Indian J. Pure Appl. Math., **49**(1): 51-69, March 2018

DOI: 10.1007/s13226-018-0255-x

ON THE MÖBIUS FUNCTION OF A POINTED GRADED LATTICE

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(*Received* 7 *November* 2015; *accepted* 22 *May* 2017)

In this paper, we compute the Möbius function of pointed integer partition and pointed ordered set partition using topological and analytic methods. We show that the associated order complex is a wedge of spheres and compute the associated reduced homology group for each subposet. In addition, we compute the Möbius function of pointed graded lattice and use our method to compute the Möbius function of pointed direct sum decomposition of vector spaces.

Key words : Partition; pointed partition; Möbius function; poset map; simplicial complex; cone; contractible; homology group; simplicial map.

1. Definitions and Preliminaries

Let n be a non-negative integer. A multiset $u = \{u_1, u_2, \cdots, u_r\}$ of integers is an integer partition of n provided that either $n = 0$ and $u = \{0\}$ or $n \geq 1$ and

$$(a)\ \sum_{i=1}^{r} u_i \ = n$$

$$(b)\ u_i \geq\ 1,\ \text{for all}\ i = 1, 2, \ldots r.$$

We use multiplicities and write $\{u_1^{i_1} \ldots u_e^{i_e}\}$ where $u_1 > \ldots > u_e$ and $i_1 \ldots i_e \geq 1$ as a superscript of each u_i in their decreasing order to give the multiset u [12]. Thus, for instance, $\{6, 4, 4, 3, 2, 2, 1, 1\}$=$\{6, 4^2, 3, 2^2, 1^2\}$.

Definition 1.1 — A pair $\{u,\ \underline{m}\} = \{u_1,\ u_2,\ \ldots,\ u_r,\ \underline{m}\}$ is called a pointed integer partition of n if $u =\ \{u_1,\ u_2,\ \ldots,\ u_r\}$ is an integer partition of $n - m$ where m is a non-negative integer $\leq n$.

The integer m is called the pointed part and we write $u_1u_2\dots u_r\underline{m}$ to denote a pointed integer partition $\{u, \underline{m}\}$.

Let $I_n^\bullet$ denote the set of all pointed integer partitions of the non-negative integer n. On this set define an order relation by the two cover relations:

(1) $\{u_1,\ \cdots,\ u_i, \cdots, u_j, \cdots, u_r,\ \underline{m}\}\ \leq\ \{u_1,\ \dots,\ \hat{u_i}, \cdots, \hat{u_j},\ \cdots,\ u_r,$
$u_i+u_j,\ \underline{m}\}$, and

(2) $\{u_1,\ u_2,\ \cdots,\ u_i, \cdots, u_r,\ \underline{m}\}\ \leq\ \{u_1,\ u_2,\ \cdots,\ \widehat{u_i},\ \dots,\ u_r,\ \underline{u_i+\ m}\}.$

Here $\hat{u_i}$ and $\hat{u_j}$ means that the corresponding elements are omitted.

In [12] the poset of pointed integer partition $I_n^\bullet$ of n was introduced and used for studying pointed set partition lattices. Even though $I_n^\bullet$ appears as an interval in I_n, the global structure of the poset $I_n^\bullet$ was not clarified. In this paper we will be able to clarify this.

For instance if $n = 3$, $I_3^\bullet = \{111\underline{0},\ 12\underline{0},\ 11\underline{1},\ 2\underline{1},\ 1\underline{2},\ 3\underline{0},\ \underline{3}\}$. It can be easily shown that $(I_n^\bullet,\ \leq)$ is a poset whose Hasse diagram is given in Figure 1, and this poset is graded, has unique minimal and maximal elements $\hat{0} = 111\underline{0}$ and $\hat{1} = \underline{3}$, respectively, and is not lattice for $n \geq 3$.

The Möbius function μ is a function that assigns to each interval in a poset P an integer and its recursive formulation [13] is given by:

$$\mu(x,y) = \begin{cases} 1 & \text{for all } x = y, \\ -\sum_{z:x\leq z<y} \mu(x,z) & \text{for all } x < y. \end{cases}$$

Example 1.2 : Let us consider $I_3^\bullet = \{111\underline{0}, 12\underline{0}, 11\underline{1}, 3\underline{0}, 2\underline{1}, 1\underline{2}, \underline{3}\}$ and its Hasse diagram shown in Figure 1; we compute the Möbius function for $I_3^\bullet$ recursively as follow.

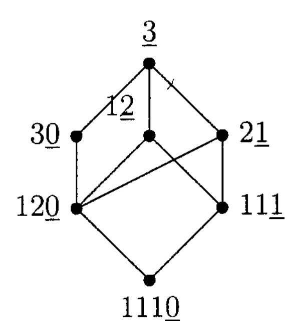


Figure 1 : Hasse diagram of $\bar{I}_3^\bullet$

$$\mu(\hat{0},\hat{0}) = 1$$

$$\mu(\hat{0},12\underline{0}) = -1$$

$$\mu(\hat{0},11\underline{1}) = -1$$

$$\begin{aligned}\mu(1\underline{2}) &= -(\mu(\hat{0},\hat{0}) + \mu(\hat{0},12\underline{0}) + \mu(\hat{0},11\underline{1})) \\ &= -(1 + (-1) + (-1)) = 1 \\ &= \mu(2\underline{1})\end{aligned}$$

Similarly,

$$\begin{aligned}\mu(3\underline{0}) &= -(\mu(\hat{0},\hat{0}) + \mu(\hat{0},12\underline{0})) \\ &= -(1 + (-1)) = 0\end{aligned}$$

Therefore,

$$\begin{aligned}\mu(I_3^\bullet) &= \mu(\hat{1}) \\ \mu(\hat{1}) &= -(\mu(\hat{0},\hat{0}) + \mu(1\underline{2}) + \mu(2\underline{1}) + \mu(3\underline{0}) + \mu(\hat{0},12\underline{0}) + \mu(\hat{0},11\underline{1})) \\ &= -(1 + 1 + 1 + 0 - 1 - 1) = -1 = (-1)^3\end{aligned}$$

Hence, $\mu(I_3^\bullet) = (-1)^3$.

