

Research Paper

Free Edge Effects in Sandwich Laminates Under Tension, Bending and Twisting Loads

N DHANESH¹ and SANTOSH KAPURIA^{1,2*}

¹Department of Applied Mechanics, Indian Institute of Technology Delhi, New Delhi 110 016, India

²CSIR-Structural Engineering Research Centre, Taramani, Chennai 600 113 India

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An accurate analytical solution for predicting the free edge effects in sandwich laminates under tension, bending and twisting loading is presented. The recently developed mixed-field multiterm extended Kantorovich method (MMEKM) has been used to obtain the solution of the governing equations, which are developed using the Reissner-type variational principle. The present mixed-field approach enables the exact and point-wise satisfaction of traction-free edge and interlaminar continuity conditions for displacements and stresses. The numerical results presented for different loadings and lay-up show rapid convergence of the iterative series solution. The comparison of the present results with the detailed FE solution shows good agreement. The present solution captures the singularity of stresses in the free edge region by showing the rise in its peak magnitude with the number of terms in the solution. The presented accurate 3D elasticity based solution can act as a useful benchmark for assessing the accuracy of solutions obtained from other approximate methods.

Keywords: Free Edge Effects; Sandwich Laminates; Extended Kantorovich Method; 3D Elasticity; Mixed Formulation

Introduction

With the wide use of laminated composite and sandwich structures in many advanced applications (aerospace, automobile, naval, civil etc.), the vulnerability of such structures to delamination damage initiating from the edge region has been a serious concern among the designers. The occurrence of localized interlaminar stresses near the free edge/boundary region is known to be the main reason for initiation of such damage, and is caused by the material and geometric discontinuities that exist at the interlayer regions at the free edge boundaries. The development of such three dimensional (3D) stresses in the vicinity of free edges under various loading conditions are commonly known as the *free edge effect*, and has been a topic of intense research since the work of Hayashi (1967). In this work, an accurate analytical 3D elasticity based solution for the free edge stress field in sandwich laminates under axial extension, bending and twisting loadings is presented.

Comprehensive reviews of various methodologies used by researchers for studying the free edge effects have been reported by Mittelstedt and Becker (2004, 2007) and Kant and Swaminathan (2000). Subsequent to the initial work of Hayashi (1967) and Puppo and Evensen (1970), where they presented approximate solutions for transverse interlaminar shear stresses by neglecting the transverse normal stress, Pipes and Pagano (1970) presented a finite difference (FD) solution of the complete system of 3D elasticity equations for the free edge problem. Thereafter, there has been a continuous effort to obtain accurate solutions for the free edge problem based on 3D elasticity, satisfying all boundary and interfacial conditions exactly at all points.

Various numerical methods such as the finite element (FE) method (Wang and Crossman, 1977; Raju and Crews, 1981; Lessard *et al.*, 1996), the boundary element method (Davi and Milazzo, 1999)

*Author for Correspondence: E-mail: kapuria@am.iitd.ac.in; Tel.: +91-11-26591218

and the scaled boundary finite element method (Lindemann and Becker, 2000) have been employed for the free edge problems. The limitations of such numerical methods in accurately predicting the stress field in presence of sharp gradients and possible singularities are well known. Various approximate analytical/semi-analytical solutions have been presented, to overcome these issues. Cho and Yoon (1999) extended the Lekhnitskii stress function based solution of Flanagan (1994) for the free edge stresses in composite laminates under extension loading, employing the iterative extended Kantorovich method (EKM) (Kerr, 1968). This method has been further extended to obtain the free edge stress solution for symmetric laminates under bending, twisting and thermal loadings (Cho and Kim, 2000). In another development, Andakhshideh and Tahani (2013a,b) adopted a displacement based formulation in conjunction with the multiterm EKM for the free edge stress analysis of finite rectangular plates under extension, shear, bending, twisting and thermal loadings. The stress based formulations fail to satisfy pointwise interlaminar continuity conditions for displacements. On the other hand, in case of displacement based formulations, the interlaminar stress continuity and traction free edge conditions are not satisfied exactly at all points, but in an average sense. In both formulations, therefore the accuracy of predicted interlaminar stresses becomes questionable. Recently, the author group has presented a mixed-field multiterm EKM (MMEKM) solution for the free edge stress analysis of composite laminates under thermomechanical loadings (Dhanesh *et al.*, 2016). The governing equations are developed using the Reissner-type mixed variational principle for composite laminates, considering both displacements and stresses as unknown variables. This approach allows exact satisfaction of the free edge traction free conditions as well as interlaminar continuity conditions of displacements and stresses in a point-wise sense. It also ensures the same degree of accuracy of the displacements and stresses.

