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# THE COMPRESSED ANNIHILATOR GRAPH OF A COMMUTATIVE RING

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Let $R$ be a commutative ring. In this paper, we introduce and study the compressed annihilator graph of $R$. The compressed annihilator graph of $R$ is the graph $\mathrm{AG_E}(R)$, whose vertices are equivalence classes of zero-divisors of $R$ and two distinct vertices $[x]$ and $[y]$ are adjacent if and only if $\mathrm{ann}(x) \cup \mathrm{ann}(y) \subset \mathrm{ann}(xy)$. For a reduced ring $R$, we show that compressed annihilator graph of $R$ is identical to the compressed zero-divisor graph of $R$ if and only if $0$ is a 2-absorbing ideal of $R$. As a consequence, we show that an Artinian ring $R$ is either local or reduced whenever $0$ is a 2-absorbing ideal of $R$.

**Key words** : Annihilator graph; compressed annihilator graph; zero divisor graph; 2-Absorbing ideal.

## 1. INTRODUCTION

Let $R$ be a commutative ring with nonzero identity, and let $Z(R)$ denote the set of zero-divisors of $R$. The zero-divisor graph of $R$, denoted by $\Gamma(R)$, is the graph with vertices $Z(R)^* = Z(R) \setminus \{0\}$, the set of nonzero zero-divisors of $R$, and for distinct $x, y \in Z(R)^*$, the vertices $x$ and $y$ are adjacent if and only if $xy = 0$, see [2, 6]. The compressed zero-divisor graph of $R$, that is constructed from equivalence classes of zero-divisors by annihilator ideals, rather than individual zero-divisors themselves was introduced by Mulay in [8] and studied in some literature, for examples [1, 7, 11]. It will be denoted by $\Gamma_E(R)$. This graph has some advantages over the zero-divisor graph. In many cases, the compressed zero-divisor graph of $R$ is finite when the zero-divisor graph is infinite and another important aspect of the compressed zero-divisor graph is the connection to the associated primes of $R$.

The annihilator graph $AG(R)$ for a commutative ring $R$ was introduced and investigated by Badawi in [5]. Let $a \in Z(R)$ and let $\mathrm{ann}(a) = \{r \in R | ra = 0\}$. $AG(R)$ is the graph with vertices $Z(R)^*$, and two distinct $x$ and $y$ are adjacent if and only if $\mathrm{ann}(x) \cup \mathrm{ann}(y) \subset \mathrm{ann}(xy)$. In this paper, inspired by ideas from Mulay in [8], we introduce the compressed annihilator graph of $R$, which is constructed from classes of zero-divisors determined by annihilator ideals, and two distinct classes $[x]$ and $[y]$ are adjacent if and only if $\mathrm{ann}(x) \cup \mathrm{ann}(y) \subset \mathrm{ann}(xy)$. It will be denoted by $\mathrm{AG_E}(R)$.

In section two, we determine the girth and the diameter of $\mathrm{AG_E}(R)$ and we show that if $\mathrm{AG_E}(R)$ is complete, then $Z(R)$ is a prime ideal of $R$ and $|\,\mathrm{Ass}(R)| = 1$. In section three, we show that $AG(R) = \Gamma(R)$ if and only if $0$ is a 2-absorbing ideal of $R$. We investigate the compressed annihilator graph of the direct product of two rings. As a consequence, we show that an Artinian ring $R$ is either local or reduced whenever $0$ is a 2-absorbing ideal of $R$.

Let $G$ be a graph. Recall that $G$ is connected if there is a path between any two distinct vertices of $G$. For vertices $x$ and $y$ of $G$, let $\mathrm{d}(x, y)$ be the length of a shortest path from $x$ to $y$ ($\mathrm{d}(x, x) = 0$ and $\mathrm{d}(x, y) = \infty$ if there is no such path). The diameter of $G$ is $\mathrm{diam}(G)$ = $\sup\{\mathrm{d}(x, y)|\ x$ and $y$ are vertices of G$\}$. The girth of $G$, denoted by $\mathrm{gr}(G)$, is the length of a shortest cycle in $G$ ($\mathrm{gr}(G) = \infty$ if $G$ contains no cycles). A cycle with $n$ vertices will be denoted by $C_n$. The degree of a vertex $v$, denoted by $\mathrm{deg}v$, is the number of edges incident to $v$.

A graph $G$ is complete if any two distinct vertices are adjacent. The complete graph with $n$ vertices will be denoted by $K_n$. A complete bipartite graph is a graph $G$ which may be partitioned into two disjoint nonempty vertex sets $A$ and $B$ such that two distinct vertices are adjacent if and only if they are in distinct vertex sets. If one of the vertex set is a singleton, then we call $G$ a star graph. Let $G'$ be an another graph. If $G$ is identical to $G'$, then we write $G = G'$; otherwise, we write $G \neq G'$.