For $n = 4$, the Hasse diagram is shown in Figure 2 and we see that

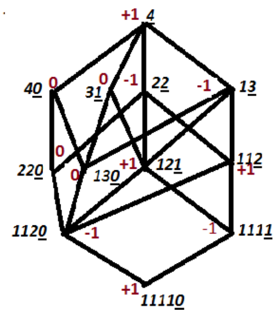
$$\begin{aligned}\mu(I_4^\bullet) &= \mu(\hat{1}) \\ \mu(\hat{1}) &= -(\mu(\hat{0}) + \mu(112\underline{0}) + \mu(111\underline{1}) + \mu(12\underline{1}) + \mu(11\underline{2}) + \mu(2\underline{2}) + \mu(1\underline{3})) \\ &= -(1 - 1 - 1 + 1 + 1 - 1 - 1) \\ &= 1 = (-1)^4\end{aligned}$$

Hence, $\mu(I_4^\bullet) = (-1)^4$.

Our main result is:

***Theorem* 1.3** — *For* $n \geq 1$ *we have,* $\mu(I_n^\bullet) = (-1)^n$.

In the next section, we review some known poset constructions and use these constructions to prove our claim that $\mu(I_n^\bullet) = (-1)^n$ for all n.



$\mu(I_4^{\bullet}) = \; 1 \; = (-1)^4.$

Figure 2 : Hasse diagram of $\bar{I}_4^{\bullet}$

2. New Posets from Old

If P and Q are posets on disjoint sets, then the ordinal sum of P and Q is the poset $P \oplus Q$ on the union $P \cup Q$ such that
$s \leq t$ in $P \oplus Q$ if

(a) $s, t \in P$ and $s \leq t$ in P, or

(b) $s, t \in Q$ and $s \leq t$ in Q, or

(c) $s \in P$ and $t \in Q$.

Now we define some simple examples of posets.

- We denote the trivial poset consisting of a single element by $\mathbf{1} = \bullet$.
- The disjoint union of n copies of P is denoted by nP. An n-element antichain (a subset A of a poset P such that any two distinct elements of A are incomparable) is isomorphic to $n\mathbf{1}$ and the n-element chain is the ordinal sum $\underbrace{\mathbf{1} \oplus \mathbf{1} \oplus \ldots \oplus \mathbf{1}}_{n \text{ times}}$ of n trivial posets.
- We denote the adjoin of a poset P and $\{\hat{0}, \hat{1}\}$ by $\hat{P}$, and similarly $\widehat{P \oplus Q} = P \oplus Q \cup \{\hat{0}, \hat{1}\}$.

Example 2.1 : Consider $P = Q = \mathbf{11}$, $P \oplus Q = \mathbf{11} \oplus \mathbf{11}$.
The Hasse diagram of $\widehat{P \oplus Q}$ is shown in Figure 5.

Proposition 2.2 — Let P and Q be finite posets. Then

$$\mu_{\widehat{P \oplus Q}}(\hat{0}, \hat{1}) = -\mu_{\hat{P}}(\hat{0}, \hat{1}) \cdot \mu_{\hat{Q}}(\hat{0}, \hat{1}).$$

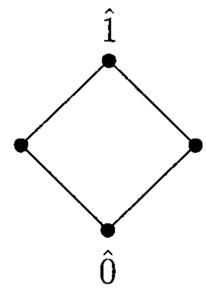


Figure 3 : Hasse diagram of $\hat{P}$ for $P = \mathbf{11}$

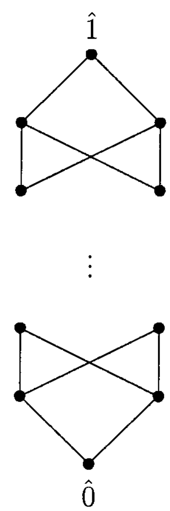


Figure 4 : Hasse diagram of $n\hat{P}$ for $P = \mathbf{11}$

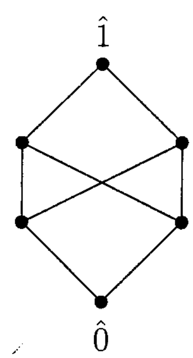


Figure 5 : Hasse diagram of $\widehat{P \oplus Q}$ for $P = Q = \mathbf{11}$

PROOF : We use induction on the cardinality of Q.

If $Q=\emptyset$ then $\hat{Q}=\{\hat{0},\hat{1}\}$ and $\mu_{\hat{Q}}(\hat{0},\hat{1})=-1$. It also holds that $P\oplus Q=P$. Hence,

$$\mu_{\widehat{P\oplus Q}}(\hat{0},\hat{1})=\mu_{\hat{P}}(\hat{0},\hat{1})=-\mu_{\hat{P}}(\hat{0},\hat{1})\cdot\mu_{\hat{Q}}(\hat{0},\hat{1}).$$

Let $q\in Q$ thus we can have that, $(P\oplus Q)_{\leq q}=P\oplus Q_{\leq q}$, where $Q_{\leq q}$ denotes the set $\{x\in Q|x\leq q\}$. $(P\oplus Q)_{\leq q}$ is defined the same way.

Now assume $|Q|\geq 1$ and that the hypothesis has been verified for all posets of smaller cardinality.

$$\begin{aligned}
\mu_{\widehat{P\oplus Q}}(\hat{0},\hat{1}) &= -\sum_{\hat{0}\leq x<\hat{1}}\mu_{\widehat{P\oplus Q}}(\hat{0},x)\\
&= -\mu_{\widehat{P\oplus Q}}(\hat{0},\hat{0})-\sum_{p\in P}\mu_{\widehat{P\oplus Q}}(\hat{0},p)-\sum_{q\in Q}\mu_{\widehat{P\oplus Q}}(\hat{0},q)\\
&= -\sum_{p\in P}\mu_{\widehat{P\oplus Q}}(\hat{0},p)-\sum_{q\in Q}\mu_{\widehat{P\oplus Q_{<q}}}(\hat{0},q)-1\\
&= -\sum_{p\in P}\mu_{\widehat{P\oplus Q}}(\hat{0},p)+\sum_{q\in Q}\mu_{P}(\hat{0},\hat{1})\cdot\mu_{\hat{Q}}(\hat{0},q)-1\\
&= -(\sum_{p\in P\cup\{\hat{0}\}}\mu_{\widehat{P\oplus Q}}(\hat{0},p))+\mu_{P}(\hat{0},\hat{1})\cdot\sum_{q\in Q}\mu_{\hat{Q}}(\hat{0},q)\\
&= \mu_{P}(\hat{0},\hat{1})(1+\sum_{q\in Q}\mu_{\hat{Q}}(\hat{0},q))\\
&= -\mu_{\hat{P}}(\hat{0},\hat{1})\cdot\mu_{\hat{Q}}(\hat{0},\hat{1}).
\end{aligned}$$
□

Corollary 2.3 — Let $P=\mathbf{11}$.Then $\mu_{\widehat{P\oplus\cdots\oplus P}}(\hat{0},\hat{1})=(-1)^{n}$.

