. Then the generating function for $c_N(n)$ is given by

$$\sum_{n=0}^{\infty} c_N(n)q^n = \frac{1}{(q;q)_\infty(q^N;q^N)_\infty}. \tag{1.10}$$

Congruences for $c_2(n)$ modulo powers of 3 was proved by Chan [1] and modulo powers of 5 was proved by Chan and Toh [2] and Xiong [13]. Soon after, Liu and Zhang [6] discovered several infinite families of congruences modulo 3 for $c_5(n)$. Most recently, Wang [12] established several Ramanujan-type congruences modulo powers of 5 for $c_5(n)$.

In this paper, we prove Ramanujan-type congruences for $b_{25}(n)$, $B_{25}(n)$, $c_{25}(n)$ modulo powers of 5 and for $c_{49}(n)$ modulo powers of 7. The main results of this paper can be stated as follows:

Theorem 1.1 — *For $j \geq 1$ and $n \geq 0$, we have*

$$c_{25}\left(5^{2j-1}n + \frac{7 \cdot 5^{2j-1} + 13}{12}\right) \equiv 0 \pmod{5^j} \tag{1.11}$$

and

$$c_{25}\left(5^{2j}n + \frac{11 \cdot 5^{2j} + 13}{12}\right) \equiv 0 \pmod{5^{j+1}}. \tag{1.12}$$

Theorem 1.2 — *For $j \geq 1$ and $\alpha, n \geq 0$, we have*

$$b_{25}\left(5^j n + 5^j - 1\right) \equiv 0 \pmod{5^j}, \tag{1.13}$$

$$B_{25}\left(5^j n + 5^j - 2\right) \equiv 0 \pmod{5^j} \tag{1.14}$$

and

$$b_{25}\left(3^{2\alpha+3} \cdot 5^{2j-1} \cdot n + 2 \cdot 3^{2\alpha+2} \cdot 5^{2j-1} - 1\right) \equiv 0 \pmod{3 \cdot 5^{2j-1}}. \tag{1.15}$$

Theorem 1.3 — *For $j \geq 1$ and $n \geq 0$, we have*

$$c_{49}\left(7^{2j-1}n + \frac{5 \cdot 7^{2j-1} + 25}{12}\right) \equiv 0 \pmod{7^j} \tag{1.16}$$

and

$$c_{49}\left(7^{2j}n + \frac{11 \cdot 7^{2j} + 25}{12}\right) \equiv 0 \pmod{7^{j+1}}. \tag{1.17}$$

2. Preliminaries

In this section, we collect number of lemmas which are essential in the proofs of main results of this paper. Let $g(q) = \sum_{n=-\infty}^{\infty} g_n q^n$ in the annulus $0 < |q| < k$, where $k \geq 1$ or $+\infty$. In [5], Hirschhorn and Hunt introduced the "huffing" operator H modulo 5, that is,

$$Hg = \sum_{n=-\infty}^{\infty} g_{5n} q^{5n}$$

and proved that

$$H(G^j) = \sum_{k=1}^{\infty} m(j,k) u^{j-k}, \tag{2.1}$$

where $G := G(q) = \frac{(q^5;q^5)_\infty^6}{q^4(q;q)_\infty(q^{25};q^{25})_\infty^5}$, $u := u(q) = \frac{(q^5;q^5)_\infty^6}{q^5(q^{25};q^{25})_\infty^6}$ and the matrix $M = \{m(j,k)\}_{j,\,k\geq 1}$ is defined as follows:
The first five rows of M are

$$\begin{bmatrix} 5 & 0 & \cdot & \cdot & \cdot & \cdot & \dots \\ 10 & 125 & 0 & \cdot & \cdot & \cdot & \dots \\ 9 & 375 & 3125 & 0 & \cdot & \cdot & \dots \\ 4 & 550 & 12500 & 78125 & 0 & \cdot & \dots \\ 1 & 500 & 25000 & 390625 & 1953125 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

and for $j \geq 6$, $m(j,1) = 0$ and for $k \geq 2$,

$$\begin{aligned} m(j,k) =& 25m(j-1,k-1) + 25m(j-2,k-1) + 15m(j-3,k-1) + 5m(j-4,k-1) \\ &+ m(j-5,k-1). \end{aligned}$$

By induction, we can show that $m(j,k)$ vanishes for all j greater than $5k$ and less than k, that is,

$$m(j,k) = \ 0 \quad \text{for} \quad j \geq 5k+1 \quad \text{or} \quad j \leq k-1. \tag{2.2}$$

Lemma 2.1 — ([5, Lemma (2.9)]). For $j \geq 1$, we have

$$H(G^{6j}) = \sum_{k=1}^{\infty} m(6j, j+k) u^{5j-k}. \tag{2.3}$$

In view of (2.1) and (2.2), we have the following lemma.

