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THREE SOLUTIONS OF A KIRCHHOFF TYPE PROBLEM INVOLVING CRITICAL GROWTH AND NEAR RESONANCE

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The purpose of this work is to study a Kirchhoff type equation involving critical and singular nonlinearities. Based on variational methods, we obtain the existence of three nontrivial solutions for this problem.

Key words : Kirchhoff type equation; critical exponents; near resonance; variational methods.

1. Introduction and Main Result

In this article we will focus on the multiplicity of nontrivial solutions to the following Kirchhoff type equation

$$\begin{cases} -\left(1+b\displaystyle\int_\Omega |\nabla u|^2\right)\Delta u = (\alpha_1+\lambda)k(x)u + (u^+)^5, & \text{in } \Omega, \\ u=0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^3$, $b>0$, λ is positive parameter, and the function k is assumed to be divergent at the origin. Recently, Chaudhuri and Ramaswamy [1] studied the following eigenvalue problem

$$\begin{cases} -\Delta u = \alpha k(x) u & \text{in } \Omega, \\ u=0, & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

$\alpha>0$ and k is a positive measurable function. Denote

$$\mathcal{K} = \left\{ k:\Omega\to\mathbb{R}^+ |\lim_{|x|\to 0}|x|^2k(x)=0 \text{ with } \mathrm{k}\in \mathrm{L}^\infty_{\mathrm{loc}}(\Omega\backslash\{0\})\right\}$$

and

$$\mathcal{K}_\beta = \left\{ k \in \mathcal{K} : 0 < \lim_{|x|\to 0} |x|^\beta k(x) < +\infty \right\},$$

where $0 \leq \beta < 2$. They proved, among other results, that

(i) Assume $k \in \mathcal{K}$. Then the embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega, kdx)$ is compact.

(ii) Assume $k \in \mathcal{K}$. Then problem (1.2) has a sequence of eigenvalues

$$0 < \alpha_1 < \alpha_2 \leq \cdots \leq \alpha_n \leq \cdots \to +\infty.$$

Moreover, the first eigenvalue is characterized by

$$\alpha_1 := \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2}{\int_\Omega k(x)|u|^2}.$$

(iii) Assume $k \in \mathcal{K}_\beta$ with $0 \leq \beta < 2$. Then the embedding $H_0^1(\Omega) \hookrightarrow L^p(\Omega, kdx)$ is continuous if $2 \leq p \leq 2^*_\beta$ and compact if $2 \leq p < 2^*_\beta$, where $2^*_\beta = \frac{2(N-\beta)}{N-2}$.

Let S be the best Sobolev constant, i.e.,

$$S := \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2}{\left(\int_\Omega |u|^6\right)^{\frac{1}{3}}}. \tag{1.3}$$

In recent years there has been a lot of research on the existence of solutions for the problem:

$$\begin{cases} -\left(a + b\displaystyle\int_\Omega |\nabla u|^2\right)\Delta u = f(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

we refer the reader to [2-7] and the reference therein. Critical boundary problems of the type (1.1) have been studied extensively in recent years (see [8-17]) and the references therein). In particular, Naimen in [16] investigated the following Kirchhoff type equation:

$$\begin{cases} -\left(1 + b\displaystyle\int_\Omega |\nabla u|^2\right)\Delta u = \lambda u + u^5, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

In the case when $\lambda > \lambda_1$ (here $\lambda_1 > 0$ is the principal eigenvalue of $-\Delta$ on the open ball), based on the variational methods, the author proved the above problem admits a positive solution provided $b > B_4(\lambda)$ (where $B_4(\lambda)$ is a positive constant). Thus, inspired by [16], if $b > 0$ is arbitrary positive constant, an interesting question now is whether multiplicity of solutions can be established for (1.1)? We now give a positive answer.

Now our main result can be described as follows:

***Theorem* 1.1** — *Assume $b > 0$, $k \in \mathcal{K}_\beta$ with $2 - \frac{1}{3} < \beta < 2$, there exists $\lambda_* > 0$ such that $0 < \lambda < \lambda_*$, then problem (1.1) has at least two positive solutions, and one ground state negative solution.*

Throughout this paper, we make use of the following notation:

On $H_0^1(\Omega)$, we denote the norm $\|u\|^2 = \int_\Omega |\nabla u|^2$.

The norm in $L^p(\Omega)$ is denoted by $|u|_p^p = \int_\Omega |u|^p$.
$u^+(x) = \max\{u(x), 0\}$, $u^-(x) = \max\{-u(x), 0\}$.
$c, c_0, c_1, c_2, \ldots$ denote various positive constants, which may vary from line to line.