PROOF : Use induction on n (the number of times we take the ordinal sum). □

A mapping $t\to\bar{t}$ on a poset P is called a closure operator (or closure) if for all $s,t\in P$,

- $t\leq\bar{t}$
- $s\leq t\Leftrightarrow\bar{s}\leq\bar{t}$
- $\bar{\bar{t}}=\bar{t}=t$

An element t of P is closed if $t=\bar{t}$. The set of closed elements of P is denoted $\bar{P}$, called the quotient of P relative to the closure. If $s\leq t\ \text{ in }\ P$, then define $\bar{s}\leq\bar{t}\ \text{ in }\ \bar{P}$. Thus it is easy to see that $\bar{P}$ is a poset.

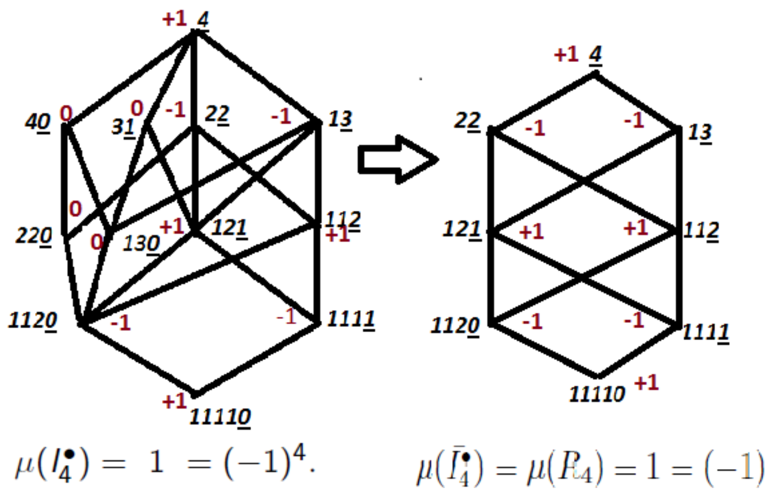


Figure 6 : Hasse digram of $I_4^{\bullet}$ and $R\,4$

Proposition 2.4 (Stanley [13]) — Let P be poset with closure $t \to \bar{t}$ and quotient $\bar{P}$. Then for all $s,\ t \in P$,

$$\sum_{u \in P, \bar{u}=\bar{\bar{t}}} \mu(s,u) = \begin{cases} \mu_{\overline{P}}(\bar{s},\bar{t}) & \text{if } s = \bar{s}, \\ 0 & \text{if } s < \bar{s}. \end{cases}$$

Example 2.5 : For example, you can consider $\mu(I_4^{\bullet}) = 1 = (-1)^4$ and $\mu(R_4) = 1 = (-1)^4$, where $R_4 = \{1 \cdots 12\underline{i} : 0 \leq i \leq 3\} \cup \{1 \cdots 1\underline{i} : 0 \leq i \leq 3\}$, whose Hasse diagram is shown in Figure 6.

Proposition 2.6 — Let P be a poset with $\hat{0}$ *and* $\hat{1}$ and $x \in P \setminus \hat{1}$ such that $\mu_P(\hat{0}, x) = 0$. Then $\mu_P(\hat{0},\hat{1}) = \mu_{P\setminus\{x\}}(\hat{0},\hat{1})$.

PROOF : We use induction on the cardinality l of the longest chain between x and $\hat{1}$.

Assume $l = 0$, that is, x is coatom (an element that $\hat{1}$ covers)

$$\begin{aligned} \mu(\hat{0},\hat{1}) &= -\sum_{\hat{0} \leq y < \hat{1}} \mu_P(\hat{0}, y) \\ &= -\sum_{\hat{0} \leq y < \hat{1}, y \neq x} \mu_P(\hat{0}, y) - \mu_P(\hat{0}, x) \\ &= -\sum_{\hat{0} \leq y < \hat{1}, y \neq x} \mu_{P\setminus\{x\}}(\hat{0}, y) = \mu_{P\setminus\{x\}}(\hat{0},\hat{1}). \end{aligned}$$

Now assume $l > 0$, then

$$\begin{aligned}
\mu_P(\hat{0},\hat{1}) &= -\sum_{\hat{0}\le y<\hat{1}} \mu_P(\hat{0},y)\\
&= -\sum_{\hat{0}\le y<\hat{1},x \not< y} \mu_P(\hat{0},y) - \sum_{x<y<\hat{1}} \mu_P(\hat{0},y)\\
&= -\sum_{\hat{0}\le y<\hat{1},x \not< y} \mu_P(\hat{0},y) - \sum_{x<y<\hat{1}} \mu_{P\setminus\{x\}}(\hat{0},y)\\
&= -\sum_{\hat{0}\le y<\hat{1},x \not< y} \mu_{P\setminus\{x\}}(\hat{0},y)\\
&= \mu_{P\setminus\{x\}}(\hat{0},\hat{1}).
\end{aligned}$$

Therefore, $\mu_P(\hat{0},\hat{1}) = \mu_{P\setminus\{x\}}(\hat{0},\hat{1})$. □

3. Proof of Theorem 1.3

***Theorem* 3.1** — *Let* $R_n = \{1\cdots 12\underline{i} : 0 \le i \le n-1\} \cup \{1\cdots 1\underline{i} : 0 \le i \le n-1\} \subseteq I_n^\bullet$. *Then* $\mu_{R_n}(\hat{0},\hat{1}) = (-1)^n$.

***Theorem* 3.2** — *Let* $\lambda \in \overline{I_n^\bullet} = \{\mu \in I_n^\bullet : \mu \notin R_n\}$. *Then* $\mu_{\overline{I_n^\bullet}}(\hat{0},\lambda) = 0$.