Lemma 2.2 — For $j \geq 1$, we have

$$H(G^{6j+2}) = \sum_{k=1}^{\infty} m(6j+2, j+k) u^{5j+2-k}. \tag{2.4}$$

Let $\pi_5(n)$ (respectively $\pi_7(n)$) denote the power of 5 (respectively 7) in the unique prime factorisation of $n \geq 2$. For convention, we let $\pi_5(0) = \pi_7(0) = +\infty$. The following lemma was proved by Hirschhorn and Hunt [5].

Lemma 2.3 — ([5, Lemma (4.1)]). For $j, k \geq 1$, we have

$$\pi_5(m(j,k)) \geq \left[\frac{5k-j-1}{2}\right]. \tag{2.5}$$

The "huffing" operator H_0 modulo 7 is defined as follows.

$$H_0 g = \sum_{n=-\infty}^{\infty} g_{7n} q^{7n}.$$

In [4], Garvan discovered the following lemma.

Lemma 2.4 — ([4, Lemma (3.1)]). For $j \geqq 1$, we have

$$H_0(\xi^{-j}) = \sum_{k=1}^{\infty} m'(j,k) T^{-k}, \tag{2.6}$$

where

$$\xi = \frac{(q;q)_\infty}{q^2(q^{49};q^{49})_\infty}, \qquad T = \frac{(q^7;q^7)_\infty^4}{q^7(q^{49};q^{49})_\infty^4} \tag{2.7}$$

and the $m'(j,k)$ are as defined in [4, pp.318-319].

Lemma 2.5 — We have

$$m'(j,k) = \ 0 \quad \text{for} \quad j \geq 4k \quad \text{or} \quad k \geq 2j+1. \tag{2.8}$$

As this can be easily proved by induction, we omit the proof.

Lemma 2.6 — ([4, Lemma (3.4)]). For $j \geq 1$, we have

$$H_0(\xi^{4j}) = \sum_{k=1}^{\infty} m'(4j, j+k) T^{-j-k}. \tag{2.9}$$

The following lemma follows from (2.6) and (2.8).

Lemma 2.7 — For $j \geq 1$, we have

$$H_0(\xi^{-4j-2}) = \sum_{k=1}^{\infty} m'(4j+2, j+k) T^{-j-k}. \tag{2.10}$$

Lemma 2.8 — (4, [Lemma (5.1)]). For any $j, k \geq 1$, we have

$$\pi_7(m'(j,k)) \geq \left[\frac{7k-2j-1}{4}\right]. \tag{2.11}$$

3. Congruences Modulo Powers of 5 for $c_{25}(n)$

In this section, we prove the Theorem 1.1.

We define the matrix $M_1 = \{a(j,k)\}_{j,\, k\geq 1}$ as follows:

$$a(1,1) = 5 \quad \text{and} \quad a(1,k) = 0 \quad \text{for} \quad k \geq 2 \tag{3.1}$$

and

$$a(j+1,k) = \begin{cases} \sum_{i=1}^{\infty} a(j,i)m(6i, i+k), & \text{if } j \text{ is odd,} \\ \sum_{i=1}^{\infty} a(j,i)m(6i+2, i+k), & \text{if } j \text{ is even.} \end{cases} \tag{3.2}$$

***Theorem* 3.1** — *For $j \geq 1$, we have*

$$\sum_{n=0}^{\infty} c_{25}\left(5^{2j-1}n + \frac{7\cdot 5^{2j-1}+13}{12}\right)q^{n+1} = \frac{1}{(q^5;q^5)_\infty^2}\sum_{k=1}^{\infty} a(2j-1,k)\cdot u^{-5k}\cdot G^{6k} \tag{3.3}$$

and

$$\sum_{n=0}^{\infty} c_{25}\left(5^{2j}n + \frac{11\cdot 5^{2j}+13}{12}\right)q^{n-2} = \frac{(q^{25};q^{25})_\infty^4}{(q^5;q^5)_\infty^6}\sum_{k=1}^{\infty} a(2j,k)\cdot u^{-5k-1}\cdot G^{6k+2}. \tag{3.4}$$