2. PROOF OF THEOREM 1.1

2.1 *Existence of a positive solution of (1.1).*

Consider the energy functional $I^+ : H_0^1(\Omega) \to \mathbb{R}$ given by

$$I^+(u) = \frac{1}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{\alpha_1 + \lambda}{2}\int_\Omega k(x)(u^+)^2 - \frac{1}{6}\int_\Omega (u^+)^6.$$

In general, a function u is called a weak solution of (1.1) if $u \in H_0^1(\Omega)$ and for all $\varphi \in H_0^1(\Omega)$ it holds

$$(1 + b\|u\|^2)\int_\Omega (\nabla u, \nabla\varphi) = (\alpha_1 + \lambda)\int_\Omega k(x)u^+\varphi + \int_\Omega (u^+)^5\varphi.$$

We begin by showing that there exists a solution of (1.1), which is a local minimizer of the functional I^+.

Lemma 2.1 — Assume $b > 0$, there exist $\lambda_0 > 0$ and $r, \rho > 0$ such that, for every $\lambda \in (0, \lambda_0)$, we have

$$I^+(u)|_{u\in\partial B_r} \geq \rho, \quad \text{and} \quad \inf_{u\in B_r} I^+(u) < 0. \tag{2.1}$$

PROOF : By Hölder's inequality and (1.3), one has

$$\begin{aligned} I^+(u) &= \frac{1}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{1}{6}\int_\Omega (u^+)^6 - \frac{\alpha_1 + \lambda}{2}\int_\Omega k(x)(u^+)^2 \\ &\geq \frac{b}{4}\|u\|^4 - \frac{1}{6}S^{-3}\|u\|^6 - \frac{\lambda}{2\alpha_1}\|u\|^2 \\ &= \|u\|^2\left(\frac{b}{4}\|u\|^2 - \frac{1}{6}S^{-3}\|u\|^4 - \frac{\lambda}{2\alpha_1}\right). \end{aligned}$$

Set $g(t) = \frac{b}{4}t^2 - \frac{1}{6}S^{-3}t^4 - \frac{\lambda}{2\alpha_1}$ with $t > 0$. We know that there exists a constant $r = \left(\frac{3bS^3}{4}\right)^{\frac{1}{2}} > 0$, such that $\max_{t>0} g(t) = g(r) = \frac{b}{8}r - \frac{\lambda}{2\alpha_1} > \frac{br}{16}$ provided $0 < \lambda < \lambda_0 \doteq \frac{br\alpha_1}{8}$. Consequently, there is a constant $\rho = \frac{br^3}{16} > 0$, such that $I^+(u)|_{\partial B_r} \geq \rho$ for every $\lambda \in (0, \lambda_0)$. Moreover, for given r, let φ_1 be correspondent eigenfunction for α_1, it holds that

$$\lim_{t \to 0^+} \frac{I^+(t\varphi_1)}{t^2} = -\frac{\lambda}{2} \int_\Omega k(x)\varphi_1^2 < 0.$$

So we have $I^+(t\varphi_1) < 0$ for t small enough, thus there exists u small enough such that $I^+(u) < 0$. Define

$$m := \inf_{u \in B_r} I^+(u).$$

Then, from the previous inequality, we obtain $m < 0$. The proof is complete. □

The following theorem is similar to that in [17] (see Theorem 2.2 in [17]).

***Theorem* 2.2** — *Assume $b > 0, 0 < \lambda < \lambda_0$ (λ_0 defined in Lemma 2.1). Then Problem 1.1 has a positive solution $u_1 \in H_0^1(\Omega)$, enjoying $I^+(u_1) < 0$.*

2.2 *Existence of a second positive solution of (1.1).*

In order to obtain a second positive solution of (1.1), we first show that the functional I^+ satisfies the mountain-pass lemma.

Lemma 2.3 — Assume $b > 0$ and $0 < \lambda < \lambda_0$, there exist positive constants $r, \rho > 0$ such that the functional I^+ satisfies $I^+(0) = 0$ and

(i) $I^+(u) > \rho$ if $\|u\| = r$.

(ii) There exists $e \in H_0^1(\Omega)$ such that $I^+(e) < \rho$.

PROOF : (i) The conclusion follows from Lemma 2.1.

(ii) Let $u \in H_0^1(\Omega), u^+ \neq 0$ and $t > 0$, there holds

$$\begin{aligned} I^+(tu) &\leq \frac{t^2}{2}\|u\|^2 + \frac{bt^4}{4}\|u\|^4 - \frac{(\alpha_1 + \lambda)t^2}{2} \int_\Omega k(x)(u^+)^2 - \frac{t^6}{6} \int_\Omega (u^+)^6 \\ &\to -\infty \end{aligned}$$

as $t \to +\infty$. Therefore we can easily find $e \in H_0^1(\Omega)$ with $\|e\| > r$, such that $I^+(e) < \rho$. The proof is complete. □

Define

$$\Lambda = \frac{bS^3}{4} + \frac{b^3S^6}{24} + \frac{(b^2S^4 + 4S)^{\frac{3}{2}}}{24}.$$

Lemma 2.4 — Assume $b > 0$, for all $\lambda > 0$, the functional I^+ satisfies the $(PS)_c$ condition at level c with $c < \Lambda - D\lambda^{\frac{3}{2}}$, where D (independent of λ) is positive constant.