For the proof of Theorem 3.1, observe that R_n is isomorphic to the ordinal sum of $n-1$ copies of $P =$ ***11***. Thus by applying Proposition 2.2 we get the result.

To prove Theorem 3.2, first we observe the following:

Proposition 3.3 — If $\lambda \in \bar{I}_n^\bullet$, then $\{\eta \in R_n : \eta \le \lambda\}$ has a unique maximal element.

For instance, consider
$I_4^\bullet = \{1111\underline{0}, 112\underline{0}, 13\underline{0}, 4\underline{0}, 22\underline{0}, 111\underline{1}, 12\underline{1}, 11\underline{2}, 3\underline{1}, 2\underline{2}, 1\underline{3}, \underline{4}\}$.

It is easy to check that,

$$\begin{aligned}
\bar{I}_4^\bullet &= \{22\underline{0}, 13\underline{0}, 4\underline{0}, 3\underline{1}\}.\\
R_4 &= \{1111\underline{0}, 112\underline{0}, 111\underline{1}, 12\underline{1}, 11\underline{2}, 2\underline{2}, 1\underline{3}, \underline{4}\}.
\end{aligned}$$

Now, for $13\underline{0} \in I_4^\bullet$,

$$\{\eta \in R_4 : \eta \le 13\underline{0}\} = \{1111\underline{0}, 112\underline{0}\} = \{\eta \in R_4 : \eta \le 22\underline{0}\} = \{\eta \in R_4 : \eta \le 4\underline{0}\}$$

For this set the unique maximal element is $112\underline{0} \in R_4$.

Consider $3\underline{1} \in \bar{I}_4^\bullet$, so that $\{\eta \in R_4 : \eta \leq 3\underline{1}\} = \{1111\underline{0}, 112\underline{0}, 111\underline{1}, 12\underline{1}\}$.

The unique maximal element for this set is $12\underline{1}$.

PROOF OF PROPOSITION 3.3 :

Case 1 : The pointed part of $\lambda \in \bar{I}_n^\bullet$ is 0.

In this case the set $\{\eta \in R_n : \eta \leq \lambda\}$ contains only two elements, namely, $11\cdots 1\underline{0}$, and $11\cdots 12\underline{0}$, of which $\eta = 11\cdots 12\underline{0}$ is the maximal element.

Case 2 : The pointed part of $\lambda \in \bar{I}_n^\bullet$ is nonzero.

To determine the unique maximal element of $\{\eta \in R_n : \eta \leq \lambda\}$, note that

$$\begin{aligned} R_n &= \{1\cdots 12\underline{i} : 0 \leq i \leq n-1\} \cup \{1\cdots 1\underline{i} : 0 \leq i \leq n\} \\ &= \{1\cdots 1\underline{0}, \cdots 1\underline{n-1}, \underline{n}, 1\cdots 12\underline{0}, \cdots, 2\underline{n-2}\}. \end{aligned}$$

Let $\lambda = \lambda_1 \ldots \lambda_n \underline{m} \notin R_n$, where $1 \leq m \leq n-3$ and $n \geq 4$. Then, there exists i such that

$$\lambda_i \geq 3 \text{ or there exists, } 1 \leq i < j \leq n \text{ and } \lambda_i, \lambda_j \geq 2. \tag{3.1}$$

If $11\cdots 1\underline{i} < \lambda_1 \ldots \lambda_n \underline{m}$ and $11\cdots 2\underline{i} < \lambda_1 \ldots \lambda_n \underline{m}$, then $i \leq m$.

Now, $\{11\cdots 1\underline{i}, 11\cdots 12\underline{i} : 0 < i \leq m\} \subseteq R_n$ has $11\cdots 2\underline{m}$ as its unique maximal element. Using equation (3.1) above, $11\cdots 12\underline{m} < \lambda_1 \ldots \lambda_n \underline{m}$, and this shows that $11\cdots 12\underline{m}$ is the unique maximal element.

Therefore, if $\lambda \in \bar{I}_n^\bullet$, then $\{\eta \in R_n : \eta \leq \lambda\}$ has a unique maximal element. This completes the proof of the proposition.

Thus, we can use induction on the rank of λ to prove that $\mu_{\bar{I}_n^\bullet}(\hat{0}, \lambda) = 0$.

If λ is of minimal rank, then the set $\{\eta \in I_n^\bullet : \eta \leq \lambda\} = \{\eta \in R_n : \eta \leq \lambda\} \cup \{\lambda\}$, and so $\mu_{\bar{I}_n^\bullet}(\hat{0}, \lambda) = 0$.

Assume λ is not of minimal rank. By induction hypothesis for all $\eta \in \bar{I}_n^\bullet$ and $\eta < \lambda$ we have $\mu_{I_n^\bullet}(\hat{0}, \eta) = 0$. Then $\mu_{I_n^\bullet}(\hat{0}, \lambda) = \mu_{R_n \cup \{\lambda\}}(\hat{0}, \lambda)$, but $\{\eta \leq \lambda : \eta \in R_n\}$ has unique maximal element η_{Top} (the pointed partition at the top). □

We now use topological methods to prove Theorem 1.2. For the notation of poset topology and background concept we follow papers by Wachs [11], Jonsson [7], Jung [8], Björner [1] and Björner and Wachs [2].

Definition 3.4 — An abstract simplicial complex $\triangle$ on finite vertex set V is a nonempty collection of subsets of V satisfying:

(i) If $v \in V$, then $\{v\} \in \triangle$, and

(ii) If $G \in \triangle \text{ and } F \subseteq G, \text{ then } F \in \triangle.$

The elements of $\triangle$ are called **faces** (or simplices) of $\triangle$ and the maximal faces are called **facets**. We say that a face F has dimension d and write $\dim F = d$, when $d = |F| - 1$. Faces of dimension d are referred to as $d - faces$. The dimension of $\triangle$, denoted by $\dim\ \triangle$ is defined to be

$$\dim \triangle = \max_{F \in \triangle}(\dim F). \tag{3.2}$$

We also allow the $(-1)-$dimensional complex $\{\emptyset\}$, which we refer to as the empty simplicial complex. We say $\triangle$ is **pure** if all facets have the same dimension.