Throughout, $R$ will denote a commutative ring with non-zero identity, $Z(R)$ its set of zero-divisors, $\mathrm{Nil}(R)$ its set of nilpotent elements, $\mathrm{Ass}(R)$ its set of associated primes and $\mathrm{Min}(R)$ its set of minimal prime ideals. We say $R$ is reduced if $\mathrm{Nil}(R) = 0$ and that $R$ is local if it has a unique maximal ideal. As usual, $\mathbb{Z}$ and $\mathbb{Z}_n$ will denote the integers and integers modulo $n$, respectively. For notations and terminologies not given in this article, the reader is referred to [10].

## 2. Basic Results

For $x, y \in R$, we say that $x \sim y$ if and only if $\mathrm{ann}(x) = \mathrm{ann}(y)$. As noted in [8], $\sim$ is an equivalence relation. Let $[x]$ denote the equivalence class of $x$. Note that $[0] = \{0\}$, $[1] = R - Z(R)$ and the other equivalence classes form a partition of $Z(R)^*$.

*Definition* 2.1 — The compressed annihilator graph of $R$, denoted $\mathrm{AG_E}(R)$, is the graph associated to $Z(R)$ whose vertices are the classes of elements in $Z(R)^*$, and two distinct classes $[x]$ and $[y]$ are adjacent if and only if $\mathrm{ann}(x) \cup \mathrm{ann}(y) \subset \mathrm{ann}(xy)$.

*Remark* 2.2 : Adjacency is well-defined: let $[x_1] = [x_2]$ and $[y_1] = [y_2]$; that is $\mathrm{ann}(x_1) = \mathrm{ann}(x_2)$ and $\mathrm{ann}(y_1) = \mathrm{ann}(y_2)$. If $z \in \mathrm{ann}(x_1y_1) \setminus (\mathrm{ann}(x_1) \cup \mathrm{ann}(y_1))$, then $zx_1y_1 = 0$ and so $zy_1 \in \mathrm{ann}(x_1) = \mathrm{ann}(x_2)$. Thus $zy_1x_2 = 0$ and so $zx_2 \in \mathrm{ann}(y_1) = \mathrm{ann}(y_2)$. Therefore, $z \in \mathrm{ann}(x_2y_2) \setminus (\mathrm{ann}(x_2) \cup \mathrm{ann}(y_2))$. Hence, $\mathrm{ann}(x_1) \cup \mathrm{ann}(y_1) \subset \mathrm{ann}(x_1y_1)$ if and only if $\mathrm{ann}(x_2) \cup \mathrm{ann}(y_2) \subset \mathrm{ann}(x_2y_2)$.

The graph $\mathrm{AG_E}(R)$ has same vertices as $\Gamma_E(R)$. It is obvious that each edge of $\Gamma_E(R)$ is an edge of $\mathrm{AG_E}(R)$. It is shown that $\Gamma_E(R)$ is connected and $\mathrm{diam}(\Gamma_E(R)) \leq 3$, see [11, Proposition 1.4].

*Lemma* 2.3 — Let $y_1, y_2 \in R$ be such that $\mathrm{ann}(y_1)$ and $\mathrm{ann}(y_2)$ are distinct associated primes of $R$. Then $[y_1]$ and $[y_2]$ are adjacent in $\mathrm{AG_E}(R)$. If $R$ is Noetherian and $[x]$ is a vertex of $\mathrm{AG_E}(R)$, then either $\mathrm{ann}(x) \in \mathrm{Ass}(R)$ or there is a vertex $[x']$ adjacent to $[x]$ such that $\mathrm{ann}(x') \in \mathrm{Ass}(R)$.

PROOF : The proof is similar to that of Lemma 1.2 in [11]. □

***Theorem* 2.4** — *The graph* $\mathrm{AG_E}(R)$ *is connected and* $\mathrm{diam}\,\mathrm{AG_E}(R) \leq 2$.

PROOF : Let $[x]$ and $[y]$ be two non-adjacent vertices. Then $\mathrm{ann}(x) \cup \mathrm{ann}(y) = \mathrm{ann}(xy)$. It is clear that $xy \neq 0$ and also $\mathrm{ann}(x) \subseteq \mathrm{ann}(y)$ or $\mathrm{ann}(y) \subseteq \mathrm{ann}(x)$. Assume that $\mathrm{ann}(y) \subseteq \mathrm{ann}(x)$. There is a non-zero element $w \in \mathrm{ann}(y)$. Thus $wy = wx = 0$. Furthermore, $[w] \neq [x]$ and $[w] \neq [y]$ since $x \in \mathrm{ann}(w) \setminus \mathrm{ann}(y)$ and $y \in \mathrm{ann}(w) \setminus \mathrm{ann}(x)$. Hence, $[x] - [w] - [y]$ is a path in $\mathrm{AG_E}(R)$. □