PROOF : Let $\{u_n\} \subset H_0^1(\Omega)$ be such that

$$I^+(u_n) \to c, \quad (I^+)'(u_n) \to 0, \quad \text{as } n \to \infty. \tag{2.2}$$

Then the sequence $\{u_n\}$ is bounded in $H_0^1(\Omega)$. Otherwise, we can assume that $\|u_n\| \to \infty$ as $n \to \infty$. It follows from (2.2) that

$$\begin{aligned} 1 + c + o(1)\|u_n\| &\geq I^+(u_n) - \frac{1}{6}\langle (I^+)'(u_n), u_n\rangle \\ &= \frac{1}{3}\|u_n\|^2 + \frac{b}{12}\|u_n\|^4 - \frac{\alpha_1 + \lambda}{3}\int_\Omega k(x)(u_n^+)^2 \\ &\geq \frac{b}{12}\|u_n\|^4 - \frac{\lambda}{3\alpha_1}\|u_n\|^2, \end{aligned}$$

which implies that the last inequality is an absurd. So $\{u_n\}$ is bounded in $H_0^1(\Omega)$, there then is a subsequence (still denoted by $\{u_n\}$) and $u \in H_0^1(\Omega)$ such that

$$\begin{cases} u_n \rightharpoonup u, & \text{weakly in } \mathrm{H}_0^1(\Omega), \\ u_n \to u, & \text{strongly in } \mathrm{L}^p(1 \leq p < 6), \\ u_n(x) \to u(x), & \text{a.e. in } \Omega. \end{cases}$$

Furthermore, based on the concentration compactness principle (see [18]), there exists a subsequence, say $\{u_n\}$, such that

$$\int_\Omega |\nabla u_n|^2 \rightharpoonup d\eta \geq \int_\Omega |\nabla u|^2 + \sum_{j\in J} \eta_j \delta_{x_j},$$

$$\int_\Omega |u_n|^6 \to d\nu = \int_\Omega |u|^6 + \sum_{j\in J} \nu_j \delta_{x_j},$$

where J is an at most countable index set, δ_{x_j} is the Dirac mass at x_j, and let $x_j \in \Omega$ in the support of η, ν. Moreover, we have

$$\eta_j \geq S\nu_j^{\frac{1}{3}} \ \text{ for all } j \in J. \tag{2.3}$$

For $\varepsilon > 0$, let $\phi_{\varepsilon,j}(x)$ be a smooth cut-off function centered at x_j such that $0 \leq \phi_{\varepsilon,j}(x) \leq 1$, and

$$\phi_{\varepsilon,j}(x) = 1 \ \text{ in } B(x_j, \varepsilon), \quad \phi_{\varepsilon,j}(x) = 0 \ \text{ in } \Omega \backslash B(x_j, 2\varepsilon), \ \ |\nabla \phi_{\varepsilon,j}(x)| \leq \frac{2}{\varepsilon}.$$

Since $k \in \mathcal{K}_\beta$, there exist $R > 0$ and $C_1, C_2 > 0$ such that $B_{2R} \subset \Omega$ and

$$C_1|x|^{-\beta} \leq k(x) \leq C_2|x|^{-\beta}, \ \ \forall x \in B_R(0).$$

By Hölder inequality and (1.3), it holds that

$$\begin{aligned}
\left|\int_\Omega k(x)u_n^+\phi_{\varepsilon,j}u_n\right| &\leq C\int_{B(x_j,\varepsilon)}\frac{|u_n|^2}{|x|^\beta}\\
&\leq \left(\int_{B(x_j,\varepsilon)}|u_n|^6\right)^{\frac{1}{3}}\left(\int_{B(x_j,\varepsilon)}\frac{dx}{|x|^{\frac{3\beta}{2}}}\right)^{\frac{2}{3}}\\
&\leq C\|u_n\|^2\varepsilon^{\frac{6-3\beta}{2}}.
\end{aligned}$$

Note that $\{u_n\}$ is bounded in $H_0^1(\Omega)$, then

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_\Omega k(x)u_n^+\phi_{\varepsilon,j}u_n = 0.$$