A $d-$dimensional geometric simplex in $\mathbb{R}^n$ is defined to be the convex hull of $d + 1$ affinely independent points or vertices in $\mathbb{R}^n$. The convex hull of any subset of the vertices is called a face of the geometric simplex. A geometric simplicial complex Γ in $\mathbb{R}^n$ is a nonempty collection of geometric simplices in $\mathbb{R}^n$ such that

(i) Every face of a simplex in Γ is in Γ,

(ii) The intersection of any two simplices of Γ is a face of both of them.

From a geometric simplicial complex Γ, one gets an abstract simplicial complex $\triangle(\Gamma)$ by letting the faces of $\triangle(\Gamma)$ be the vertex sets of the simplices of Γ. Every abstract simplicial complex $\triangle$ can be obtained in this way, i.e., given a geometric simplicial complex Γ we can have that $\triangle(\Gamma) = \triangle$. Although Γ is not unique, the underlying topological space, obtained by taking the union of the simplices of Γ under the usual topology on $\mathbb{R}^n$, is unique up to homeomorphism [4]. We refer to this space as the geometric realization of $\triangle$ (methods for turning abstract simplical complex into topological space) and denote it by $\|\triangle\|$. We will usually drop the $\|\ \|$ and let $\triangle$ denote an abstract simplicial complex as well as its geometric realization.

Let $P = (P, \leq)$ be a poset. A chain (or totally ordered set or linearly ordered set) is a poset in which any two elements are comparable. A subset C of P is called a chain if C is a chain when regarded as a subposet of P. The length $\ell(C)$ of a finite chain is defined by

$$\ell(C) = |C| - 1, \text{ where } |C| \text{ is the cardinality of } C.$$

The length (or rank) of a finite poset P is defined by

$$\ell(P) := \max\{\ell(C) :\ C \text{ is a chain of } P\}.$$

The length of an interval $[x, y]$ of P is denoted by $\ell(x, y)$. A chain $x_0 < x_1 < \ldots < x_n$ in poset P is maximal or saturated if x_{i+1} covers x_i for $1 \leq i \leq n$ and x_0 is minimum and x_n is maximum. If every maximal chain of P has the same length n, then we say that P is **graded of rank** n (or P is pure). In this case there is a unique rank function $\rho : P \longrightarrow \{0, 1, 2, \ldots, n\}$ such that $\rho(x) = 0$ if x is a minimal element of P, and $\rho(y) = \rho(x) + 1$ if y covers x in P.

To every poset P, we can associate an abstract simplicial complex $\triangle(P)$ called the **order complex** of P. The vertices of $\triangle(P)$ are the elements of P and the $d-$faces of $\triangle(P)$ are the $d-$ chains (i.e., totally ordered subsets) of P. Note that the order complex of the empty poset is the empty simplicial complex $\{\emptyset\}$. Thus, for this simplicial complex we can always have a topological space called geometric realization. Conversely to every simplicial complex $\triangle$, one can associate a poset $P(\triangle)$ called the **face poset** of $\triangle$, which is defined to be the poset of nonempty faces ordered by inclusion. The face lattice $L(\triangle)$ is $P(\triangle)$ with a smallest element $\hat{0}$ and a largest element $\hat{1}$ attached.

If Δ is finite, then let f_i denote the number of $i-$dimensional faces of Δ. Define the **reduced Euler characteristic** $\widetilde{\chi}(\Delta)$ by

$$\widetilde{\chi}(\Delta) = -f_{-1} + f_0 - f_1 + \cdots$$

Note that $f_{-1} = 1$ unless $\Delta = \emptyset$. The simplicial complexes $\Delta_1 = \emptyset$ and $\Delta_2 = \{\emptyset\}$ are not the same; in particular, $\widetilde{\chi}(\Delta_1) = 0$ and $\widetilde{\chi}(\Delta_2) = -1$. Recall that the **ordinary Euler characteristic** $\chi(\Delta)$ is defined as $f_0 - f_1 + f_2 - \cdots$. Hence

$$\widetilde{\chi}(\Delta) = \chi(\Delta) - 1.$$

Recall that the **Möbius function** μ is a function which assigns to each interval in a poset P an integer and its recursive formulation is given by:

$$\mu(x, y) = \begin{cases} 1 & \text{for all } x = y, \\ -\sum_{z:x \leq z < y} \mu(x, z) & \text{for all } x < y. \end{cases}$$

In Figure 7 the values of $\mu(\hat{0}, x)$ are shown for each element x of a given poset.

Philip Hall's formula to calculate the Möbius function μ on a poset P is given by

$$\mu(x, y) = \sum_{x = z_0 < z_1 < \cdots < z_i = y} (-1)^i$$

for all $x < y$ in P [13] and its connection to the reduced Euler characteristic $\widetilde{\chi}(\Delta)$ is given in Theorem 3.5.

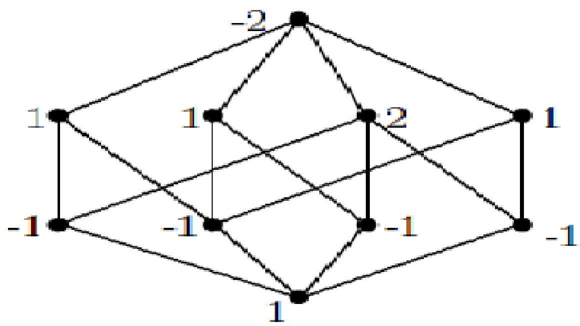


Figure 7 : The values of $\mu(\hat{0}, x)$ for each element x of a given poset

Proposition 3.5 — [**Philip Hall Theorem**] For any poset P,

$$\mu(\hat{P}) = \widetilde{\chi}(\Delta(P)).$$

As stated in [9], there is a standard result of algebraic topology [5] that the Euler characteristic of a complex can be computed from its homology groups. Thus

$$\mu(P) = \sum_{n} (-1)^n \; rank \; \widetilde{H_n}(\Delta(P)),$$

where $\widetilde{H_n}(\Delta(P))$ represents reduced simplicial homology with integer coefficients. This relationship between Möbius numbers and homology is one of the main reasons for studying the geometric realizations of posets.