***Theorem* 2.5** — *If* $R$ *is Noetherian, then* $\mathrm{gr}(\mathrm{AG_E}(R)) \in \{3, 4, \infty\}$.

PROOF : If $\mid \mathrm{Ass}(R) \mid \geq 3$, then in view of Lemma 2.3, we have a cycle with length 3. Therefore, assume that $\mid \mathrm{Ass}(R) \mid \leq 2$. By Theorem 2.4 $\mathrm{diam}\,\mathrm{AG_E}(R) \leq 2$ which implies that $\mathrm{gr}(\mathrm{AG_E}(R)) \leq 5$. Let $[x_1] - \cdots - [x_5] - [x_1]$ be a cycle. If $\mid \mathrm{Ass}(R) \mid = 1$ and $\mathfrak{p} = \mathrm{ann}(y)$ is a prime ideal of $R$, then any vertex of $\mathrm{AG_E}(R)$ is adjacent to $[y]$ and so we have a cycle with length 3. Now, assume that $\mid \mathrm{Ass}(R) \mid = 2$ and $\mathfrak{p}_1 = \mathrm{ann}(y_1)$ and $\mathfrak{p}_2 = \mathrm{ann}(y_2)$ are prime ideals of $R$. If $[y_1] = [x_1]$, then $[x_3], [x_4]$ are adjacent to $[y_1]$ or $[y_2]$ and we have a cycle with length 3. By a similar argument one can show that, in the other cases, we have a cycle with length 3. Therefore, $\mathrm{gr}(\mathrm{AG_E}(R)) \leq 4$, if $\mathrm{AG_E}(R)$ has a cycle. □

*Example* 2.6 : Let $R = \mathbb{Z}_{p^2q}$, where $p, q$ are two prime numbers. Then $[p]$, $[p^2]$, $[q]$ and $[pq]$ are vertices of $\mathrm{AG_E}(R)$ and $\mathrm{AG_E}(R) = C_4$. Therefore, $\mathrm{gr}(\mathrm{AG_E}(R)) = 4$.

*Example* 2.7 : Let $R = \mathbb{Z}_4[X, Y]/(X^2, YX, 2X)$ and let $x$ and $y$ denote the image of $X$ and $Y$ in $R$, respectively. The proof of Corollary 1.6 in [11] shows that $\Gamma_E(R)$ is the path $[2] - [x] - [y]$. It is clear that $\mathrm{ann}(2y) = \mathrm{ann}(2)$ and so $[2] - [y]$ is not an edge of $\mathrm{AG_E}(R)$. Therefore, $\mathrm{AG_E}(R) = \Gamma_E(R)$ and $\mathrm{gr}(\mathrm{AG_E}(R)) = \infty$.

It is shown that there are no complete graph $\Gamma_E(R)$ with three or more vertices, see [11, Proposition 1.5]. But the following example shows that for any integer $n$ there is a ring for which the compressed annihilator graph is complete with $n$ vertices.

*Example* 2.8 : Suppose that $p$ is a prime number and $n \geq 2$. Then $[p], [p^2], \cdots, [p^n]$ are all distinct vertices of $\mathrm{AG_E}(\mathbb{Z}_{p^{n+1}})$. Assume that $1 \leq i, j \leq n$. It is obvious that $[p^i][p^j] = 0$ whenever $i+j \geq n$, so that $[p^i], [p^j]$ are adjacent. If $i+j < n$, then $p^{n-(i+j)} \in \mathrm{ann}(p^{i+j}) \setminus (\mathrm{ann}(p^i) \cup \mathrm{ann}(p^j))$ so that $[p^i], [p^j]$ are adjacent. Hence, $\mathrm{AG_E}(\mathbb{Z}_{p^{n+1}})$ is a complete graph with $n$ vertices.

***Theorem* 2.9** — *Let $\mathrm{AG_E}(R)$ be a complete graph with at least three vertices. Then $Z(R)$ is an ideal of $R$.*

PROOF : Suppose that $x, y \in Z(R)^*$. It is enough to show that $x + y \in Z(R)$. If $[x] = [y]$, then there is a non-zero element $z \in R$ such that $z(x + y) = 0$ and in this case we are done. Now, assume that $[x] \neq [y]$. There is $w \in Z(R)^*$ such that $[w] \neq [x], [y]$ and $[w] - [y]$ is an edge of $\Gamma_E(R)$ since $\Gamma_E(R)$ is connected. Thus $wy = 0$. If $wx = 0$, then $w(x + y) = 0$ and we are done. Otherwise $wx \neq 0$ and $[w] - [x]$ is an edge of $\mathrm{AG_E}(R)$, so there is $z \in \mathrm{ann}(wx) \setminus (\mathrm{ann}(w) \cup \mathrm{ann}(x))$. Hence, $zw \neq 0$ and $zw(x + y) = 0$. Therefore, $Z(R)$ is an ideal of $R$. □

*Lemma* 2.10 — Let $x \in R$ and $\mathfrak{p} = \mathrm{ann}(x)$ be a prime ideal of $R$ and let $y \in Z(R)^*$ be such that $[x] \neq [y]$. If $[x] - [y]$ is not an edge of $\Gamma_E(R)$, then $[x] - [y]$ is not an edge of $\mathrm{AG_E}(R)$.