Proof : Setting $N = 25$ in (1.10), we see that

$$\sum_{n=0}^{\infty} c_{25}(n)q^{n+1} = \frac{1}{(q^{25};q^{25})_\infty^2}\cdot u^{-1}\cdot G.$$

In view of operator H, we have

$$\sum_{n=0}^{\infty} c_{25}(5n+4)q^{5n+5} = H\left(\frac{u^{-1}\cdot G}{(q^{25};q^{25})_\infty^2}\right) = \frac{u^{-1}}{(q^{25};q^{25})_\infty^2}\cdot H(G) = 5\frac{q^5(q^{25};q^{25})_\infty^4}{(q^5;q^5)_\infty^6}. \tag{3.5}$$

Replacing q^5 by q in (3.5), we see that (3.3) holds for $j=1$. Suppose that (3.3) holds for some j.

Applying operator H on both sides of (3.3) and then using (2.3), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} c_{25}\left(5^{2j}n+\frac{11\cdot 5^{2j}+13}{12}\right)q^{5n+5} =& H\left(\frac{1}{(q^5;q^5)_\infty^2}\sum_{k=1}^{\infty} a(2j-1,k)\cdot u^{-5k}\cdot G^{6k}\right)\\ =&\frac{1}{(q^5;q^5)_\infty^2}\sum_{k=1}^{\infty} a(2j-1,k)\cdot u^{-5k}\cdot H(G^{6k})\\ =&\frac{1}{(q^5;q^5)_\infty^2}\sum_{k=1}^{\infty} a(2j-1,k)u^{-5k}\sum_{i=1}^{\infty} m(6k,i+k)u^{5k-i}\\ =&\frac{1}{(q^5;q^5)_\infty^2}\sum_{i=1}^{\infty}\sum_{k=1}^{\infty} a(2j-1,k)m(6k,i+k)\cdot u^{-i}\\ =&\frac{1}{(q^5;q^5)_\infty^2}\sum_{i=1}^{\infty} a(2j,i)\left(\frac{q^5(q^{25};q^{25})_\infty^6}{(q^5;q^5)_\infty^6}\right)^i. \qquad (3.6)\end{aligned}$$

Replacing q^5 by q in (3.6) and then utilizing the definitions of u and G, we can rewrite the resulting identity as

$$\sum_{n=0}^{\infty} c_{25}\left(5^{2j}n+\frac{11\cdot 5^{2j}+13}{12}\right)q^{n-2} = \frac{(q^{25};q^{25})_\infty^4}{(q^5;q^5)_\infty^6}\sum_{i=1}^{\infty} a(2j,i)\cdot u^{-5i-1}\cdot G^{6i+2}. \qquad (3.7)$$

Applying the operator H on both sides of (3.7) and then using (2.4), we deduce that

$$\begin{aligned}\sum_{n=0}^{\infty} c_{25}\left(5^{2j+1}n+\frac{7\cdot 5^{2j+1}+13}{12}\right)q^{5n} =& H\left(\frac{(q^{25};q^{25})_\infty^4}{(q^5;q^5)_\infty^6}\sum_{i=1}^{\infty} a(2j,i)\cdot u^{-5i-1}\cdot G^{6i+2}\right)\\ =&\frac{(q^{25};q^{25})_\infty^4}{(q^5;q^5)_\infty^6}\sum_{i=1}^{\infty} a(2j,i)\cdot u^{-5i-1}\cdot H(G^{6i+2})\\ =&\frac{(q^{25};q^{25})_\infty^4}{(q^5;q^5)_\infty^6}\sum_{i=1}^{\infty} a(2j,i)u^{-5i-1}\sum_{k=1}^{\infty} m(6i+2,k+i)u^{5i+2-k}\\ =&\frac{(q^{25};q^{25})_\infty^4}{(q^5;q^5)_\infty^6}\sum_{k=1}^{\infty}\sum_{i=1}^{\infty} a(2j,i)m(6i+2,k+i)u^{1-k}\\ =&\frac{(q^{25};q^{25})_\infty^4}{(q^5;q^5)_\infty^6}\sum_{k=1}^{\infty} a(2j+1,k)\left(\frac{q^5(q^{25};q^{25})_\infty^6}{(q^5;q^5)_\infty^6}\right)^{k-1}.\end{aligned}$$