Let Δ_0 be a simplicial complex, and let $'a'$ be a vertex not in Δ_0. The cone,

$$Cone(\Delta_0) = \; Cone_a(\Delta_0)$$

over Δ_0 is the simplicial complex obtained from Δ_0 by adding $\sigma \cup \{a\}$ for each $\sigma \in \Delta_0$. Equivalently, Δ is a cone with apex a if $\sigma \cup \{a\}$ is a face of Δ whenever σ is a face of Δ. In particular, if a poset P has some element which is comparable to every other element, then $\Delta(P)$ is a cone. It is well known that any realization of a cone is contractible (complex which is homotopy equivalent to a point). Since a homotopy equivalence induces homology isomorphism, any contractible space is acyclic (has trivial homology groups). It follows that if P is acyclic, $\mu(P) = 0$.

If we start with a simplicial complex $\triangle$, take its face poset $P(\triangle)$, and then take the order complex $\triangle(P(\triangle))$, we get a simplicial complex known as the (first) ***barycentric subdivision*** of $\triangle$. The geometric realizations are always homeomorphic, i.e., $\triangle \cong \triangle(P(\triangle))$. Therefore, from a topological point of view simplicial complexes and posets can be considered to be essentially equivalent notions. Thus, when we attribute a topological property to a poset, we mean that the geometric realization of the order complex of the poset has that property. For instance, if we say that the poset P is homeomorphic to the $n-$sphere $\mathbb{S}^n$ we mean that $\|\triangle(P)\|$ is homeomorphic to $\mathbb{S}^n$. Then for a poset P we have that $\mu_P(\widehat{0},\widehat{1}) = \widetilde{\chi}(P(\triangle))$. Let A and B be two finite sets such that $A \cap B = \emptyset$. Let Δ be a family of subsets of A, and let Γ be a family of subsets of B. The **join** of Δ and Γ is the family $\Delta * \Gamma = \{\delta \cup \gamma : \delta \in \Delta, \gamma \in \Gamma\}$. Let Δ be a simplicial complex, and let $\sigma \in \Delta$. The deletion of Δ with respect to σ is the subcomplex $del_\triangle(\sigma) = \{\tau \in \Delta : \tau \cap \sigma = \emptyset\}$. The **link** of Δ with respect to σ is the subcomplex $lk_\triangle(\sigma) = \{\tau \in \Delta : \tau \cap \sigma = \emptyset, \text{ and } \tau \cup \sigma \in \Delta\}$, and star $st_\triangle(\sigma) = \{\tau \in \Delta : \tau \cup \sigma \in \Delta\}$. Clearly, $del_\triangle(\sigma) \cap st_\triangle(\sigma) = lk_\triangle(\sigma)$ and $\sigma * lk_\triangle(\sigma) = st_\triangle(\sigma)$.

An (order-preserving) poset map between two posets $P = (A, \leq)$ and $Q = (B, \leq)$ is a function $f : A \longrightarrow B$ such that $f(x) \leq_Q f(y)$ whenever $x \leq_P y$. We will often write $f : P \longrightarrow Q$. We obtain a poset structure on a simplicial complex $\triangle$ of sets by defining $\sigma \leq \tau$ whenever $\sigma \subseteq \tau$.

A simplicial map f from a simplicial complex $\triangle$ to a poset P sends vertices of $\triangle$ to elements of P and faces of the simplicial complex $\triangle$ to chains of P.

For two continuous maps f and g of the space X into the space Y, if there exists a continuous map $F : X \times [0,1] \longrightarrow Y$ such that $F(x,0) = f(x)$ and $F(x,1) = g(x)$ for each x, then we call the map F a homotopy between two maps f and g. When a homotopy F between f and g exists, we write $f \approxeq g$ and say that f is homotopic to g.

The spaces X and Y are of the same homotopy type (or are called **homotopy equivalent**) if there exist mappings $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$ such that the map $f \circ g : X \longrightarrow X$ is homotopic to the identity map of the space X and the map $g \circ f : Y \longrightarrow Y$ is homotopic to the identity map of the space Y. Each of the maps f and g is called a homotopy equivalence. If there exists a homotopy equivalence between two spaces X and Y then we say the spaces X and Y are homotopy equivalent, and two spaces X and Y are said to have the same **homotopy type**. A space with the homotopy type of a point is **contractible**. Equivalently, X is contractible if and only if identity map of the space X is homotopic to a constant map. A homotopy from identity map of the space X to a constant is called a **contraction** of X. If two spaces X and Y have the same homotopy type then they have the same homology groups. Hence the Euler characteristic χ, which is defined as the sum $\chi(\triangle) = f_0 - f_1 + f_2 - \ldots$, where f_i is the number of $i-$faces of the complex $\triangle$, is an invariant under homotopy.

***Theorem* 3.6 — [*Order Homotopy Theorem, Quillen 1978*].** *Let $f, g : \Delta \longrightarrow P$ be simplicial* *maps from a simplicial complex Δ to a poset P. If $f(x) \leq g(x)$ for every x in ground set of Δ, then f and g are homotopic.*

Corollary 3.7 — Let $f : P \longrightarrow P$ be an order-preserving map such that $f(x) \geq x$ for all $x \in P$. Then f induces homotopy equivalence between P and $f(P)$.

Theorem* 3.8** — [Quillen's Fiber Lemma***]. *Let f be a simplicial map from the simplicial complex Γ to the poset P such that for all elements x in the poset P, the subcomplex $\Delta(f^{-1}(P_{\geq x}))$ is contractible. Then the order complex $\Delta(P)$ and the simplicial complex Γ are homotopy equivalent.*

4. Pointed Integer Partition and Pointed Set Partition

In this section we determine $\widetilde{H}_i(\Delta(I_n^\bullet))$ and show that $\Delta(O_n^\bullet)$ is contractible and $\mu_{O_n^\bullet}(\hat{0}, \hat{1}) = 0$.

***Theorem* 4.1** — *Let $I_n^\bullet$ denote the set of all pointed integer partitions of a non-negative integer n . Then the order complex of $I_n^\bullet$, $(\Delta(I_n^\bullet))$, has the homotopy type of sphere of dimension $n-1$. In particular the $M\ddot{o}bius$ function of the poset $I_n^\bullet$ is given by $(-1)^n$.*

Proof : Let $f : I_n^\bullet \longrightarrow I_n^\bullet$ be an order-preserving map such that $f(\lambda) = \max\{\tau \in R_n : \tau \leq \lambda\}$ and $R_n = \{1 \cdots 12\underline{i} : 0 \leq i \leq n-1\} \cup \{1 \cdots 1\underline{i} : 0 \leq i \leq n-1\} \subseteq I_n^\bullet$. By Proposition 3.3 $\max\{\tau \in R_n : \tau \leq \lambda\}$ has a unique maximal element, and hence f is well defined.