As $\phi_{\varepsilon,j}u_n$ is bounded in $H_0^1(\Omega)$, taking the test function $\varphi = \phi_{\varepsilon,j}u_n$ in (2.2), it follows

$$\begin{aligned}
0 &= \lim_{\varepsilon\to 0}\lim_{n\to\infty}\langle (I^+)'(u_n),\phi_{\varepsilon,j}u_n\rangle\\
&= \lim_{\varepsilon\to 0}\lim_{n\to\infty}\Big\{(1+b\|u_n\|^2)\int_\Omega(\nabla u_n,\nabla(\phi_{\varepsilon,j}u_n))\\
&\quad -(\alpha_1+\lambda)\int_\Omega k(x)u_n^+\phi_{\varepsilon,j}u_n-\int_\Omega(u_n^+)^5\phi_{\varepsilon,j}u_n\Big\}\\
&\geq \lim_{\varepsilon\to 0}\lim_{n\to\infty}\Big\{(1+b\|u_n\|^2)\int_\Omega\Big(|\nabla u_n|^2\phi_{\varepsilon,j}+u_n\nabla u_n\nabla\phi_{\varepsilon,j}-\int_\Omega|u_n|^5\phi_{\varepsilon,j}u_n\Big)\\
&\geq \lim_{\varepsilon\to 0}\lim_{n\to\infty}\Big\{(1+b\|u_n\|^2)\int_\Omega|\nabla u_n|^2\phi_{\varepsilon,j}-\int_\Omega|u_n|^5\phi_{\varepsilon,j}u_n\Big\}\\
&\geq (1+b\eta_j)\eta_j-\nu_j,
\end{aligned}$$

so that

$$\nu_j\geq\eta_j+b\eta_j^2.$$

Applying (2.3), we deduce that

$$\nu_j^{\frac{2}{3}}\geq S+bS^2\nu_j^{\frac{1}{3}},\ \text{ or } \eta_{\rm j}=\nu_{\rm j}=0. \tag{2.4}$$

Set $X=\nu_j^{\frac{1}{3}}$, it follows from (2.4) that

$$X^2\geq S+bS^2X,$$

that is

$$X\geq\frac{bS^2+\sqrt{b^2S^4+4S}}{2}.$$

Using (2.4) again, consequently

$$\eta_j\geq SX\geq\frac{bS^3+\sqrt{b^2S^6+4S^3}}{2}\triangleq K.$$

Next we show that $\eta_j \geq \frac{bS^3+\sqrt{b^2S^6+4S^3}}{2}$ is impossible. Assume the contrary, there exists some $j_0 \in J$ such that $\eta_{j_0} \geq \frac{bS^3+\sqrt{b^2S^6+4S^3}}{2}$. By (2.2) and Young inequality, one has

$$\begin{aligned}
c &= \lim_{n\to\infty} I^+(u_n) \\
&= \lim_{n\to\infty} \left\{ \frac{1}{2}\|u_n\|^2 + \frac{b}{4}\|u_n\|^4 - \frac{\alpha_1+\lambda}{2}\int_\Omega k(x)(u_n^+)^2 - \frac{1}{6}\int_\Omega (u_n^+)^6 \right\} \\
&\geq \lim_{n\to\infty} \left\{ \left(\frac{1}{2}-\frac{1}{4}\right)\|u_n\|^2 + b\left(\frac{1}{4}-\frac{1}{4}\right)\|u_n\|^4 \right. \\
&\quad \left. + \left(\frac{1}{4}-\frac{1}{6}\right)\int_\Omega |u_n|^6 - \frac{\alpha_1+\lambda}{4}\int_\Omega k(x)|u_n|^2 \right\} \\
&\geq \left\{ \left(\frac{1}{2}-\frac{1}{4}\right)\left(\|u\|^2 + \sum_{j\in J}\eta_j\right) + b\left(\frac{1}{4}-\frac{1}{4}\right)\left(\|u\|^2 + \sum_{j\in J}\eta_j\right)^2 \right. \\
&\quad \left. + \left(\frac{1}{4}-\frac{1}{6}\right)\left(\int_\Omega |u|^6 + \sum_{j\in J}\nu_j\right) - \frac{\alpha_1+\lambda}{4}\int_\Omega k(x)|u|^2 \right\} \\
&\geq \left(\frac{1}{2}-\frac{1}{4}\right)\eta_{j_0} + \left(\frac{1}{4}-\frac{1}{4}\right)b\eta_{j_0}^2 + \left(\frac{1}{4}-\frac{1}{6}\right)\nu_{j_0} + \frac{1}{12}\int_\Omega |u|^6 \\
&\quad + \frac{1}{4}\left(\|u\|^2 - \alpha_1\int_\Omega k(x)|u|^2\right) - \frac{\lambda}{4}\int_\Omega k(x)|u|^2 \\
&\geq \left(\frac{1}{2}-\frac{1}{4}\right)K + \left(\frac{1}{4}-\frac{1}{4}\right)bK^2 + \left(\frac{1}{4}-\frac{1}{6}\right)\frac{K^3}{S^3} \\
&\quad + \frac{1}{12}\int_\Omega |u|^6 - \frac{C_2\lambda}{4}\int_\Omega \frac{|u|^2}{|x|^\beta}dx \\
&\geq \frac{K}{2} + \frac{b}{4}K^2 - \frac{K^3}{6S^3} - \frac{1}{4}\left(K + bK^2 - \frac{K^3}{S^3}\right) - D\lambda^{\frac{3}{2}},
\end{aligned}$$