Changing q^5 to q in the above equation and then employing the definitions of u and G in the resulting identity, we see that (3.3) holds for $j+1$. This completes the proof of (3.3). The proof of (3.4) follows from (3.3) and (3.7). □

Lemma 3.2 — For $j \geq 1$ and $k \geq 1$, we have

$$\pi_5(a(2j-1,k)) \geq j + \left[\frac{5k-5}{2}\right] \tag{3.8}$$

and

$$\pi_5(a(2j,k)) \geq j + 1 + \left[\frac{5k-5}{2}\right]. \tag{3.9}$$

PROOF : Since $\pi_5(a(1,1)) = 1$ and $\pi_5(a(1,k)) = +\infty$ for $k \geq 2$, we have (3.8) holds for $j = 1$. Suppose that (3.8) holds for some j. From (2.5) and (3.2), we have

$$\begin{aligned}\pi_5(a(2j,k)) =& \pi_5\Big(\sum_{i=1}^{\infty} a(2j-1,i)m(6i,i+k)\Big)\\ &\geq \min_{i\geq 1}\{\pi_5(a(2j-1,i)) + \pi_5(m(6i,i+k))\}\\ &\geq \min_{i\geq 1}\left\{j + \left[\frac{5i-5}{2}\right] + \left[\frac{5k-i-1}{2}\right]\right\}\\ &\geq j + \left[\frac{5k-2}{2}\right] \geq j+1+\left[\frac{5k-5}{2}\right].\end{aligned} \tag{3.10}$$

Again in the view of (2.5) and (3.2), we have

$$\begin{aligned}\pi_5(a(2j+1,k)) =& \pi_5\Big(\sum_{i=1}^{\infty} a(2j,i)m(6i+2,i+k)\Big)\\ &\geq \min_{i\geq 1}\{\pi_5(a(2j,i)) + \pi_5(m(6i+2,i+k))\}\\ &\geq \min_{i\geq 1}\left\{j + 1 + \left[\frac{5i-5}{2}\right] + \left[\frac{5k-i-3}{2}\right]\right\}\\ &\geq j + 1 + \left[\frac{5k-4}{2}\right] \geq j+1+\left[\frac{5k-5}{2}\right],\end{aligned}$$

which implies that (3.8) holds for $j+1$. Proof of (3.9) follows from (3.8) and (3.10). □

Lemma 3.3 — For $j \geq 1$, we have

$$a(2j-1,1) \equiv 2^{j-1} \cdot 5^j \cdot 11^{j-1} \pmod{5^{j+2}} \tag{3.11}$$

and

$$a(2j,1) \equiv 13 \cdot 2^{j-1} \cdot 5^{j+1} \cdot 11^{j-1} \pmod{5^{j+3}}. \tag{3.12}$$

PROOF : By (3.2), we have

$$a(2j+1,1) = a(2j,1)m(8,2) + \sum_{i=2}^{\infty} a(2j,i)m(6i+2,i+1), \tag{3.13}$$

$$a(2j,1) = a(2j-1,1)m(6,2) + \sum_{i=2}^{\infty} a(2j-1,i)m(6i,i+1). \tag{3.14}$$

In view of (2.5), (3.8) and (3.9), we find that

$$\pi_5\left(\sum_{i=2}^{\infty} a(2j,i)m(6i+2,i+1)\right) \geq \min_{i\geq 2}\left\{\pi_5(a(2j,i)) + \pi_5(m(6i+2,i+1))\right\}$$
$$\geq \min_{i\geq 2}\left\{j+1+\left[\frac{5i-5}{2}\right]+\left[\frac{2-i}{2}\right]\right\} \geq j+3 \tag{3.15}$$

and

$$\pi_5\left(\sum_{i=2}^{\infty} a(2j-1,i)m(6i,i+1)\right) \geq \min_{i\geq 2}\left\{\pi_5(a(2j-1,i)) + \pi_5(m(6i,i+1))\right\}$$
$$\geq \min_{i\geq 2}\left\{j+\left[\frac{5i-5}{2}\right]+\left[\frac{4-i}{2}\right]\right\} \geq j+3. \tag{3.16}$$

From (3.13) to (3.16), we deduce

$$a(2j+1,1) \equiv a(2j,1)m(8,2) \pmod{5^{j+3}}, \tag{3.17}$$

$$a(2j,1) \equiv a(2j-1,1)m(6,2) \pmod{5^{j+3}}. \tag{3.18}$$

From these two congruences, we have

$$a(2j+1,1) \equiv a(2j-1,1)m(6,2)m(8,2) = 13860a(2j-1,1)$$
$$\equiv 2\cdot 5\cdot 11\cdot a(2j-1,1) \pmod{5^{j+3}}$$

and since $a(1,1)=5$, by induction, we obtain (3.11). From (3.11) and (3.18), arrive at (3.12). □

Theorem 1.1 now follows from Theorem 3.1, (3.11) and (3.12).