All of the above mentioned studies on free edge stress analysis deal with composite laminates. Very few studies, however exist on the free edge effect in sandwich structures, which consist of relatively thin and stiff face sheets separated by a relatively soft thick and lightweight core. Such structures are preferred in applications where a higher bending

stiffness is required, maintaining the light weight of the structure. Because of widely different material properties of the face sheet and core, the 3D elasticity solution may face numerical difficulties in solving for sandwich laminates. Lovinger and Frostig (2004) presented a hybrid approach for the study of free edge effects in soft core sandwich plates which is supported only at the lower face sheet, employing the classical laminate theory (CLT) approximations for the face sheets and 3D elasticity theory for the core. The analytical solution for bending, thermal, moisture loading conditions was obtained by using the EKM. Afshin *et al.* (2010) employed Reddy's layerwise theory (LWT) to study the free edge effects in cylindrical sandwich panel. Recently, a closed-form solution for the free edge stress field in sandwich structure subjected to differential temperature and mechanical loading has been presented by Wong (2015, 2016), following the strength of material approach. In this work, the face sheets are modelled as beam elements and the soft core as an elastic medium. The approach leads to a discontinuous peeling stress at the interface between core and face sheet layers. Such simplified 2D theory based solutions generally lead to inaccurate prediction of the 3D free edge stress field. In the present work, an accurate solution for the free edge stress field in sandwich laminates under tension, bending and twisting loadings is presented using the recently developed technique, MMEKM, of the author group (Dhanesh *et al.*, 2016). The convergence of the iterative series solution and its comparison with the detailed FE analysis are presented. The results are obtained for sandwich laminates having both cross-ply and angle-ply lay-ups for the face sheets.

Governing Equations

Reissner-type Mixed Variational Principle

To study the free edge effect, an elastic sandwich panel having stiff unidirectional composite faces and a soft core is considered. The infinitely long (y -direction) panel has a width a in the x -direction and thickness h in the z -direction. The panel has free edges at $x = 0$ and $x = a$. It is subjected to a uniform axial strain (ε_0), bending (χ_0), and twisting curvature (Θ) as shown in Fig. 1. The reference xy -plane of the L -layered panel is located at the mid-surface of the panel. The layers of the panel are numbered from