where D (independent of λ) is positive constant. Easy computations show that

$$\begin{cases} \dfrac{K}{2} + \dfrac{b}{4}K^2 - \dfrac{K^3}{6S^3} = \Lambda, \\ K + bK^2 - K^3S^{-3} = 0. \end{cases}$$

Thereby, we get $\Lambda - D\lambda^{\frac{3}{2}} \leq c < \Lambda - D\lambda^{\frac{3}{2}}$. This is a contradiction. It implies that J is empty, which means that $\int_\Omega u_n^6 \to \int_\Omega u^6$ as $n \to \infty$. Now, set $\lim\limits_{n\to\infty} \|u_n\| = l$. By (2.2), it holds that

$$(1+b\|u_n\|^2)\|u_n\|^2 - (\alpha_1+\lambda)\int_\Omega k(x)(u_n^+)^2 - \int_\Omega (u_n^+)^6 = o(1), \tag{2.5}$$

and

$$(1+b\|u_n\|^2)\int_\Omega (\nabla u_n, \nabla\varphi) = (\alpha_1+\lambda)\int_\Omega k(x)u_n^+\varphi + \int_\Omega (u_n^+)^5\varphi + o(1) \tag{2.6}$$

for any $\varphi \in H_0^1(\Omega)$. Let $n \to \infty$, then from (2.5), one gets

$$(1+bl^2)l^2 - (\alpha_1+\lambda)\int_\Omega k(x)(u^+)^2 - \int_\Omega (u^+)^6 = 0.$$

Similarly, from (2.6), we have

$$(1+bl^2)\int_\Omega (\nabla u, \nabla\varphi) = (\alpha_1+\lambda)\int_\Omega k(x)u^+\varphi + \int_\Omega (u^+)^5\varphi. \tag{2.7}$$

Taking the test function $\varphi = u$ in (2.7), it follows

$$(1+bl^2)\|u\|^2 - (\alpha_1+\lambda)\int_\Omega k(x)(u^+)^2 - \int_\Omega (u^+)^6 = 0.$$

So, we obtain $l = \|u\|$, consequently $u_n \to u$ in $H_0^1(\Omega)$. The proof is complete. □

Now, it is well known that the function

$$U_\varepsilon(x) = 3^{\frac{1}{4}}\frac{\varepsilon^{\frac{1}{2}}}{(\varepsilon^2+|x|^2)^{\frac{1}{2}}}, \quad x\in\mathbb{R}^3, \ \varepsilon>0$$

satisfies

$$-\Delta U_\varepsilon = U_\varepsilon^5 \quad \text{in } \mathbb{R}^3,$$

and

$$\int_{\mathbb{R}^3} |U_\varepsilon|^6 = \int_{\mathbb{R}^3} |\nabla U_\varepsilon|^2 = S^{\frac{3}{2}}.$$

Let $\eta\in C_0^\infty(\Omega)$ be a cut-off function such that $0\le\eta\le1$, $|\nabla\eta|\le C$ and $\eta(x)=1$ for $|x|<R$ and $\eta(x)=0$ for $|x|>2R$, we set $u_\varepsilon(x)=\eta(x)U_\varepsilon(x)$. It is known (see [19], Lemma 1.46) that

$$\begin{cases} \|u_\varepsilon\|^2 = S^{\frac{3}{2}} + O(\varepsilon); \\ |u_\varepsilon|_6^6 = S^{\frac{3}{2}} + O(\varepsilon^3). \end{cases} \tag{2.8}$$

We can also deduce

$$\begin{cases} \|u_\varepsilon\|^4 \le S^3 + O(\varepsilon); \\ \|u_\varepsilon\|^6 \le S^{\frac{9}{2}} + O(\varepsilon); \\ \|u_\varepsilon\|^8 \le S^6 + O(\varepsilon); \\ \|u_\varepsilon\|^{12} \le S^9 + O(\varepsilon). \end{cases} \tag{2.9}$$