4. CONGRUENCES MODULO POWERS OF 5 FOR $b_{25}(n)$ AND $B_{25}(n)$

To prove our Theorem 1.2, we need the following results whose proofs are analogues to that of Theorem 3.1 and thus proofs are omitted.

***Theorem* 4.1** — *For $j \geq 1$, we have*

$$\sum_{n=0}^{\infty} b_{25}\Big(5^j n + 5^j - 1\Big) q^{n+1} = \sum_{k=1}^{\infty} a(j,k) \cdot u^{-5k} \cdot G^{6k}, \tag{4.1}$$

where

$$a(1,1) = 5, \;\; a(1,k) = 0 \;\; \text{for} \;\; k \geq 2 \;\; \text{and} \;\; a(j+1,k) = \sum_{i=1}^{\infty} a(j,i) m(6i, i+k). \tag{4.2}$$

***Theorem* 4.2** — *For $j \geq 1$, we have*

$$\sum_{n=0}^{\infty} B_{25}\Big(5^j n + 5^j - 2\Big) q^{n+1} = \sum_{k=1}^{\infty} d(j,k) \cdot u^{-5k} \cdot G^{6k}, \tag{4.3}$$

where

$$d(1,1) = 10, \; d(1,2) = 125, \; d(1,k) = 0 \text{ for } k \geq 3 \text{ and } d(j+1,k) = \sum_{i=1}^{\infty} d(j,i) m(6i, i+k). \tag{4.4}$$

Lemma 4.3 — For $j \geq 1$ and $n \geq 0$, we have

$$b_{25}\Big(5^j n + 5^j - 1\Big) \equiv 5^j \cdot 13^{j-1} \cdot b_{25}(n) \pmod{5^{j+1}} \tag{4.5}$$

and

$$B_{25}\Big(5^j n + 5^j - 2\Big) \equiv 2 \cdot 5^j \cdot 13^{j-1} \cdot b_{25}(n) \pmod{5^{j+1}}. \tag{4.6}$$

PROOF : Using (2.5) and by induction on j, we can show that

$$\pi_5(a(j,k)) \geq j + \left[\frac{5k-5}{2}\right] \tag{4.7}$$

and

$$\pi_5(d(j,k)) \geq j + \left[\frac{5k-5}{2}\right], \qquad \text{for} \quad j,k \geq 1. \tag{4.8}$$

From (2.5) and (4.7), it follows that

$$\begin{aligned} \pi_5\left(\sum_{i=2}^{\infty} a(j,i) m(6i, i+1)\right) &\geq \min_{i\geq 2}\{\pi_5(a(j,k)) + \pi_5(m(6i,i+1))\} \\ &\geq \left\{ j + \left[\frac{5i-5}{2}\right] + \left[\frac{4-i}{2}\right] \right\} \\ &\geq j + 3. \end{aligned} \tag{4.9}$$

In view of (4.2) and (4.9), we have

$$a(j+1,1) \equiv 315a(j,1) \pmod{5^{j+3}}. \tag{4.10}$$

Because $a(1,1)=5$, from (4.10) and by induction on j, we find that

$$a(j,1) \equiv 5^j \cdot 13^{j-1} \pmod{5^{j+1}}.$$

Applying the above congruence in (4.1), we obtain

$$\sum_{n=0}^{\infty} b_{25}\Big(5^j n + 5^j - 1\Big) q^n \equiv 5^j \cdot 13^{j-1} \cdot u^{-5} \cdot G^6 = 5^j \cdot 13^{j-1} \cdot \frac{(q^5;q^5)_\infty^6}{(q;q)_\infty^6} \pmod{5^{j+1}}. \tag{4.11}$$