PROOF : By assumption $[x] - [y]$ is not an edge of $\Gamma_E(R)$ so that $xy \neq 0$. Assume that $[x] - [y]$ is an edge of $\mathrm{AG_E}(R)$. Thus $\mathrm{ann}(x) \cup \mathrm{ann}(y) \subset \mathrm{ann}(xy)$ and so there is $z \in \mathrm{ann}(xy) \setminus (\mathrm{ann}(x) \cup \mathrm{ann}(y))$. Hence, $zxy = 0$ and $zx \neq 0$. Therefore, $zy \in \mathrm{ann}(x) = \mathfrak{p}$ but $z \notin \mathfrak{p}$ and $y \notin \mathfrak{p}$ which is a contradiction. Thus $[x] - [y]$ is not an edge of $\mathrm{AG_E}(R)$. □

*Corollary* 2.11 — Let $\mathrm{AG_E}(R)$ be a complete graph with $n$ vertices. If $\mathrm{ann}(x)$ is a prime ideal of $R$ for some $x \in Z(R)^*$, then $\deg[x] = n - 1$ in $\Gamma_E(R)$.

PROOF : Assume that $[y]$ is an arbitrary vertex of $\Gamma_E(R)$, by assumption $[x] - [y]$ is an edge of $\mathrm{AG_E}(R)$. In view of Lemma 2.10, $[x] - [y]$ is an edge of $\Gamma_E(R)$. Hence, $\deg[x] = n - 1$ in $\Gamma_E(R)$. □

***Theorem* 2.12** — *Let $R$ be Noetherian and $\mathrm{AG_E}(R)$ be a complete graph with $n \geq 3$ vertices. Then $Z(R)$ is a prime ideal of $R$ and $|\operatorname{Ass}(R)| = 1$.*

PROOF : Assume that $z \in Z(R)^*$ is such that $\operatorname{ann}(z)$ is a prime ideal of $R$. We show that $\operatorname{ann}(z) = Z(R)$. Let $x \in Z(R)^*$. Then either $xz = 0$ or $\operatorname{ann}(x) = \operatorname{ann}(z)$ since $deg[z] = n-1$ in $\Gamma_E(R)$, by Corollary 2.11. If $xz = 0$ we are done. Now, assume that $\operatorname{ann}(x) = \operatorname{ann}(z)$. It is enough to show that $z^2 = 0$. Let $[x], [x_1], \cdots, [x_{n-1}]$ be all vertices of $\mathrm{AG_E}(R)$. In view of [11, Proposition 1.5] $\Gamma_E(R)$ is not complete, so we can assume that $[x_1]$ is not adjacent to $[x_2]$ in $\Gamma_E(R)$. Thus $x_1x_2 \neq 0$. By Theorem 2.9, $Z(R)$ is an ideal of $R$ so $z + x_1 \in Z(R)^*$. Therefore, $\operatorname{ann}(z+x_1) = \operatorname{ann}(z)$ or $\operatorname{ann}(z+x_1) = \operatorname{ann}(x_i)$ for some $i$ with $1 \leq i \leq n-1$. In the first case, the assumption $zx_2 = 0$ implies that $x_1x_2 = 0$ which is a contradiction. Thus $\operatorname{ann}(z+x_1) = \operatorname{ann}(x_i)$ for some $i$ with $1 \leq i \leq n-1$. We have $zx_j = 0$ for all $j$ with $1 \leq j \leq n-1$, in particular for $j = 1, i$. Hence, $z^2 = 0$. □

***Theorem* 2.13** — *Let $R$ be Noetherian and $\Gamma_E(R)$ be a star graph with at least four vertices. Then $\mathrm{AG_E}(R)$ is a complete graph.*

PROOF : In view of [11, Proposition 2.4], $\operatorname{Ass}(R) = \{\mathfrak{p}\}$ and $\mathfrak{p}^3 = 0$. Let $[y]$ be the vertex with maximal degree and let $[x_1], [x_2], \cdots$ be the other vertices of $\Gamma_E(R)$. If $i \neq j$, then $x_ix_j \neq 0$ and annihilated by $x_i$ and $x_j$ since $\mathfrak{p}^3 = 0$. Hence, the only choice for $[x_ix_j]$ is $[y]$. If $[x_i] - [x_j]$ is not an edge of $\mathrm{AG_E}(R)$, then $\operatorname{ann}(x_i) \cup \operatorname{ann}(x_j) = \operatorname{ann}(x_ix_j) = \operatorname{ann}(y)$. So $\operatorname{ann}(x_i) = \operatorname{ann}(y)$ or $\operatorname{ann}(x_j) = \operatorname{ann}(y)$, which is a contradiction. Thus $[x_i] - [x_j]$ is an edge of $\mathrm{AG_E}(R)$ and therefore, $\mathrm{AG_E}(R)$ is a complete graph. □

*Example* 2.14 : Let $R = \mathbb{Z}_2[X, Y, Z]/(X^2, Y^2)$. By [11, Example 2.6] $\Gamma_E(R)$ is an infinite star graph. Therefore, $\mathrm{AG_E}(R)$ is an infinite complete graph.