Thus, $f(\lambda) \leq \lambda$ for all $\lambda \in I_n^\bullet$ and $f(I_n^\bullet) = R_n$. Therefore, by Corollary 3.7 $I_n^\bullet \simeq R_n$. Now it follows that $\Delta(I_n^\bullet) \simeq \Delta(R_n)$. Using the fact that $\Delta(\bullet\bullet) * \Delta(\bullet\bullet) = S^0 * S^0 = S^1$ and proof of Theorem 3.1 $\Delta(I_n^\bullet)$ has homotopy type of a wedge of spheres consisting of n copies of the $0-sphere$. Thus, $\Delta(I_n^\bullet) = S^{n-1}$.

Let $\widetilde{\chi}(\Delta(I_n^\bullet))$ be the reduced Euler characteristic of the order complex of $I_n^\bullet$. Thus by Hall's theorem we have that $\mu_{I_n^\bullet}(\hat{0}, \hat{1}) = \widetilde{\chi}(\Delta(I_n^\bullet))$. Recall that, given two simplical complex Δ and Γ, we have that $\mid \Delta \mid \; \simeq \; \mid \Gamma \mid \Longrightarrow \widetilde{\chi}(\Delta) = \widetilde{\chi}(\Gamma)$.

Hence, $\mu_{I_n^\bullet}(\hat{0}, \hat{1}) = \widetilde{\chi}(\Delta(I_n^\bullet)) = (-1)^{(n-1)-1} = (-1)^n$. □

***Theorem* 4.2** — *Let $I_n^\bullet$ denote the set of all pointed integer partitions of non-negative integer n. Then $\widetilde{H}_i(\Delta(I_n^\bullet)) = Z$, if $i = n-1$, and 0 otherwise.*

Proof : Recall that, given two simplical complex Δ and Γ, we have that $\mid \Delta \mid \simeq \mid \Gamma \mid \Longrightarrow \widetilde{\chi}(\Delta) = \widetilde{\chi}(\Gamma) \Longrightarrow \widetilde{H}_i(\Delta, Z) = \widetilde{H}_i(\Gamma, Z)$. Thus, $\widetilde{H}_i(S^{n-1}, Z) = Z$, if $i = n-1$, and 0 otherwise. □

A pointed ordered set partition π of set $[n]$ is a list of blocks $(B_1, \ldots, \underline{B_m})$ where the blocks are subset of the set $[n]$ satisfying:

(1) All blocks except possibly the last block are non-empty.

(2) $B_i \cap B_j = \emptyset$ for $1 \leq i < j \leq m$.

(3) $\cup_{i=1}^{m} B_i = [n]$.

We can associate a pointed ordered set partition π to the pointed composition $c = (|B_1|, \ldots, |\underline{B_m}|)$. Where $|B_i|$ for each i denotes the number of parts of the block i. Let $O_n^{\bullet}$ be the collection of the all pointed ordered set partitions of $[n]$. Partially order the elements of $O_n^{\bullet}$ by the cover relations:

(1). $(B_1, \ldots, B_{j-1}, B_j, \ldots, B_{m-1}, \underline{B_m}) < (B_1, \ldots, B_{j-1}, B_j \cup B_{j+1}, B_{j+2}, \ldots, B_{m-1}, \underline{B_m})$, and

(2). $(B_1, \ldots, B_{m-2}, B_{m-1}, \underline{B_m}) < (B_1, \ldots, B_{m-2}, \underline{B_{m-1} \cup \underline{B_m}})$.

Clearly, $(O_n^{\bullet}, \leq)$ is a poset.

***Theorem* 4.3** — *Let $O_n^{\bullet}$ be the collection of the all pointed ordered set partitions of $[n]$. Then $\Delta(O_n^{\bullet})$ is contractible(complex which is homotopy equivalent to a point).*

PROOF : Let $\overline{O_n^{\bullet}}$ consists of the pointed ordered partition whose pointed part is $\emptyset$. $\overline{O_n^{\bullet}}$ is a subposet of $O_n^{\bullet}$. Then the simplicial complex $\Delta(\overline{O_n^{\bullet}})$ is a cone with apex $[n]/\emptyset$. Hence contractible.

Note : Let $\triangle_0$ be a simplicial complex, and let $'a'$ be a vertex not in $\triangle_0$. The cone, $Cone(\triangle_0)$= $Cone_a(\triangle_0)$ over $\triangle_0$ with apex $'a'$ is the simplicial complex obtained from $\triangle_0$ by adding $\sigma \cup \{a\}$ for each $\sigma \in \triangle_0$. Equivalently, $\triangle$ is a cone with apex a if $\sigma \cup \{a\}$ is a face of $\triangle$ whenever σ is a face of $\triangle$.

Let $f : O_n^{\bullet} \longrightarrow O_n^{\bullet}$ be an order-preserving map such that

$$f(\pi) = \max_{\pi \in O_n^{\bullet}} \{B_1 \mid \ldots B_r \mid \emptyset : B_1 \mid \ldots B_r \mid \emptyset \leq \pi\}.$$

Since for any $B = B_1 \mid \ldots B_r \mid \emptyset$ there is $\pi = \pi_1 \mid \ldots \pi_r \mid \underline{\tau}$, which is obtained when the last block of B is merged with the pointed block we get τ and it becomes the new pointed block of π, f is well defined.

For $\pi \leq \sigma \implies \max\{B : B \leq \pi\} \leq \max\{B' : B' \leq \sigma\} \implies f(\pi) \leq f(\sigma)$. Thus, f is a poset map. By definition of f, we have that $f(\pi) \leq \pi$ for all $\pi \in O_n^{\bullet}$ and $f(O_n^{\bullet}) = \overline{O_n^{\bullet}}$.