Lemma 2.5 — Assume $b>0$, when $\varepsilon>0$ is enough small, there holds

$$\sup_{t\ge0} I^+(tu_\varepsilon) < \Lambda - D\lambda^{\frac{3}{2}}.$$

PROOF : Since $I^+(tu_\varepsilon)\to-\infty$ as $t\to\infty$, then there exists $t_\varepsilon>0$ such that $I^+(t_\varepsilon u_\varepsilon) = \max_{t>0} I^+(tu_\varepsilon)$. By Lemma 2.3, we know that $\max_{t>0} I^+(tu_\varepsilon)\ge\rho>0$, it follows that $I^+(t_\varepsilon u_\varepsilon)\ge$

$\rho > 0$. Hence from the continuity of I^+, we may assume that there exist positive constants t_0, T_0 (independent of ε) such that $0 < t_0 \le t_\varepsilon \le T_0$. Let $I^+(t_\varepsilon u_\varepsilon) = A(t_\varepsilon u_\varepsilon) + B(t_\varepsilon u_\varepsilon)$, where

$$A(t_\varepsilon u_\varepsilon) = \frac{t_\varepsilon^2}{2}\|u_\varepsilon\|^2 + \frac{bt_\varepsilon^4}{4}\|u_\varepsilon\|^4 - \frac{1}{6}t_\varepsilon^6\int_\Omega u_\varepsilon^6,$$

and

$$B(t_\varepsilon u_\varepsilon) = -\frac{(\alpha_1+\lambda)t_\varepsilon^2}{2}\int_\Omega k(x)u_\varepsilon^2.$$

First, we claim that there exists a constant $c_7 > 0$, such that $A(t_\varepsilon u_\varepsilon) \le \Lambda + c_7\varepsilon^{\frac{1}{2}}$. Indeed, we set

$$h(t) = \frac{1}{2}t^2\|u_\varepsilon\|^2 + \frac{b}{4}t^4\|u_\varepsilon\|^4 - \frac{t^6}{6}\int_\Omega u_\varepsilon^6.$$

Since $\lim_{t\to\infty} h(t) = -\infty, h(0) = 0$, and $\lim_{t\to 0^+} h(t) > 0$, which means that $\sup\limits_{t\ge 0} h(t)$ attained at $T_\varepsilon > 0$, i.e.,

$$h'|_{t=T_\varepsilon} = T_\varepsilon\|u_\varepsilon\|^2 + bT_\varepsilon^3\|u_\varepsilon\|^4 - T_\varepsilon^5\int_\Omega u_\varepsilon^6 = 0,$$

so that

$$T_\varepsilon^4\int_\Omega u_\varepsilon^6 - \|u_\varepsilon\|^2 - bT_\varepsilon^2\|u_\varepsilon\|^4 = 0.$$

Then $h(t)$ achieves its maximum at T_ε with

$$T_\varepsilon = \left(\frac{b\|u_\varepsilon\|^4 + \sqrt{b^2\|u_\varepsilon\|^8 + 4\|u_\varepsilon\|^2\int_\Omega u_\varepsilon^6}}{2\int_\Omega u_\varepsilon^6}\right)^{\frac{1}{2}}.$$

As $h(t)$ is increasing in the interval $[0, T_\varepsilon]$, then by (2.8) and (2.9), one has

$$\begin{aligned}
A(t_\varepsilon u_\varepsilon) &\le h(T_\varepsilon) = \frac{1}{2}T_\varepsilon^2\|u_\varepsilon\|^2 + \frac{b}{4}T_\varepsilon^4\|u_\varepsilon\|^4 - \frac{T_\varepsilon^6}{6}\int_\Omega u_\varepsilon^6 \\
&= \frac{b\|u_\varepsilon\|^6}{6\int_\Omega u_\varepsilon^6} + \frac{b\|u_\varepsilon\|^6}{12\int_\Omega u_\varepsilon^6} + \frac{\|u_\varepsilon\|^2\sqrt{b^2\|u_\varepsilon\|^8 + 4\|u_\varepsilon\|^2\int_\Omega u_\varepsilon^6}}{6\int_\Omega u_\varepsilon^6} \\
&\quad + \frac{b^3\|u_\varepsilon\|^{12}}{24(\int_\Omega u_\varepsilon^6)^2} + \frac{b^2\|u_\varepsilon\|^8\sqrt{b^2\|u_\varepsilon\|^8 + 4\|u_\varepsilon\|^2\int_\Omega u_\varepsilon^6}}{24(\int_\Omega u_\varepsilon^6)^2} \\
&\le \frac{b(S^{\frac{9}{2}} + O(\varepsilon))}{4(S^{\frac{3}{2}} + O(\varepsilon))} + \frac{b^3(S^9 + O(\varepsilon))}{24(S^{\frac{3}{2}} + O(\varepsilon))^2} \\
&\quad + \frac{[b^2(S^6 + O(\varepsilon)) + 4(S^{\frac{3}{2}} + O(\varepsilon))(S^{\frac{3}{2}} + O(\varepsilon))]^{\frac{3}{2}}}{24(S^{\frac{3}{2}} + O(\varepsilon))^2} \\
&\le \frac{bS^3}{4} + \frac{b^3S^6}{24} + \frac{(b^2S^6 + 4S^3)^{\frac{3}{2}}}{24S^3} + c_1\varepsilon \\
&= \Lambda + c_1\varepsilon.
\end{aligned}$$

Therefore, we finish the claim.