*Corollary* 2.15 — Let $R$ be Noetherian. Then there is no star graph $\mathrm{AG_E}(R)$ with more than three vertices.

*Proposition* 2.16 — Let $R$ be Noetherian. Then $\operatorname{gr}(\mathrm{AG_E}(R)) = 4$ if and only if $\mathrm{AG_E}(R) = C_4$.

PROOF : If the girth of $\Gamma_E(R)$ is finite, then it must be three by Theorem 6.6, in [7]. Thus the girth of $\mathrm{AG_E}(R)$ is three contrary to assumption. Hence, $\Gamma_E(R)$ is acyclic and therefore, $\Gamma_E(R)$ is a star graph or a path of length three, by Proposition 6.4, in [7]. Now, Theorem 2.13 implies that $\Gamma_E(R)$ is not a star graph. Thus $\Gamma_E(R)$ is a path of length three and so $\mathrm{AG_E}(R) = C_4$. □

***Theorem* 2.17** — *Let $R$ be reduced. Then $|\mathrm{AG_E}(R)| = 2$ if and only if $\Gamma(R)$ is a complete bipartite graph.*

PROOF : Let $x, y \in Z(R)^*$ and $[x] \neq [y]$. Then $\mathrm{ann}(x) \neq \mathrm{ann}(y)$. Suppose that $z \in \mathrm{ann}(x) \setminus \mathrm{ann}(y)$. Thus $\mathrm{ann}(z) = \mathrm{ann}(x)$ or $\mathrm{ann}(z) = \mathrm{ann}(y)$ and so $x^2 = 0$ or $xy = 0$. Since $R$ is reduced, we have $xy = 0$. Furthermore, for any $w \in Z(R)^*$, we have $\mathrm{ann}(w) = \mathrm{ann}(x)$ or $\mathrm{ann}(w) = \mathrm{ann}(y)$. If $\mathrm{ann}(w) = \mathrm{ann}(x)$, then $wy = 0$ and $wx \neq 0$ and if $\mathrm{ann}(w) = \mathrm{ann}(y)$, then $wx = 0$ and $wy \neq 0$. It follows that $\Gamma(R)$ is a complete bipartite graph. Conversely, assume that $\Gamma(R)$ is a complete bipartite graph. Thus $\Gamma_E(R)$ is the graph $K_2$ since $R$ is reduced. Hence, $|\,\mathrm{AG_E}(R)| = 2$. □

***Theorem* 2.18** — *If $R$ is Noetherian and $|\,\mathrm{AG_E}(R)| = 3$, then $R$ is not reduced.*

PROOF : Assume that $[x]$, $[y]$ and $[z]$ are distinct vertices of $\mathrm{AG_E}(R)$. It follows that $\Gamma_E(R)$ is the graph $[x] - [y] - [z]$ by Corollary 1.6, in [11]. So we have $\mathrm{AG_E}(R) = \Gamma_E(R)$ or $\mathrm{AG_E}(R) = K_3$. If $\mathrm{AG_E}(R) = \Gamma_E(R)$, then $\mathrm{ann}(x) \cup \mathrm{ann}(z) = \mathrm{ann}(xz)$. Without loss of generality, assume that $\mathrm{ann}(x) = \mathrm{ann}(xz)$. Thus $\mathrm{ann}(z) \subset \mathrm{ann}(x)$. Select $s \in \mathrm{ann}(x) \setminus \mathrm{ann}(z)$. Then $\mathrm{ann}(s) = \mathrm{ann}(x)$ and so $x^2 = 0$, so $R$ is not reduced. Now, assume that $\mathrm{AG_E}(R) = K_3$. Thus $[x] - [z]$ is an edge of $\mathrm{AG_E}(R)$. Hence $\mathrm{ann}(x) \cup \mathrm{ann}(z) \subset \mathrm{ann}(xz)$ and therefore $\mathrm{ann}(xz) = \mathrm{ann}(y)$. We have $xy = 0$ and $x^2z = 0$. So that $(xz)^2 = 0$ thus $R$ is not reduced. □

*Corollary* 2.19 — If $R$ is Noetherian and $\mathrm{gr}(\mathrm{AG_E}(R)) = \infty$, then $\mathrm{AG_E}(R)$ is a path of length $\leq$ 2.

