

Research Paper

Scaling Laws of Rough Turbulent Flows from Turbulence Phenomenology: An Overview and a New Application

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In this paper, the scaling laws of rough turbulent flows stemming from the phenomenological theory of turbulence are reviewed. It is revealed that the scaling of the well-known empirical formulae can also be obtained by means of the phenomenological theory. The phenomenological theory thus enables to provide an insight into the physics of the flows of the problems arising from turbulence. As a new thought, the famous scaling laws of the velocity distributions, the logarithmic law and the power law of the wall, are also sought applying the same theory.

Keywords: Fluid Mechanics; Hydrodynamics; Turbulence; Turbulent Flow

Introduction

Turbulence, the unrest phenomenon of fluid in nature, is believed to be one of the most challenging and intricate aspects of fluid mechanics and applied hydrodynamics. Although considerable endeavors were made to study the behavioural phenomena of turbulent flows over rough boundaries (Raupach *et al.*, 1991; Jiménez, 2004), the turbulence phenomena remain the least understood aspects in fluid mechanics. In many physical problems concerning the rough turbulent flows, researchers are interested to seek different scaling laws. In doing so, several empirical and semi-empirical laws based on the experimental data were proposed. However, there remains an important question that to what extent these empirical or semi-empirical laws are legitimate from the perspective of turbulence. This question can now be answered by applying the phenomenological theory of turbulence.

The phenomenological theory of turbulence originated from Kolmogorov's (1941) philosophical thoughts on the scaling laws of fully developed steady state turbulence. Although Kolmogorov's original ideas were based on unconfirmed, but plausible doctrines of the homogeneous and isotropic turbulence, several

studies after Kolmogorov's (1941) theory revealed that the Kolmogorov's scaling laws are still valid for the non-homogeneous and anisotropic turbulent flows (Knight and Sirovich, 1990; Moser, 1993; Saddoughi and Veeravalli, 1994; Gioia and Chakraborty, 2006). Frisch (1995) defined the phenomenology of turbulence as a shorthand system that allows to obtain the scaling laws of fully developed steady state turbulence in a much simpler way at the price of less heuristic arguments. Another advantage of the phenomenological theory is that it associates the dependent and the independent variables by means of turbulence scaling laws, in which each of the independent variables possesses a single unique free exponent. In other words, the phenomenological theory controls the degrees of freedom of the free exponents. However, in case of empirical or semi-empirical laws describing the turbulent flow phenomena, the free exponents of the independent variables are determined from the regression analyses using the experimental data. As a consequence, when the empirical or semi-empirical laws proposed by different researchers are applied to analyze the same problem, discrepancies in results among the different formulae are often found. Although the phenomenological theory was primarily used to model the turbulent flows (Frisch,

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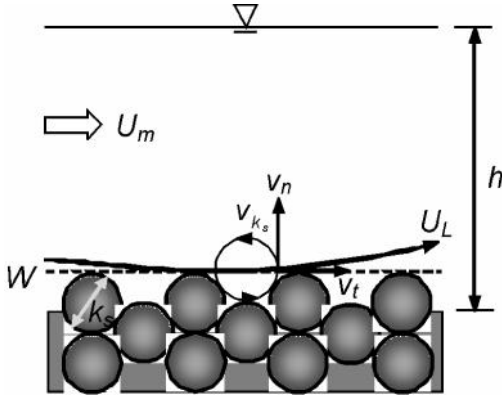


Fig. 1: Schematic of rough turbulent flow in an open channel

specifically the hydraulic radius R_b . If v_l is the characteristic velocity scale that corresponds to l , the circulation time t_l is $t_l \sim l/v_l$, where the symbol \sim denotes the scales with. Subsequently, the energy flux E_l transferred from the scales l to the smaller ones is given by $E_l \sim v_l^2/t_l \sim v_l^3/l$. Since in the inertial subrange, the TKE production rate and the TKE dissipation rate $\delta\epsilon$ balance each other, one can obtain $E_l = \delta\epsilon \delta l^3/l$. However, for $l \sim L$, it yields $\delta\epsilon \delta U_L^3/L$, where U_L is the velocity scale corresponding to L . It may be noted that the U_L can be scaled with the external velocity scale, for example, the mean flow velocity U_m . Thus, the above phenomenological consideration yields $v_l \sim U_L (l/L)^{1/3} \sim U_m (l/R_b)^{1/3}$.