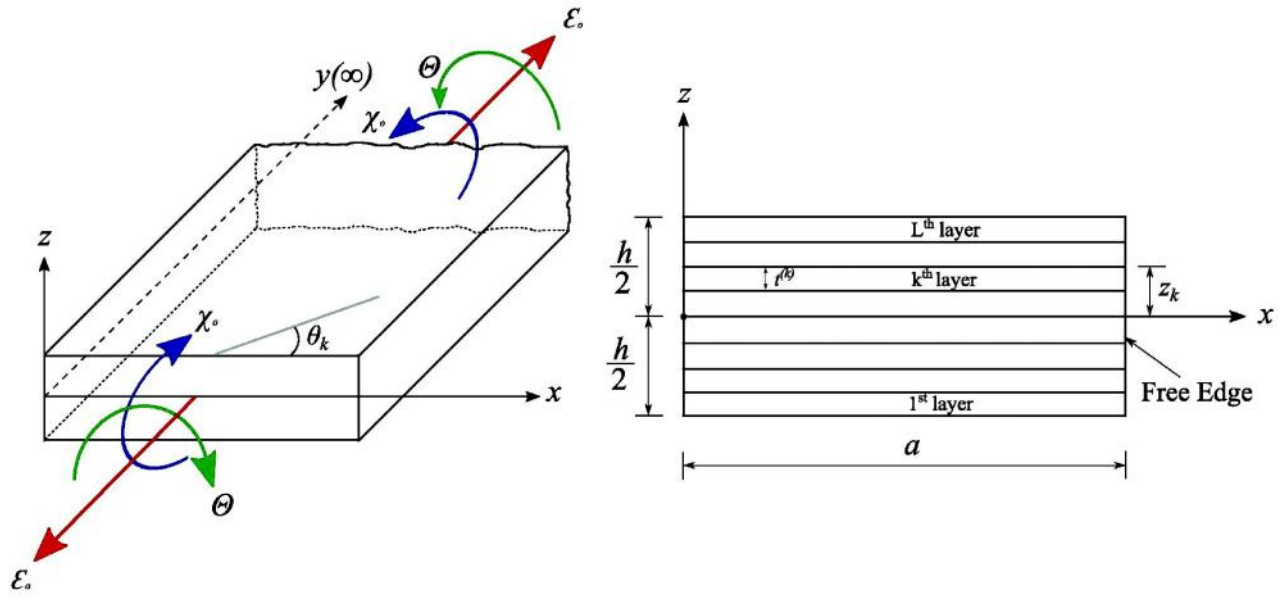


Fig. 1: Geometry of the sandwich laminated panel with free edges subjected to extension, bending and twisting loading

bottom to top, and the z -coordinate of the upper surface of the k th layer with respect to the xy -plane is denoted as z_k . The thickness of each layer can be different, and for k th layer it is denoted as $t^{(k)}$. The fibres of the unidirectional composite laminas are oriented at an angle θ with the x -axis. The principal material axis x_3 of all the layers is oriented along the z -direction.

Since the problem falls under the class of generalized plane deformation as described by Lekhnitskii (1963), the displacement field $u_i(x, y, z)$ in the laminate under extension, bending and twisting loading can be written as

$$\begin{aligned} u_1 &= u(x, z) - \Theta yz + \omega_2 z - \omega_3 y + u_0 \\ u_2 &= v(x, z) - (\epsilon_0 - \chi_0 z)y + \omega_3 x - \omega_1 z + v_0 \quad (1) \\ u_3 &= w(x, z) + \frac{\chi_0 y^2}{2} + \Theta xy + \omega_1 y - \omega_2 x + w_0 \end{aligned}$$

where u_i ($i=1, 2, 3$) are the displacement components in x, y and z direction, respectively. u, v and w are the unknown displacements, which are functions of x and z coordinates. The constants u_0, v_0, w_0 and $\omega_1, \omega_2, \omega_3$ characterize the rigid body translations and rotations of the panel, respectively.

Using the displacement field given in Eq. (1), the normal and shear strains, ϵ_i and γ_{ij} can be obtained as

$$\begin{aligned} \epsilon_x &= u_{,x}, \epsilon_y = \epsilon_0 - \chi_0 z, \epsilon_z = w_{,z}, \\ \gamma_{yz} &= v_{,z} + \Theta x, \gamma_{zx} = w_{,x} + u_{,z}, \gamma_{xy} = v_{,x} - \Theta z \quad (2) \end{aligned}$$

where a subscript comma followed by x , for example, denotes partial differentiation with respect to x .

For the k th layer, the constitutive relationship between the strain (ϵ_{ij}) and stress (σ_{ij}) components in the plate coordinate system (x, y, z) can be expressed as

$$\begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \bar{s}_{11} & \bar{s}_{12} & \bar{s}_{13} & 0 & 0 & \bar{s}_{16} \\ \bar{s}_{12} & \bar{s}_{22} & \bar{s}_{23} & 0 & 0 & \bar{s}_{26} \\ \bar{s}_{13} & \bar{s}_{23} & \bar{s}_{33} & 0 & 0 & \bar{s}_{36} \\ 0 & 0 & 0 & \bar{s}_{44} & \bar{s}_{45} & 0 \\ 0 & 0 & 0 & \bar{s}_{45} & \bar{s}_{55} & 0 \\ \bar{s}_{16} & \bar{s}_{26} & \bar{s}_{36} & 0 & 0 & \bar{s}_{66} \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{zx} \\ \tau_{xy} \end{bmatrix} \quad (3)$$

where \bar{s}_{ij} are the transformed elastic compliances, which can be expressed in terms of the engineering properties, namely, Young's moduli Y_i , shear moduli G_{ij} and major Poisson's ratios ν_{ij} (Jones, 1999). Upon

substitution of ε_y from Eq. (2) into the corresponding constitutive relation in Eq. (3), σ_y is obtained as

$$\begin{aligned} \sigma_y &= (\varepsilon_0 - \chi_0 z) / \bar{s}_{22} - (\bar{s}_{12} / \bar{s}_{22}) \sigma_x \\ &\quad - (\bar{s}_{23} / \bar{s}_{22}) \sigma_z - (\bar{s}_{26} / \bar{s}_{22}) \tau_{xy} \end{aligned} \quad (4)$$