PROOF : By hypothesis we have $\Gamma_E(R) = \mathrm{AG_E}(R)$. Hence, $\Gamma_E(R)$ is a star graph or a path of length three by Proposition 6.4, in [7]. On the other hand, Theorem 2.13 implies that $\mathrm{AG_E}(R)$ is not a star graph with more than three vertices. We have to show that $\mathrm{AG_E}(R)$ is not a path of length three. Suppose that $\mathrm{AG_E}(R)$ is the path $[z] - [x] - [y] - [w]$. It follows that $\mathrm{ann}(z) \cup \mathrm{ann}(w) = \mathrm{ann}(zw)$ and therefore one can assume that $\mathrm{ann}(z) = \mathrm{ann}(zw)$. Now, we have $yzw = 0$ since $yw = 0$ and so $yz = 0$ but $yz \neq 0$. Hence $\mathrm{AG_E}(R)$ is not a path of length three. □

*Proposition* 2.20 — Let $R, S$ be commutative rings and $\varphi : R \longrightarrow S$ be a flat ring monomorphism. Then $\mathrm{AG_E}(R)$ is isomorphic to an induced subgraph of $\mathrm{AG_E}(S)$.

PROOF : By the proof of Proposition 4.1, in [7], $Z(R)^*$ maps into $Z(S)^*$ and $\mathrm{ann}(x) = \mathrm{ann}(y)$ if and only if $\mathrm{ann}(\varphi(x)) = \mathrm{ann}(\varphi(y))$ for every $x, y \in Z(R)^*$. On the other hand, $z \in \mathrm{ann}(xy) \setminus (\mathrm{ann}(x) \cup \mathrm{ann}(y))$ if and only if $\varphi(z) \in \mathrm{ann}(\varphi(xy)) \setminus (\mathrm{ann}(\varphi(x)) \cup \mathrm{ann}(\varphi(y)))$ since $\varphi$ is a monomorphism. This means that $\mathrm{AG_E}(R)$ is isomorphic to an induced subgraph of $\mathrm{AG_E}(S)$. □

*Corollary* 2.21 — If $T = R \setminus Z(R)$, then $\mathrm{AG_E}(R) \cong \mathrm{AG_E}(T^{-1}R)$.

PROOF : It is enough to show that any vertex of $\mathrm{AG_E}(T^{-1}R)$ is corresponding to a vertex of $\mathrm{AG_E}(R)$. Assume that $0 \neq x/t \in Z(T^{-1}R)$. It is easy to show that $\mathrm{ann}_{T^{-1}R}(x/t) = \mathrm{ann}_{T^{-1}R}(x/s) =$

$\mathrm{ann}_{T^{-1}R}(x/1)$, for every $s \in T$. Also, we have $\mathrm{ann}(x) \neq \mathrm{ann}(y)$ if and only if $\mathrm{ann}_{T^{-1}R}(x/1) \neq \mathrm{ann}_{T^{-1}R}(y/1)$, for all $x, y \in Z(R)$. It follows that $\mathrm{AG_E}(R) \cong \mathrm{AG_E}(T^{-1}R)$ by Proposition 2.15. □

## 3. 2-ABSORBING IDEAL AND ANNIHILATOR GRAPH

The concept of 2-absorbing ideal is a generalization of prime ideals and introduced in [4]. A proper ideal $I$ of $R$ is called 2-absorbing if whenever $abc \in I$ for $a, b, c \in R$, then $ab \in I$ or $ac \in I$ or $bc \in I$. In the following we investigate the connection between $\Gamma(R)$ and $AG(R)$, when $0$ is a 2-absorbing ideal of $R$.

***Theorem* 3.1** — *$\Gamma(R) = AG(R)$ if and only if $0$ is a 2-absorbing ideal of $R$.*

PROOF : Suppose that $0$ is a 2-absorbing ideal of $R$. Let $x, y \in Z(R)^*$ and $x - y$ is an edge of $AG(R)$. Then $\mathrm{ann}(x) \cup \mathrm{ann}(y) \subset \mathrm{ann}(xy)$. Therefore, there is $z \in \mathrm{ann}(xy) \setminus (\mathrm{ann}(x) \cup \mathrm{ann}(y))$ and so we have $zxy = 0$, $zx \neq 0$ and $zy \neq 0$. Thus $xy = 0$ since $0$ is a 2-absorbing ideal of $R$. Hence, $x - y$ is an edge of $\Gamma(R)$. Conversely, assume that $\Gamma(R) = AG(R)$. We show that $0$ is a 2-absorbing ideal of $R$. Let $x, y, z \in R$, $xyz = 0$ with $xy \neq 0$ and $xz \neq 0$. Then $x \in \mathrm{ann}(yz) \setminus (\mathrm{ann}(y) \cup \mathrm{ann}(z))$ and thus $y - z$ is an edge of $AG(R)$. Therefore, $y - z$ is an edge of $\Gamma(R)$ and so $yz = 0$. □

*Corollary* 3.2 — If $\Gamma(R) = AG(R)$, then $\{\mathrm{ann}(x) | x \in R\}$ is a totally ordered set or is the union of two totally ordered sets.

PROOF : By Theorem 3.1, $0$ is a 2-absorbing ideal of $R$. Now the result follows by Theorem 2.1, in [9]. □

The following corollary is a generalization of Theorem 3.5, in [5].