Since $(q;q)_\infty^5 \equiv (q^5;q^5)_\infty \pmod{5}$, we have

$$\sum_{n=0}^{\infty} b_{25}\Big(5^j n + 5^j - 1\Big) q^n \equiv 5^j \cdot 13^{j-1} \cdot \frac{(q^{25};q^{25})_\infty}{(q;q)_\infty} = 5^j \cdot 13^{j-1} \cdot \sum_{n=0}^{\infty} b_{25}(n) q^n \pmod{5^{j+1}}, \tag{4.12}$$

which implies (4.5). Proof of (4.6) follows in a similar way, except that in the place of (4.2) and (4.7), (4.4) and (4.8) are used, respectively and the numbers $d(j,1)$ satisfies the congruence $d(j,1) \equiv 2 \cdot 5^j \cdot 13^{j-1} \pmod{5^{j+1}}$. □

Now, congruences (1.13) and (1.14) follow from (4.5) and (4.6), respectively. Changing j to $2j-1$ and n to $3^{2\alpha+3} \cdot n + 3^{2\alpha+2} - 1$ in (1.13), we deduce that

$$b_{25}\Big(5^{2j-1} \cdot \Big(3^{2\alpha+3} \cdot n + 3^{2\alpha+2} - 1\Big) + 5^{2j-1} - 1\Big) \equiv 0 \pmod{5^{2j-1}},$$

which is same as

$$b_{25}\left(3^{2\alpha+3} \cdot \Big(5^{2j-1} \cdot n + \frac{5^{2j-1}-2}{3}\Big) + 2 \cdot 3^{2\alpha+2} - 1\right) \equiv 0 \pmod{5^{2j-1}}. \tag{4.13}$$

If we replace n by $5^{2j-1} \cdot n + \frac{5^{2j-1}-2}{3}$ in (1.8), we obtain

$$b_{25}\left(3^{2\alpha+3} \cdot \Big(5^{2j-1} \cdot n + \frac{5^{2j-1}-2}{3}\Big) + 2 \cdot 3^{2\alpha+2} - 1\right) \equiv 0 \pmod{3}. \tag{4.14}$$

Finally, congruence (1.15) follows from (4.13) and (4.14). This completes the proof of Theorem 1.2.

5. Congruences Modulo Powers of 7 for $c_{49}(n)$

In this section, we prove Theorem 1.3. We first define the matrix $M_2 = \{g(j,k)\}_{j,\,k\geq 1}$ as follows:

$$g(1,1)=7, \quad g(1,2)=49 \quad \text{and} \quad g(1,k)=0 \quad \text{for} \quad k\geq 3 \tag{5.1}$$

and

$$g(j+1,k)=\begin{cases}\sum_{i=1}^{\infty} g(j,i)m'(4i,i+k), & \text{if } j \text{ is odd,}\\ \sum_{i=1}^{\infty} g(j,i)m'(4i+2,i+k), & \text{if } j \text{ is even.}\end{cases} \tag{5.2}$$

In view of (2.8), we note that the summation in (5.2) is indeed finite.

***Theorem* 5.1** — *For $j \in \mathbb{N}$, we have*

$$\sum_{n=0}^{\infty} c_{49}\left(7^{2j-1}n+\frac{5\cdot 7^{2j-1}+25}{12}\right)q^{n+1}=\frac{1}{(q^7;q^7)_\infty^2}\sum_{k=1}^{\infty} g(2j-1,k)\cdot T^k\cdot \xi^{-4k} \tag{5.3}$$

and

$$\sum_{n=0}^{\infty} c_{49}\left(7^{2j}n+\frac{11\cdot 7^{2j}+25}{12}\right)q^{n+5}=\frac{1}{(q^{49};q^{49})_\infty^2}\sum_{k=1}^{\infty} g(2j,k)\cdot T^k\cdot \xi^{-(4k+2)}. \tag{5.4}$$

Proof : We proceed by induction on j. Setting $N=49$ in (1.10), we have

$$\sum_{n=0}^{\infty} c_{49}(n)q^{n+2}=\frac{1}{(q^{49};q^{49})_\infty^2}\cdot \xi^{-1}. \tag{5.5}$$

Extracting the terms involving q^{7n} on both sides of (5.5), we obtain

$$\sum_{n=0}^{\infty} c_{49}(7n+5)q^{7n+7}=\frac{1}{(q^{49};q^{49})_\infty^2}\left(7\frac{q^7(q^{49};q^{49})_\infty^4}{(q^7;q^7)_\infty^4}+49\frac{q^{14}(q^{49};q^{49})_\infty^8}{(q^7;q^7)_\infty^8}\right). \tag{5.6}$$