Substituting the above expression, σ_y can be eliminated from the other constitutive relations in Eq. (3) as

$$\begin{aligned} \varepsilon_x &= p_{11} \sigma_x + p_{13} \sigma_z + p_{16} \tau_{xy} + \tilde{s}_{21} (\varepsilon_0 - \chi_0 z), \\ \varepsilon_z &= p_{31} \sigma_x + p_{33} \sigma_z + p_{36} \tau_{xy} + \tilde{s}_{23} (\varepsilon_0 - \chi_0 z) \quad (5) \\ \gamma_{xy} &= p_{61} \sigma_x + p_{63} \sigma_z + p_{66} \tau_{xy} + \tilde{s}_{26} (\varepsilon_0 - \chi_0 z) \end{aligned}$$

where $p_{ij} = \bar{s}_{ij} - \bar{s}_{2i} \bar{s}_{2j} / \bar{s}_{22}$, $\tilde{s}_{2j} = \bar{s}_{2j} / \bar{s}_{22}$, for $i, j = 1, 3, 6$. The governing equations for the free edge problem are developed using the Reissner-type variational principle (Shames and Dym, 1985) for linear elastic medium, which can be written as

$$\begin{aligned} &\int_V [\delta u (\tau_{zx,z} + \sigma_{x,x}) \\ &\quad + \delta v (\tau_{yz,z} + \tau_{xy,x}) + \delta w (\sigma_{z,z} + \tau_{zx,x}) \\ &\quad + \delta \sigma_x (\varepsilon_x - u_{,x}) + \delta \sigma_z (\varepsilon_z - w_{,z}) \\ &\quad + \delta \tau_{yz} (\gamma_{yz} - v_{,z}) \\ &\quad + \delta \tau_{zx} (\gamma_{zx} - u_{,z} - w_{,x}) \\ &\quad + \delta \tau_{xy} (\gamma_{xy} - v_{,x})] dV \\ &- \int_{A_T} (T_i^n - \bar{T}_i^n) \delta u_i dA - \int_{A_u} T_i^n \delta u_i dA = 0, \forall \delta u_i, \delta \sigma_i, \delta \tau_{ij} \end{aligned} \quad (6)$$

where V denotes the volume of the panel per unit length. The summation convention for repeated indices holds for i and j . A_T and A_u denote, respectively, the surface boundaries where tractions \bar{T}_i^n and displacements \bar{u}_i are prescribed. T_i^n are the components of the traction T_i , given by $T_i^n = \sigma_{ij} n_j$, where n_j denotes the direction cosines of the outward normal \bar{n} to the surface. The area integral terms in Eq. (6) vanish, as all the surface boundary conditions are sought to be satisfied exactly. Upon substitution of the strain field obtained from strain-displacement relations in Eq. (1) and constitutive relations in Eqs. (3) and (5), the variational statement in Eq. (6) reads

$$\int_V [\delta u (\tau_{zx,z} + \sigma_{x,x})$$

$$\begin{aligned} &+ \delta v (\tau_{yz,z} + \tau_{xy,x}) + \delta w (\sigma_{z,z} + \tau_{zx,x}) \\ &+ \delta \sigma_x (p_{11} \sigma_x + p_{13} \sigma_z + p_{16} \tau_{xy} + \tilde{s}_{21} (\varepsilon_0 - \chi_0 z) - u_{,x}) \\ &+ \delta \sigma_z (p_{31} \sigma_x + p_{33} \sigma_z + p_{36} \tau_{xy} + \tilde{s}_{23} (\varepsilon_0 - \chi_0 z) - w_{,z}) \\ &+ \delta \tau_{yz} (\bar{s}_{44} \tau_{yz} + \bar{s}_{45} \tau_{zx} - v_{,z} - \Theta x) \\ &+ \delta \tau_{zx} (\bar{s}_{45} \tau_{yz} + \bar{s}_{55} \tau_{zx} - u_{,z} - w_{,x}) \\ &+ \delta \tau_{xy} (p_{61} \sigma_x + p_{63} \sigma_z + p_{66} \tau_{xy} \\ &+ \tilde{s}_{26} (\varepsilon_0 - \chi_0 z) - v_{,x} + \Theta z)] dV = 0, \forall \delta u_i, \delta \sigma_i, \delta \tau_{ij} \end{aligned} \quad (7)$$

Boundary and Interface Conditions

The boundary conditions associated with the free edge problem considered in the present study are the traction free conditions at the bottom and top surfaces of the laminate and at the free edges, and the interlaminar continuity conditions assuming a perfect bonding between the layers at the interfaces. These conditions can be written as:

(i) traction free conditions at the bottom and top surfaces ($z = \mp h/2$):

$$\sigma_z = 0, \tau_{yz} = 0, \tau_{zx} = 0 \quad (8)$$

(ii) continuity of displacements and stresses at the interface between the k th and $(k+1)$ th layers

$$\begin{aligned} &(u, v, w, \sigma_z, \tau_{yz}, \tau_{zx})|_{z=z_k}^{(k)} \\ &= (u, v, w, \sigma_z, \tau_{yz}, \tau_{zx})|_{z=z_k}^{(k+1)} \end{aligned} \quad (9)$$

(iii) traction free conditions at the free edges at $x = 0, a$:

$$\sigma_x = 0, \tau_{zx} = 0, \tau_{xy} = 0 \quad (10)$$

MMEKM Solution of Governing Equations

The MMEKM solution considers both displacements and stresses as primary variables. The field variable vector is defined as

$$\mathbf{X} = [u \ v \ w \ \sigma_x \ \sigma_z \ \tau_{xy} \ \tau_{yz} \ \tau_{zx}]^T \quad (11)$$

The following normalized coordinates ξ and $\zeta^{(k)}$ are introduced for the in-plane and local thickness coordinate for the k th layer such that they vary from 0 to 1 for $0 \leq x \leq a$ and $z_{k-1} \leq z \leq z_k$, respectively:

$$\xi = x/a, \quad \zeta^{(k)} = (z - z_{k-1})/t^{(k)} \quad (12)$$

The solution of the field variables $X_l(\xi, \zeta)$ is expressed as an n -term series of the product of two independent functions $f_l^i(\xi)$ and $g_l^i(\zeta)$ in the in-plane and thickness direction, respectively.

$$X_l(\xi, \zeta) = \sum_{i=1}^n f_l^i(\xi) g_l^i(\zeta) \quad (13)$$

Functions $g_l^i(\zeta)$ are defined separately for each layer, whereas functions $f_l^i(\xi)$ are the same for all layers. The analytical solutions for these functions are determined iteratively, satisfying all the boundary and interlaminar conditions specified in the previous section. Each iteration process involves two basic steps, which are described below.

Step 1: Solving for Functions $g_l^i(\zeta)$

In the first step of an iteration, functions $f_l^i(\xi)$ are treated as known from the previous iteration, and the functions $g_l^i(\zeta)$ for each layer are determined. In the first iteration, the following trigonometric functions are chosen as initial guess for $f_l^i(\xi)$:

$$f_1^i(\xi) = f_2^i(\xi) = f_7^i(\xi) = f_8^i(\xi) = \cos \pi \xi$$

$$f_3^i(\xi) = f_4^i(\xi) = f_5^i(\xi) = f_6^i(\xi) = \sin \pi \xi \quad (14)$$

Unlike other approximate methods of solving PDEs (e.g. Ritz and Galerkin), the EKM does not warrant the initial functions to satisfy the prescribed boundary conditions and the selection of initial functions does not have any adverse effect on the accuracy of the final solution. Since the first step considers $f_l^i(\xi)$ as known, the variation δX_l obtained from Eq. (13) reads

$$\delta X_l = \sum_{i=1}^n f_l^i(\xi) \delta g_l^i \quad (15)$$

The unknown variables in the first step, $g_l^i(\zeta)$ for each layer are divided into two groups $\bar{\mathbf{G}}$ and $\hat{\mathbf{G}}$ as follows:

$$\bar{\mathbf{G}} = [g_{1\dots 8}^i \dots g_{1\dots 8}^n \dots g_{2\dots 8}^i \dots g_{2\dots 8}^n \dots g_{3\dots 8}^i \dots g_{3\dots 8}^n \dots g_{4\dots 8}^i \dots g_{4\dots 8}^n \dots g_{5\dots 8}^i \dots g_{5\dots 8}^n \dots g_{6\dots 8}^i \dots g_{6\dots 8}^n \dots g_{7\dots 8}^i \dots g_{7\dots 8}^n \dots g_{8\dots 8}^i \dots g_{8\dots 8}^n]^T$$

$$\hat{\mathbf{G}} = [g_{4\dots 6}^i \dots g_{4\dots 6}^n \dots g_{6\dots 6}^i \dots g_{6\dots 6}^n]^T \quad (16)$$