Since $k \in \mathcal{K}_\beta$, let ε be enough small such that $0 < \varepsilon < R$. Then for $\beta < 3$, it follows that

$$\begin{aligned}
\int_\Omega k(x) u_\varepsilon^2 dx &\geq C \int_{|x|\leq R} \frac{|x|^{-\beta}\varepsilon}{\varepsilon^2 + |x|^2} dx + C \int_{|x|>R} k(x) u_\varepsilon^2 dx \\
&\geq C\varepsilon \int_0^R \frac{r^2}{r^\beta(\varepsilon^2 + r^2)} dr \\
&= C\varepsilon \int_0^{\frac{R}{\varepsilon}} \frac{t^{2-\beta}\varepsilon^{3-\beta}}{\varepsilon^2(1+t^2)} dt \\
&\geq C\varepsilon^{2-\beta} \int_0^1 \frac{t^{2-\beta}}{1+t^2} dt \\
&\geq C\varepsilon^{2-\beta} \int_0^1 \frac{t^{2-\beta}}{2} dt \\
&= c_2 \varepsilon^{2-\beta},
\end{aligned}$$

here $c_2 > 0$ (independently of ε) is real constant. Thereby, there is positive number c_3 such that

$$\begin{aligned}
I^+(t_\varepsilon u_\varepsilon) &= A(t_\varepsilon u_\varepsilon) + B(t_\varepsilon u_\varepsilon) \\
&\leq \Lambda + c_1\varepsilon - \frac{(\alpha_1 + \lambda) t_\varepsilon^2}{2} \int_\Omega k(x) u_\varepsilon^2 \\
&\leq \Lambda + c_1\varepsilon - \frac{\lambda t_0^2}{2} \int_\Omega k(x) u_\varepsilon^2 \\
&\leq \Lambda + c_1\varepsilon - \lambda c_3 \varepsilon^{2-\beta}.
\end{aligned}$$

As $2 - \frac{1}{3} < \beta < 2$, let $\varepsilon = \lambda^{\frac{3}{2}}$, when $0 < \lambda < \tilde{\lambda}_0 := \left(\frac{c_3}{c_1 + D}\right)^{\frac{2}{3\beta - 5}}$, one has

$$\begin{aligned}
c_1\varepsilon - c_3\lambda\varepsilon^{2-\beta} &= c_1\lambda^{\frac{3}{2}} - c_3\lambda^{\frac{3(2-\beta)+2}{2}} \\
&= \lambda^{\frac{3}{2}}\left(c_1 - c_3\lambda^{\frac{5-3\beta}{2}}\right) \\
&< -D\lambda^{\frac{3}{2}}.
\end{aligned}$$

Consequently, $I^+(t_\varepsilon u_\varepsilon) < \Lambda - D\lambda^{\frac{3}{2}}$. The proof is complete. □

***Theorem* 2.6** — *Assume $b > 0$ and $\lambda > 0$ is sufficiently small, problem (1.1) has at least a positive solution u_2 with $I^+(u_2) > 0$.*

PROOF : There exists $\delta > 0$ such that $\Lambda - D\lambda^{\frac{3}{2}} > 0$ for $\lambda < \delta$. Set $\lambda^* = \min\{\lambda_0, \tilde{\lambda}_0, \delta\}$, then Lemma 2.1-2.5 hold for $0 < \lambda < \lambda_*$. By Lemma 2.3, I^+ satisfies the geometry of the mountain-pass lemma [20]. Applying the mountain-pass lemma, there exists a sequence $\{u_n\} \subset H_0^1(\Omega)$, such that

$$I^+(u_n) \to c > \rho, \quad \text{and} \quad (I^+)'(u_n) \to 0, \tag{2.10}$$

where

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I^+(\gamma(t)),$$

and

$$\Gamma = \left\{\gamma \in C([0,1], H_0^1(\Omega)) : \gamma(0) = 0, \gamma(1) = e\right\}.$$

Moreover, by Lemmas 2.3 and 2.5, it holds that

$$0 < \rho < c \le \max_{t\in[0,1]} I^+(te) \le \sup_{t\ge 0} I^+(te) < \Lambda - D\lambda^{\frac{3}{2}}. \tag{2.11}$$