Therefore, by Corollary 3.7 $O_n^{\bullet} \simeq \overline{O_n^{\bullet}}$. Hence, $O_n^{\bullet}$ is contractible. From which we get that $\widetilde{H_n}(\Delta(O_n^{\bullet})) = 0$ for all n. It follows that it is $Z - acyclic$. Thus, $\widetilde{\chi}(\Delta(O_n^{\bullet})) = 0$. Therefore, by Philip Hall's theorem $\mu_{O_n^{\bullet}}(\hat{0}, \hat{1}) = 0$. □

5. MÖBIUS FUNCTION FOR POINTED GRADED LATTICES AND VECTOR SPACE DECOMPOSITIONS

A graded lattice L is a lattice with the minimal element $\widehat{0}$, maximal element $\widehat{1}$, and a rank function $r_k : L \longrightarrow \{0,1,2,\dots,n\}$ satisfying $r_k(x) = 0$ if x is a minimal element of L, and $r_k(y) = r_k(x)+1$ if y covers x in L. The rank of a graded lattice L is defined as the rank of the maximal element $\widehat{1}$, that is, $r_k(\widehat{1})$.

Let L be graded lattice with rank function r_k , minimal element $\widehat{0}$ and maximal element $\widehat{1}$. Consider the set $D(L^\bullet)$ with minimal element denoted by $\widehat{0}_{D(L^\bullet)}$, maximal element denoted by $\widehat{1}$ and $r_k(\widehat{1}) = r_k(A_1)+\cdots+r_k(A_1)$.

$$D(L^\bullet) = \left\{ \; A_1 \mid A_2 \mid \cdots \mid A_r \mid \underline{A_{r+1}} \quad : A_1 \vee \cdots \vee A_{r+1} = \widehat{1}, \; \right\} \cup \left\{ \widehat{0}_{D(L^\bullet)} \right\},$$

with the order relation $\leq$ defined by:

(1). $A_1 \mid \cdots \mid A_r \mid \underline{A_{r+1}} \leq A_1 \mid A_2 \mid \dots \mid \hat{A_j} \mid \hat{A_{j+1}} \mid \cdots \mid A_r \mid A_j \vee A_{j+1} \mid \underline{A_{r+1}}$, and

(2). $A_1 \mid A_2 \mid \cdots \mid A_r \mid \underline{A_{r+1}} \; \leq A_1 \mid A_2 \mid \cdots \mid A_{r-1} \mid \underline{A_r \vee A_{r+1}}$.

$\hat{A}_i$ and $\hat{A}_j$ in (1) above means that the corresponding elements are omitted and the bar $\mid$ as in partitions stands for the defined order relation with respect to a given ground set. Clearly, $(D(L^\bullet), \leq)$ is a graded lattice. However, it is not a lattice in general. For instance, if we take a partition $L = \prod_n^r$ of all partitions on n elements where the cardinality of each block is divisible by r. Clearly this lattice does not have a minimal element if $r > 1$. However, it is join-semi lattice. Thus $\prod_n^r$ joined by an artificial new minimal element $\widehat{0}$, is a lattice. Hence, although $\prod_n^r$ lacks a minimal element, it is still called a lattice.

Example 5.1 : $L = B_n = \{A \subseteq [n]\}, \;\; r_k A = |A|$. Clearly, $\widehat{1} = [n], \widehat{0} = \emptyset$ and $D(B_n^\bullet) = \{A_1 \mid \cdots \mid A_r \mid \underline{A_{r+1}} : \; A_1 \cup \cdots \cup A_{r+1} = [n]$ and $n = |A_1| + \cdots + |A_{r+1}|\} \cup \emptyset$. Then $D(B_n^\bullet)$ is pointed ordered graded lattice.

Example 5.2 : $L_n = \{u : \; u$ is a subspace of $\mathbb{F}_q^n,$ and $r_k(u) = dim(u)\}$.

$$D(L_n^\bullet) = \left\{ \; A_1 \mid \cdots \mid A_r \mid \underline{A_{r+1}} \quad : A_1 \oplus \cdots \oplus A_{r+1} = \mathbb{F}_q^n, \; \right\} \cup \left\{ \widehat{0}_{D(L_n^\bullet)} \right\}.$$

Then $D(L_n^\bullet)$ is the poset of pointed vector space decompositions of $\mathbb{F}_q^n$.

***Theorem* 5.3** — *Let L be a graded lattice. Then $D(L^\bullet)$ is contractible, that is, it is a simplicial complex which is homotopy equivalent to a point. In particular, $\widetilde{H_n}(\Delta(D(L^\bullet)) = 0$ for all n and $\mu(D(L^\bullet)) = 0$.*

$U = U_1 \oplus \cdots \oplus U_l$ of V_n the inequality $V \leq U$ holds if and only if for all, $1 \leq j \leq k$, there exists an i, $1 \leq i \leq l$ such that $V_j \leq U_i$. Since in general there does not exist a direct sum decomposition which refines all the others, we always need to add a least element $\widehat{0}$ to $D(V_n)$ in order to make $D(V_n)$ into a bounded poset with top element $\widehat{1} = V_n$.

Let $V_n^\bullet$ be the set of all pointed direct sum decompositions of V_n.

(1) $U_1 \oplus U_2 \oplus \cdots \oplus U_j \oplus U_{j+1} \oplus \cdots \oplus U_r \oplus \underline{U_{r+1}} \leq U_1 \oplus U_2 \oplus \ldots \oplus \hat{U_j} \oplus \hat{U_{j+1}} \oplus \cdots \oplus U_r \oplus < U_j, U_{j+1} > \oplus \underline{U_{r+1}}$.

(2) $U_1 \oplus U_2 \oplus \cdots \oplus U_i \oplus \cdots \oplus U_j \oplus \cdots \oplus U_r \oplus \underline{U_{r+1}} \leq U_1 \oplus U_2 \oplus \cdots \oplus U_i \oplus \ldots \oplus U_{r-1} \oplus < \underline{U_r + U_{r+1}} >$.

Here $\hat{U_i}$ and $\hat{U_j}$ means that the corresponding elements are omitted and $< U_j, U_{j+1} >$ denotes the vector space generated by U_j and U_{j+1}.

Corollary 5.4 — The Möbius number $\mu(V_n^\bullet)$ of the poset $D(V_n^\bullet)$ is zero.

PROOF : This follows from Theorem 5.3 since $V_n^\bullet$ is a graded lattice. □

ACKNOWLEDGEMENT

The first author would like to thank Professor Volkmar Welker(Germany) for hosting him twice at his institution and providing him guidance while he worked on this project. He also acknowledges the support he received from Addis Ababa University, ISP (International Science Program) and DAAD (German Academic Exchange Service) to conduct this research. The second author acknowledges assigned release time for research provided by William Paterson University of New Jersey to work on this project.

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