where $\bar{\mathbf{G}}$ contains those displacements and stress components appearing in free edge boundary and interlaminar conditions [Eqs. (8) and (9)], and $\hat{\mathbf{G}}$ which contains the remaining two stress components, which are the dependent variables. Now, substitute X_l and its variation δX_l from (13) and (15) into the variational equation (7), and perform integration over the ξ -direction. Since the variations δg_l^i are arbitrary, the coefficients of δg_l^i in the integral must vanish individually, which results in the following first order differential-algebraic system of equations for each layer

$$\mathbf{M}\bar{\mathbf{G}}_{,\zeta} = \bar{\mathbf{A}}\bar{\mathbf{G}} + \hat{\mathbf{A}}\hat{\mathbf{G}} + \bar{\mathbf{Q}} \quad (17)$$

$$\mathbf{K}\hat{\mathbf{G}} = \tilde{\mathbf{A}}\bar{\mathbf{G}} + \tilde{\mathbf{Q}} \quad (18)$$

where matrices $\mathbf{M}, \bar{\mathbf{A}}, \hat{\mathbf{A}}, \mathbf{K}$ and $\tilde{\mathbf{A}}$ are of size $6n \times 6n$, $6n \times 6n$, $6n \times 2n$, $2n \times 2n$ and $2n \times 6n$, respectively. The nonzero elements of these matrices are identical to those presented in Dhanesh *et al.* (2016), and are omitted here for brevity. $\bar{\mathbf{Q}}, \tilde{\mathbf{Q}}$ are the load vectors of size $6n$ and $2n$, respectively. The nonzero elements of these load vectors are defined using the notation,

$\langle \dots \rangle_a = a \int_0^1 (\dots) d\xi$, which denotes the integration over the span length a :

$$\bar{Q}_{i2} = -\Theta a t \langle \xi f_7^i \rangle_a, \bar{Q}_{i3} = t \tilde{s}_{23} \langle f_5^i \rangle_a (\bar{\epsilon}_0^k - \chi_0 t \zeta),$$

$$\tilde{Q}_{i1} = -\tilde{s}_{21} \langle f_4^i \rangle_a (\bar{\epsilon}_0^k - \chi_0 t \zeta), \quad (19)$$

$$\tilde{Q}_{i2} = -\tilde{s}_{26} \langle f_6^i \rangle_a (\bar{\epsilon}_0^k - \chi_0 t \zeta) - \Theta \langle f_6^i \rangle_a (z_{k-1} + t \zeta)$$

where $\bar{\varepsilon}_0^{-k} = \varepsilon_0 - \chi_0 z_{k-1}$. All integrals appearing in the elements of the matrices in Eqs. (17)-(19) are evaluated exactly in closed form. The following $6n$ first order ODEs in ζ are obtained after eliminating $\hat{\mathbf{G}}$ (obtained from Eq. (18)) from Eq. (17)

$$\bar{\mathbf{G}} = \mathbf{A}\bar{\mathbf{G}} + \mathbf{Q} \quad (20)$$

where

$$\mathbf{A} = \mathbf{M}^{-1}[\bar{\mathbf{A}} + \hat{\mathbf{A}}\mathbf{K}^{-1}\tilde{\mathbf{A}}] \text{ and } \mathbf{Q} = \mathbf{M}^{-1}[\bar{\mathbf{Q}} + \hat{\mathbf{A}}\mathbf{K}^{-1}\tilde{\mathbf{Q}}]$$

The solution of the above first order ODEs can be obtained analytically in closed form, and the general solution can be expressed in terms of $6n$ real constants

$C_i^{(k)}$ as

$$\bar{\mathbf{G}}(\zeta) = \sum_{i=1}^{6n} \mathbf{F}_i(\zeta) C_i^{(k)} + \mathbf{U}_0 + \zeta \mathbf{U}_1 \quad (21)$$

where the elements of the column vector $F_i(\zeta)$ are expressed using the exponential and trigonometric functions of ζ in terms of the eigenvalues and eigenvectors of \mathbf{A} . \mathbf{U}_0 and \mathbf{U}_1 are the particular solution vectors corresponding to the constant and linear loading terms of \mathbf{Q} . The detailed solution procedure is omitted here for brevity and the same can be found in Kapuria and Kumari (2011). The $6n \times L$ unknown constants $C_i^{(k)}$'s for L layers are determined from the following $6n$ surface boundary and $6n \times (L-1)$ interface conditions, obtained from Eqs. (8) and (9):