According to Lemma 2.4, $\{u_n\} \subset H_0^1(\Omega)$ has a convergent subsequence, still denoted by $\{u_n\}$, we may assume that $u_n \to u_2$ in $H_0^1(\Omega)$ as $n \to \infty$. Hence, it follows from (2.10) and (2.11) that

$$I^+(u_2) = \lim_{n\to\infty} I^+(u_n) = c > \rho > 0,$$

which implies that $u_1 \not\equiv 0$. Furthermore, from the continuity of $(I^+)'$, we obtain that u_2 is a solution of (1.1), namely

$$(1 + b\|u_2\|^2)\int_\Omega (\nabla u_2, \nabla\varphi) = (\alpha_1 + \lambda)\int_\Omega k(x)u_2^+\varphi + \int_\Omega (u_2^+)^5\varphi \tag{2.12}$$

for all $\varphi \in H_0^1(\Omega)$. Taking the test function $\varphi = u_2^-$ in (2.12), one has $\|u_2^-\| = 0$. Therefore, $u_2 \ge 0$ and $u_2 \not\equiv 0$. By using the strong maximum principle we obtained that u_2 is a positive solution of (1.1). The proof is complete. □

2.2 *Existence of a ground state negative solutions of (1.1).*

Consider the energy functional $I^- : H_0^1(\Omega) \to \mathbb{R}$ given by

$$I^-(u) = \frac{1}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{\alpha_1+\lambda}{2}\int_\Omega k(x)(u^-)^2.$$

***Theorem* 2.7** — *Assume $b > 0$, problem (1.1) has a ground state negative solution for all $\lambda > 0$.*

PROOF : For $u \in H_0^1(\Omega)$, we have

$$\begin{aligned} I^-(u) &= \frac{1}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{\alpha_1+\lambda}{2}\int_\Omega k(x)(u^-)^2 \\ &\ge \frac{1}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{\alpha_1+\lambda}{2}\int_\Omega k(x)|u|^2 \\ &\ge \frac{b}{4}\|u\|^4 - \frac{\lambda}{2\alpha_1}\|u\|^2. \end{aligned}$$

It follows that I^- is coercive on $H_0^1(\Omega)$. So we can define $d := \inf\limits_{u\in H_0^1(\Omega)} I^-(u)$. Therefore, there exist a bounded subsequence, say $\{u_n\}$, and $u_* \in H_0^1(\Omega)$ such that

$$\begin{cases} u_n \rightharpoonup u_*, & \text{weakly in } H_0^1(\Omega), \\ u_n \to u_*, & \text{strongly in } L^2(k, dx), \\ u_n(x) \to u_*(x), & \text{a.e. in } \Omega. \end{cases}$$

As usually, set $w_n = u_n - u_*$, we claim that $\|w_n\| \to 0$ as $n \to \infty$. Otherwise, there exists a positive real number l, such that $\lim_{n\to\infty} \|w_n\| = l > 0$. Consequently

$$\begin{aligned} d &= I^-(u_n) + o(1) \\ &= \frac{1}{2}\|u_n\|^2 + \frac{b}{4}\|u_n\|^4 - \frac{\mu}{2}\int_\Omega k(x)(u_n^-)^2 + o(1) \\ &= I^-(u_*) + \frac{1}{2}\|w_n\|^2 + \frac{b}{4}\|w_n\|^4 + \frac{b}{2}\|w_n\|^2\|u_*\|^2 + o(1) \\ &= I^-(u_*) + \frac{1}{2}l^2 + \frac{b}{4}l^4 + \frac{b}{2}l^2\|u_*\|^2 \\ &> I^-(u_*). \end{aligned}$$

It follows that $I^-(u_*) < d$, which contradicts the fact that $I^-(u_*) \geq d$. So $l = 0$, i.e., $u_n \to u_*$ in $H_0^1(\Omega)$. Therefore, we get $\lim_{n\to\infty} I^-(u_n) = I^-(u_*) = d$. Furthermore, for $t > 0$, there holds

$$\begin{aligned} \lim_{t\to 0^+} \frac{I^-(-t\varphi_1)}{t^2} &= -\frac{\lambda}{2}\int_\Omega k(x)\varphi_1^2 \\ &< 0. \end{aligned}$$

Consequently, we obtain $d < 0$ and $u_* \not\equiv 0$. By standard discussion, we can obtain that u_* is a ground state solution of (1.1).

Note that

$$\langle (I^-)'(u_*), u_*^+\rangle = (1 + b\|u_*\|^2)\|u_*^+\|^2 = 0.$$

Then, we deduce that $u_*^+ \equiv 0$, so that $u_* \leq 0$. Therefore, u_* is a ground state negative solution of (1.1). The proof is complete. □

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