The Reynolds shear stress $\delta\tau$ developed due to the localized eddies in the immediate vicinity of the boundary at an imaginary wetted surface W tangential to the summit of the boundary roughness elements is $\delta\tau = \overline{\delta v_t v_n}$ (Pope, 2000), where v_t and v_n are the velocity fluctuations tangential and normal to W , respectively and the overbar denotes the time-averaged value. Using the scaling law $v_l \sim U_m (l/R_b)^{1/3}$, the velocity scale of the localized eddies ($l \sim k_s$) follows:

$$v_l \delta v_{k_s} \sim \delta U_m \frac{\partial k_s}{\partial R_b}^{1/3} \frac{\partial \epsilon}{\partial \phi} \quad (4)$$

To determine v_p , Gioia and Bombardelli (2002) considered $v_t \sim U_L \sim U_m$, since the eddies of all sizes can induce shear tangential to W . However, $v_n \sim v_{k_s}$ because the smaller eddies ($l \sim k_s$) provide substantial

contribution to the momentum flux normal to W , while the larger eddies ($l \gg k_s$) contribute feebly. Thus,

$$\delta\tau \delta v_{k_s} \sim \delta U_m v_{k_s} \delta v_{k_s} \sim \delta U_m^2 \frac{\partial k_s}{\partial R_b}^{1/3} \frac{\partial \epsilon}{\partial \phi} \quad (5)$$

The streamwise component of force per unit length acting on the imaginary wetted surface is $F_t = \delta\tau (b + 2h)$. Applying the momentum balance, one obtains $F_t = F_g$. Therefore,

$$\delta\tau U_m^2 \frac{\partial k_s}{\partial R_b}^{1/3} \frac{\partial \epsilon}{\partial \phi} (b + 2h) \sim \delta\tau g S_0 b h \delta P$$

$$U_m \sim \sqrt{R_b g S_0} \frac{\partial k_s}{\partial R_b}^{1/6} \frac{\partial \epsilon}{\partial \phi} \quad (6)$$

Thus, Eq. (6) resembles to the foremost term of Eq. (3) with $\delta b = 1/6$. This completes the derivation.

In this context, it is pertinent to mention that if the boundary roughness height k_s reduces, the mean velocity U_m increases due to the reduction in the momentum transfer. Subsequently, when the boundary roughness height k_s approaches the Kolmogorov dissipation length scale $\delta l(k_s \delta \epsilon) / \delta$ using $\delta\epsilon \delta U_m^3/R_b$, one obtains

$$\frac{\delta h}{R_b} \delta \text{Re}^{\delta 3/4} \quad (7)$$

where $\text{Re} (= U_m R_b / \delta \nu)$ is the Reynolds number. Therefore, Eq. (7) reduces to

$$U_m \sim \sqrt{R_b g S_0} \text{Re}^{1/8} \quad (8)$$

The presence of Re in Eq. (8) reveals that the momentum transfer is viscous in the limit $k_s \delta \epsilon \delta h$. Introducing the resistance coefficient $f = R_b g S_0 / U_m^2$, one can rewrite Eq. (8) as

$$f \delta \text{Re}^{\delta 1/4} \quad (9)$$

which is the well-known empirical scaling of Blasius for hydraulically smooth channels (Dey, 2014).

The Scaling Law of Scour Depth Caused by Localized Turbulent Flows

When a turbulent jet impinges into a pool having a uniform flow depth, a turbulent cauldron is formed under the action of the jet. Due to the energetics of the jet, the turbulent cauldron starts to scour the bed to form a pothole as depicted in Fig. 2. The depth of the pothole increases until a dynamic equilibrium is established between the sediment bed and the turbulent cauldron. For such a problem, researchers are interested to find the equilibrium depth of the pothole for a given power of the jet. An extensive survey of empirical equations for the equilibrium depth of the pothole was done by Dey (2014). These empirical equations can be written in a generalized form as

$$D = Cq^{\alpha_q} h_0^{\alpha_{h_0}} g^{\alpha_g} d^{\alpha_d} \left(\frac{\rho_f}{\rho_s - \rho_f} \right)^{\alpha_\rho} \quad (10)$$

where D is the sum of the depth of the pothole d_s and the depth of the pool h , that is $D = d_s + h$, q is the discharge of the jet per unit thickness, h_0 is the head of the jet, d is the characteristic size of the sediment particles, ρ_s is the mass density of sediment particles and α_q , α_{h_0} , α_g , α_d and α_ρ are the empirical exponents.