Cancelling q^7 on both sides of (5.6) and then replacing q^7 by q, we see that (5.3) holds for $j=1$. Suppose that (5.3) holds for some j. Applying the operator H_0 on both sides of (5.3) and then using

(2.9), we obtain

$$\sum_{n=0}^{\infty} c_{49}\left(7^{2j}n+\frac{11\cdot 7^{2j}+25}{12}\right)q^{7n+7} = \frac{1}{(q^7,q^7)_\infty^2}\sum_{k=1}^{\infty} g(2j-1,k)\cdot H_0(T^k\cdot\xi^{-4k})$$
$$= \frac{1}{(q^7,q^7)_\infty^2}\sum_{k=1}^{\infty} g(2j-1,k)\cdot H_0(T^k\cdot\xi^{-4k})$$
$$= \frac{1}{(q^7,q^7)_\infty^2}\sum_{k=1}^{\infty} g(2j-1,k)\cdot T^k\cdot\sum_{i=1}^{\infty} m'(4k,i+k)T^{-i-k}$$
$$= \frac{1}{(q^7,q^7)_\infty^2}\sum_{i=1}^{\infty}\sum_{k=1}^{\infty} g(2j-1,k)m'(4k,i+k)T^{-i}$$
$$= \frac{1}{(q^7,q^7)_\infty^2}\sum_{i=1}^{\infty} g(2j,i)\left(\frac{q^7(q^{49};q^{49})_\infty^4}{(q^7;q^7)_\infty^4}\right)^i. \tag{5.7}$$

Replacing q^7 by q in the above equation and then employing the definitions of ξ and T in the resulting identity, we find that

$$\sum_{n=0}^{\infty} c_{49}\left(7^{2j}n+\frac{11\cdot 7^{2j}+25}{12}\right)q^{n+5} = \frac{1}{(q^{49};q^{49})_\infty^2}\sum_{i=1}^{\infty} g(2j,i)\cdot T^i\cdot\xi^{-(4i+2)}. \tag{5.8}$$

Again applying the operator H_0 on both sides of (5.8) and using (2.10), we deduce that

$$\sum_{n=0}^{\infty} c_{49}\left(7^{2j+1}n+\frac{5\cdot 7^{2j}+25}{12}\right)q^{7n+7} = \frac{1}{(q^{49};q^{49})_\infty^2}\sum_{i=1}^{\infty} g(2j,i)\cdot H_0(T^i\cdot\xi^{-(4i+2)})$$
$$= \frac{1}{(q^{49};q^{49})_\infty^2}\sum_{i=1}^{\infty} g(2j,i)\cdot T^i\cdot H_0(\xi^{-(4i+2)})$$
$$= \frac{1}{(q^{49};q^{49})_\infty^2}\sum_{i=1}^{\infty} g(2j,i)\cdot T^i\cdot\sum_{k=1}^{\infty} m'(4i+2,k+i)T^{-i-k}$$
$$= \frac{1}{(q^{49};q^{49})_\infty^2}\sum_{k=1}^{\infty}\sum_{i=1}^{\infty} g(2j,i)\cdot m'(4k+2,j+k)T^{-k}$$
$$= \frac{1}{(q^{49};q^{49})_\infty^2}\sum_{k=1}^{\infty} g(2j+1,k)\left(\frac{q^7(q^{49};q^{49})_\infty^4}{(q^7;q^7)_\infty^4}\right)^k. \tag{5.9}$$

Changing q^7 to q in the above equation and then applying (2.7), we obtain

$$\sum_{n=0}^{\infty} c_{49}\left(7^{2j+1}n+\frac{5\cdot 7^{2j+1}+25}{12}\right)q^{n+1} = \frac{1}{(q^7;q^7)_\infty^2}\sum_{k=1}^{\infty} g(2j+1,k)\cdot T^k\cdot\xi^{-(4k+2)}.$$

This implies that (5.3) holds for $j+1$. This completes the proof of (5.3). The proof of (5.4) follows from (5.3) and (5.8). □