$$\begin{aligned} \text{for } k=1, \text{ at } \zeta=0: g_5^i &= 0, g_7^i = 0, g_8^i = 0, \\ \text{for } k=L, \text{ at } \zeta=1: g_5^i &= 0, g_7^i = 0, g_8^i = 0 \quad (22) \\ (g_1^i, g_2^i, g_3^i, g_5^i, g_7^i, g_8^i)|_{\zeta=1}^{(k)} \\ &= (g_1^i, g_2^i, g_3^i, g_5^i, g_7^i, g_8^i)|_{\zeta=1}^{(k+1)} \quad (23) \end{aligned}$$

for $k=1, 2, \dots, (L-1)$ and $i=1, 2, \dots, n$. The solution $\bar{\mathbf{G}}(\zeta)$ obtained after solving the system of ODEs is now substituted back into Eq. (18) to yield $\hat{\mathbf{G}}(\zeta)$.

This completes the determination of functions $g_i^i(\zeta)$ for all the L layers, and concludes the first step in an iteration process.

Step 2: Solving for Functions $f_l^i(\xi)$

In the second step of iteration, the solution for functions in the in-plane direction, $f_l^i(\xi)$ is obtained. Here, the solution for $g_l^i(\zeta)$ obtained in the previous step is considered as known, hence the variation δX_l obtained from Eq. (13) reads

$$\delta X_l = \sum_{i=1}^n g_l^i(\zeta) \delta f_l^i \quad (24)$$

Similar to $g_l^i(\zeta)$, functions $f_l^i(\xi)$ also divided into two groups $\bar{\mathbf{F}}$ and $\hat{\mathbf{F}}$ as follows:

$$\begin{aligned} \bar{\mathbf{F}} &= [f_{1\dots}^i, f_{1\dots}^n, f_{2\dots}^i, f_{2\dots}^n, f_{3\dots}^i, f_{3\dots}^n, f_{4\dots}^i, f_{4\dots}^n, f_{5\dots}^i, f_{5\dots}^n, f_{6\dots}^i, f_{6\dots}^n, f_{7\dots}^i, f_{7\dots}^n, f_{8\dots}^i, f_{8\dots}^n]^T \\ \hat{\mathbf{F}} &= [f_{5\dots}^i, f_{5\dots}^n, f_{7\dots}^i, f_{7\dots}^n]^T \quad (25) \end{aligned}$$

Now, substitute equations (13) and (24) into the variational equation (7), and perform integration over the thickness direction ζ , as $g_l^i(\zeta)$ are known. The coefficients of δf_l^i in the resulting expression are individually equated to zero, since the variations are arbitrary. This yields the following $8n$ differential-algebraic equations for unknowns f_l^i :

$$\mathbf{N}\bar{\mathbf{F}}_{,\xi} = \bar{\mathbf{B}}\bar{\mathbf{F}} + \hat{\mathbf{B}}\hat{\mathbf{F}} + \bar{\mathbf{P}} \quad (26)$$

$$\mathbf{L}\hat{\mathbf{F}} = \hat{\mathbf{B}}(\xi)\bar{\mathbf{F}} + \tilde{\mathbf{P}} \quad (27)$$

where $\mathbf{N}, \bar{\mathbf{B}}, \hat{\mathbf{B}}, \mathbf{L}$ and $\tilde{\mathbf{B}}$ are matrices of size $6n \times 6n$, $6n \times 6n$, $6n \times 2n$, $2n \times 2n$ and $2n \times 6n$, respectively. The nonzero elements of these matrices are as given in Dhanesh *et al.* (2016), and are omitted here for brevity. $\bar{\mathbf{P}}$ and $\tilde{\mathbf{P}}$ represent the load vectors of size $6n$ and $2n$, and their nonzero elements obtained as

$$\begin{aligned} \bar{P}_{i1} &= \varepsilon_0 \langle \tilde{s}_{21} g_4^i \rangle_h - \chi_0 \langle \tilde{s}_{21} (z_{k-1} + t\zeta) g_4^i \rangle_h \\ \bar{P}_{i2} &= \varepsilon_0 \langle \tilde{s}_{26} g_6^i \rangle_h - \chi_0 \langle \tilde{s}_{26} (z_{k-1} + t\zeta) g_6^i \rangle_h \\ &\quad + \Theta \langle (z_{k-1} + t\zeta) g_6^i \rangle_h \\ \tilde{P}_{i1} &= -\varepsilon_0 \langle \tilde{s}_{23} g_5^i \rangle_h + \chi_0 \langle \tilde{s}_{23} (z_{k-1} + t\zeta) g_5^i \rangle_h \end{aligned}$$