*Corollary* 3.3 — If $|Min(R)| \geq 3$, then $\Gamma(R) \neq AG(R)$.

PROOF : Assume that $\Gamma(R) = AG(R)$. Then in view of Theorem 3.1, $0$ is a 2-absorbing ideal of $R$. Thus Theorem 2.3, in [4], shows that $|Min(R)| \leq 2$ contrary with assumption. □

*Proposition* 3.4 — Let $0$ be a 2-absorbing ideal of $R$ and $Z(R) = \mathrm{Nil}(R)$. Then $\Gamma(R)$ is complete.

PROOF : Since $0$ is a 2-absorbing ideal, we have $z^2 = 0$ for every $z \in Z(R)$. Let $x$ and $y$ be distinct vertices. Thus $(x + y)xy = 0$. Hence, $(x + y)x = 0$ or $(x + y)y = 0$ or $xy = 0$. In each case, $xy = 0$. That implies $\Gamma(R)$ is complete. □

*Proposition* 3.5 — Let $R$ be a reduced ring. Then $\Gamma_E(R) = \mathrm{AG_E}(R)$ if and only if $\Gamma(R) = AG(R)$.

PROOF : In view of Theorem 3.1, it is enough to show that $\Gamma_E(R) = \mathrm{AG_E}(R)$ if and only if $0$ is a

2-absorbing ideal of $R$. Let $\Gamma_E(R) = \mathrm{AG_E}(R)$ and let $x, y, z \in R$ be such that $xyz = 0$, $xy \neq 0$ and $xz \neq 0$. Then $y, z \in Z(R)^*$. Since $R$ is reduced, we have $[y] \neq [z]$. If $yz \neq 0$, then $[y] - [z]$ is not an edge of $\Gamma_E(R)$. Therefore, $[y] - [z]$ is not an edge of $\mathrm{AG_E}(R)$. Then $\mathrm{ann}(y) \cup \mathrm{ann}(z) = \mathrm{ann}(yz)$, and so $\mathrm{ann}(y) = \mathrm{ann}(yz)$ or $\mathrm{ann}(z) = \mathrm{ann}(yz)$. It follows that $xy = 0$ or $xz = 0$ and it is a contradiction. Hence $yz = 0$ and $0$ is a 2-absorbing ideal of $R$. The converse is clear. □

In the following, we show that if $R_1, R_2$ are commutative rings such that either $Z(R_1)^* \neq \emptyset$ or $Z(R_2)^* \neq \emptyset$, then $0$ is not a 2-absorbing ideal of $R_1 \times R_2$.

***Theorem* 3.6** — *Let $R_1, R_2$ be commutative rings such that either $Z(R_1)^* \neq \emptyset$ or $Z(R_2)^* \neq \emptyset$ and $R = R_1 \times R_2$. Then*

*(i)* $\mathrm{AG_E}(R)$ *is not complete.*

*(ii)* $\mathrm{AG_E}(R) \neq \Gamma_E(R)$.

PROOF : Suppose $Z(R_1)^* \neq \emptyset$ and select $a \in Z(R_1)^*$. Then there is $b \in Z(R_1)^*$ such that $ab = 0$.

(i) It is clear that $(a, 0), (1, 0) \in Z(R)^*$ and $[(a, 0)] \neq [(1, 0)]$. We have $\mathrm{ann}(a, 0) \cup \mathrm{ann}(1, 0) = \mathrm{ann}((a, 0)(1, 0))$. Therefore, $[(a, 0)] - [(1, 0)]$ is not an edge of $\mathrm{AG_E}(R)$ and thus $\mathrm{AG_E}(R)$ is not complete.

(ii) It is easily verified that $(a, 1), (1, 0) \in Z(R)^*$ and $[(a, 1)] \neq [(1, 0)]$. On the other hand, we have $(1, 0)(a, 1) \neq 0$ and $(b, 1) \in \mathrm{ann}(a, 0) \setminus (\mathrm{ann}(1, 0) \cup \mathrm{ann}(a, 1))$. Thus $[(a, 1)] - [(1, 0)]$ is an edge of $\mathrm{AG_E}(R)$ but it is not an edge of $\Gamma_E(R)$. Hence, $\mathrm{AG_E}(R) \neq \Gamma_E(R)$. □

***Theorem* 3.7** — *Let $R = R_1 \times R_2$ and let $R_1, R_2$ be commutative rings such that either $\mathrm{AG_E}(R_1)$ or $\mathrm{AG_E}(R_2)$ has at least an edge. Then $\mathrm{gr}(\mathrm{AG_E}(R)) = 3$ and $\mathrm{diam}\,\mathrm{AG_E}(R) = 2$.*