Lemma 5.2 — For all $j, k \in \mathbb{N}$, we have

$$\pi_7(g(2j-1,k)) \geq j + \left[\frac{7k-7}{4}\right] \tag{5.10}$$

and

$$\pi_7(g(2j,k)) \geq j + 1 + \left[\frac{7k-7}{4}\right]. \tag{5.11}$$

PROOF : Since $\pi_7(g(1,1)) = 1$, $\pi_7(g(1,2)) = 2$, and $\pi_7(g(1,k)) = +\infty$ for $k \geq 3$, we have (5.10) holds for $j = 1$. Suppose that (5.10) holds for some j. From (2.11) and (5.2), we have

$$\begin{aligned}
\pi_7(g(2j,k)) =&\pi_7\Big(\sum_{i=1}^{\infty} g(2j-1,i)m'(4i,i+k)\Big)\\
&\geq \min_{i\geq 1}\{\pi_7(g(2j-1,i)) + \pi_7(m'(4i,i+k))\}\\
&\geq \min_{i\geq 1}\left\{j + \left[\frac{7i-7}{4}\right] + \left[\frac{7k-i-1}{4}\right]\right\}\\
&\geq j + \left[\frac{7k-2}{4}\right] \geq j + 1 + \left[\frac{7k-7}{4}\right].
\end{aligned} \tag{5.12}$$

Again in the view of (2.11) and (5.2), we have

$$\begin{aligned}
\pi_7(g(2j+1,k)) =&\pi_7\Big(\sum_{i=1}^{\infty} g(2j,i)m'(4i+2,i+k)\Big)\\
&\geq \min_{i\geq 1}\{\pi_7(g(2j,i)) + \pi_7(m'(4i+2,i+k))\}\\
&\geq \min_{i\geq 1}\left\{j + 1 + \left[\frac{7i-7}{4}\right] + \left[\frac{7k-i-5}{4}\right]\right\}\\
&\geq j + 1 + \left[\frac{7k-6}{4}\right] \geq j + 1 + \left[\frac{7k-7}{4}\right],
\end{aligned} \tag{5.13}$$

which implies that (5.10) holds for $j+1$. This completes the proof of (5.10). Proof of (5.11) follows from (5.10) and (5.12).

Lemma 5.3 — For $j \geq 1$, we have

$$g(2j-1,1) \equiv 2^{j-1} \cdot 7^j \pmod{7^{j+1}} \tag{5.14}$$

and

$$g(2j,1) \equiv 2^{j-1}\cdot 5\cdot 7^{j+1} \pmod{7^{j+2}}. \tag{5.15}$$

PROOF : By (5.2), we have

$$\pi_7\left(\sum_{i=2}^{\infty} g(2j-1,i)m'(4i,i+1)\right) \geq \min_{i\geq 2}\left\{\pi_7(g(2j-1,i))+\pi_7(m'(4i,i+1))\right\}$$
$$\geq \min_{i\geq 2}\left\{j+\left[\frac{7i-7}{4}\right]+\left[\frac{6-i}{4}\right]\right\} \geq j+2 \tag{5.16}$$

and

$$\pi_7\left(\sum_{i=2}^{\infty} g(2j,i)m'(4i+2,i+1)\right) \geq \min_{i\geq 2}\left\{\pi_7(g(2j,i))+\pi_7(m'(4i+2,i+1))\right\}$$
$$\geq \min_{i\geq 2}\left\{j+1+\left[\frac{7i-7}{4}\right]+\left[\frac{2-i}{4}\right]\right\} \geq j+2. \tag{5.17}$$

Using (5.16) and (5.17) in (5.2), we see that

$$g(2j,1) \equiv g(2j-1,1)m'(4,2) \pmod{7^{j+2}}, \tag{5.18}$$
$$g(2j+1,1) \equiv g(2j,1)m'(6,2) \pmod{7^{j+2}}. \tag{5.19}$$

Therefore,

$$g(2j+1,1) \equiv m'(4,2)m'(6,2)g(2j-1,1) = 15498g(2j-1,1)$$
$$\equiv 14g(2j-1,1) \pmod{7^{j+2}}. \tag{5.20}$$

From (5.20) and by induction on j, we deduce (5.14). In view of (5.14) and (5.18), we have (5.15). □

Theorem 1.3 follows from (5.1), (5.14) and (5.15).

ACKNOWLEDGEMENT

The author would like to thank Prof. Chadrashekar Adiga for his advice and guidance. The author also thankful to UGC, New Delhi for awarding UGC-BSR Fellowship, under which this work has been done.

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