$$\tilde{P}_{i2} = \Theta a \zeta \langle g_7^i \rangle_h \quad (28)$$

where the notation $\langle \dots \rangle_h = \sum_{k=1}^L t^{(k)} \int_0^1 (\dots)^{(k)} d\zeta$ represents the integration across the thickness of the laminate. Similar to the first step, all integrals in nonzero terms of the above matrices are evaluated exactly in closed form. Since the resulting differential-algebraic system of equations in this step ((26) and (27)) are of same nature as in the previous step ((17) and (18)), the solution procedure remains the same. The unknown constants are obtained from the free edge boundary conditions obtained from Eq. (10), written in terms of f_i^i as

$$f_8^i = 0, f_4^i = 0, f_6^i = 0, i = 1, 2, \dots, n$$

The solution of $\bar{\mathbf{F}}(\xi)$ and $\hat{\mathbf{F}}(\xi)$ completes the second step, and one iteration in the solution process.

The steps for computing $g_i^i(\zeta)$ and $f_i^i(\xi)$ are repeated until desired level of convergence is achieved. For a particular problem, the convergence study is performed by obtaining the solution using different number of terms (n) in the solution approximation. From the previous studies on composite laminates, it has been observed that the solution for each term converges within two to three iterations, and in most of the problems, an accurate solution can be obtained with five to six terms. The convergence of the MMEKM solution for free edge problems in sandwich laminate will be verified in the numerical results section.

Numerical Results and Discussions

Numerical results are presented for the free edge stress field in soft-core sandwich panels with the lay-up configurations as shown in Fig. 2. Five-layer sandwich panels having a thick central core and two

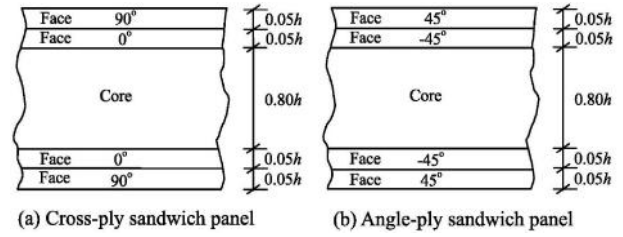


Fig. 2: Lay-ups of sandwich panel

thin composite face sheets at its bottom and top with cross-ply $[90/0/\text{core}]_s$ and angle-ply $[45/-45/\text{core}]_s$ lay-ups are considered. The material properties of the soft-core and face sheet layers are selected from Kapuria and Achary (2006) and are presented in Table 1. The span-to-thickness ratio of the panel considered is $S = a/h = 5$. The numerical results are presented for the extension (ε_0), bending (χ_0) and twisting (Θ) load cases. The results are normalized with respect to the corresponding load as follows:

Extension:

$$(\bar{\sigma}_z, \bar{\tau}_{yz}, \bar{\tau}_{zx}) = (\sigma_z, \tau_{yz}, \tau_{zx}) S / Y_2 \varepsilon_0$$

Bending:

$$(\bar{\sigma}_z, \bar{\tau}_{yz}, \bar{\tau}_{zx}) = (\sigma_z, \tau_{yz}, \tau_{zx}) S^2 / a Y_2 \chi_0$$

Twisting:

$$(\bar{\sigma}_z, \bar{\tau}_{yz}, \bar{\tau}_{zx}, \bar{\tau}_{xy}) = (\sigma_z, \tau_{yz}, \tau_{zx}, \tau_{xy}) S / a Y_2 \Theta$$

A dimensionless global thicknesses coordinate $\bar{\zeta} = z/h$ varying from -0.5 to 0.5 is introduced to present the through-thickness distributions of stresses.

Uniform Extension

First, the cross-ply sandwich panel shown in Fig. 2(a) under unit axial strain is considered. Fig. 3 shows the longitudinal distributions of the interlaminar transverse

Table 1: Material Properties

Material	Y_1	Y_2	Y_3	G_{12}	G_{23}	G_{31}	ν_{12}	ν_{13}	ν_{23}
	(GPa)								
Face ¹	131.0	6.9	6.9	3.588	2.3322	3.588	0.32	0.32	0.49
Core ¹	0.2208	0.2001	2.76	0.01656	0.4554	0.5451	0.99	3×10^{-5}	3×10^{-5}

¹Kapurria and Achary (2006)

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