PROOF : Let $a, b \in Z(R)^*$ and $[a] - [b]$ be an edge of $\mathrm{AG_E}(R_1)$. Then there is $c \in \mathrm{ann}(ab) \setminus (\mathrm{ann}(a) \cup \mathrm{ann}(b))$. Thus $(c, 0) \in \mathrm{ann}((a, 1)(b, 1))$ but $(c, 0) \notin \mathrm{ann}(a, 1) \cup \mathrm{ann}(b, 1)$. It follows that $[(a, 1)] - [(b, 1)]$ is an edge of $\mathrm{AG_E}(R)$. Now, the proof of Theorem 3.6(ii) shows that $[(1, 0)] - [(a, 1)] - [(b, 1)] - [(1, 0)]$ is a cycle in $\mathrm{AG_E}(R)$. Therefore, $\mathrm{gr}(\mathrm{AG_E}(R)) = 3$. Furthermore, $\mathrm{AG_E}(R)$ is not complete by Theorem 3.6(i). Hence, $\mathrm{diam}\,\mathrm{AG_E}(R) = 2$ by Theorem 2.4. □

*Corollary* 3.8 — Let $R$ be an Artinian ring such that $0$ is a 2-absorbing ideal of $R$. Then $R$ is either local or reduced.

PROOF : By assumption $0$ is a 2-absorbing ideal, thus $R$ has at most two maximal ideals by Theorem 2.3, in [4]. If $R$ is not local, then it has two maximal ideals $\mathfrak{m}_1, \mathfrak{m}_2$. Therefore, there is

$k > 0$ such that $\mathfrak{m}_1^k \cap \mathfrak{m}_2^k = 0$ and $R \cong R/\mathfrak{m}_1^k \times R/\mathfrak{m}_2^k$, by Proposition 8.4 and Theorem 8.7, in [3]. If $k > 1$, then $Z(R/m_1^k)^* \neq \emptyset$. Hence, $\mathrm{AG_E}(R) \neq \Gamma_E(R)$ by Theorem 3.6(ii). It follows that $0$ is not 2-absorbing which is a contradiction. Therefore, $k = 1$ and so $\mathrm{Nil}(R) = \mathfrak{m}_1 \cap \mathfrak{m}_2 = 0$. Hence, $R$ is reduced. □

*Corollary* 3.9 — Let $0$ be a 2-absorbing ideal of $R$ and let $0 = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n$ be a primary decomposition of $0$, where $\mathfrak{q}_i, \mathfrak{q}_j$ are coprime whenever $i \neq j$. Then $0$ is either primary or intersection of two prime ideals.

PROOF : Suppose that $0$ is not primary. Thus Proposition 1.10, in [3] shows that $R \cong \prod_i R/\mathfrak{q}_i$. Assume that $\mathfrak{q}_i$ is not prime for some $i$. Then $Z(R/\mathfrak{q}_i)^* \neq \emptyset$. Hence, $\mathrm{AG_E}(R) \neq \Gamma_E(R)$ by Theorem 3.8. It follows that $0$ is not 2-absorbing, contrary to assumption. Hence, $\mathfrak{q}_i$ is prime ideal of $R$, for all $i$. On the other hand, $R$ has at most two prime ideals that are minimal, by [4, Theorem 2.3]. Hence, $0 = \mathfrak{q}_1 \cap \mathfrak{q}_2$, where $\mathfrak{q}_1, \mathfrak{q}_2$ are prime ideals of $R$. □

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## REFERENCES

1. D. F. Anderson and J. D. LaGrange, Commutative Boolean monoids, reduced rings, and the compressed zero-divisor graph, *J. Pure Appl. Algebra*, **216** (2012), 1626-1636.

2. D. F. Anderson and P. S. Livingston, The zero-divisor graph of a commutative ring, *J. Algebra*, **217** (1999), 434-447.

3. M. F. Atiyah and I. G. Macdonald, *Introduction to commutative algebra*, Addison-Wesly, 1969.

4. A. Badawi, On 2-absorbing ideals of commutative rings, *Bull. Austral. Math. Soc.*, **75** (2007), 417-429.

5. A. Badawi, On the annihilator graph of a commutative ring, *Comm. Algebra*, **42** (2014), 108-121.

6. I. Beck, Coloring of commutative rings, *J. Algebra*, **116** (1988), 208-226.

7. J. Coykendall, S. Sather-Wagstaff, L. Sheppardson and S. Spiroff, *On zero divisor graphs*, in Progress in Commutative Algebra, II: Closures, finiteness and factorization, edited by (C. Francisco *et al.* Eds.), Walter Gruyter, Berlin, (2012), 241-299.

8. S. B. Mulay, Cycles and symmetries of zero-divisors, *Comm. Algebra*, **30** (2002), 3533-3558.

9. Sh. Payrovi and S. Babaei, On the 2-absorbing ideals in commutative rings, *Bull. Malaysian Math. Sci. Soc.*, **23** (2013), 1511-1526.

10. R. Y. Sharp, *Steps in commutative algebra*, Second edition, Cambridge University Press, Cambridge, 2000.

11. S. Spiroff and C. Wickham, A zero divisor graph determine by equivalence classes of zero divisors, *Comm. Algebra*, **39** (2011), 2338-2348.