Gioia and Bombardelli (2005) and Bombardelli and Gioia (2006) studied the local scour due to turbulent jet applying the phenomenology of turbulence. Before predicting the equilibrium scour

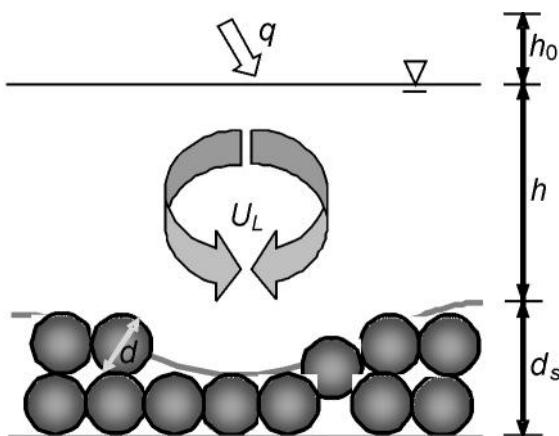


Fig. 2: Schematic of local scour due to turbulent jet

depth, it is appropriate to examine to what extent Eq. (10) can be obtained from the dimensional analysis and similarity methods. To start, six plausible variables D , ρ_f , g , ρ_s , d and P are considered, where P is the power of the jet per unit thickness. Thus, in functional form, the equation $f(D, \rho_f, g, \rho_s, d, P) = 0$ exists. Applying the Buckingham Π theorem, one obtains three Π parameters: $\Pi_1 = P/(\rho_f g^{3/2} D^{5/2})$, $\Pi_2 = \rho_s/\rho_f$ and $\Pi_3 = d/D$. Hence, one can write

$$P = \rho_f g^{3/2} D^{5/2} f_2 \left(\frac{d}{D}, \frac{\rho_s}{\rho_f} \right) \quad (11)$$

In case of a complete similarity, $f(d/D)$ asymptotically reaches a constant value as the value of d/D tends to zero. For such a scenario, D is independent of d for $d/D \ll 1$. However, this observation is contrary to the measurements since the values of the empirical exponent α_d proposed so far are different from zero. On the other hand, in case of an incomplete similarity, Eq. (11) can be written as

$$P = \rho_f g^{3/2} D^{5/2} \left(\frac{d}{D} \right)^\beta f_3 \left[\frac{\rho_s}{\rho_f} + O \left(\frac{d}{D} \right)^\beta \right] \quad (12)$$

Substituting the foremost term of Eq. (12) in Eq. (11) and rearranging yield

$$D = Cq^{\alpha_q} h_0^{\alpha_{h_0}} g^{\alpha_g} d^{\alpha_d} f_4 \left(\frac{\rho_s}{\rho_f} \right) \quad (13)$$

where $\alpha_q = \alpha_{h_0} = 2/(5 - 2\beta)$, $\alpha_g = -1/(5 - 2\beta)$, $\alpha_d = -2\alpha/(5 - 2\beta)$ and $f_4 = 1/[C f_3^{2/(5-2\beta)}]$. It may be noted that Eq. (13) has the similar form as Eq. (10). Moreover, the empirical exponents are related to each other via a single free parameter α , which can be determined from the phenomenology of turbulence.

From the phenomenological perspective as already derived before, one obtains $v_l \sim U_L(l/L)^{1/3}$. Considering $L \sim D$, yields $\varepsilon \sim U_L^3/D$. The power of the jet per unit thickness is $P = q\rho_f g h_0$. Further, P

can be expressed as $P = \varepsilon M$, where M is the mass per unit thickness of the turbulent cauldron. The M can be scaled as $M \sim \rho_f D^2$. Thus, one obtains

$$U_L \sim \left(qg \frac{h_0}{D} \right)^{1/3} \quad (14)$$

The Reynolds shear stress τ_t developed due to the localized eddies at an imaginary wetted surface W tangential to the summit of the sediment particles at the surface of the pothole is $\tau_t = \rho_f \overline{v_l v_n} \sim \rho_f U_L v_l$. The velocity scale of the localized eddies ($l \sim d$) follows: $v_l \sim v_d \sim U_L (d/D)^{1/3}$. Thus, using Eq. (14) yields

$$\tau_t \sim \rho_f \frac{(qgh_0)^{2/3} d^{1/3}}{D} \quad (15)$$

When the depth d_s of the pothole increases, the size D of the turbulent cauldron increases at the same time. As a consequence, the boundary shear stress exerted on the pothole surface reduces due to the reduced velocity resulting from the increased cross-sectional area of flow. Eventually, the scour ceases when the boundary shear stress becomes equal to the threshold boundary shear stress for the incipient motion of sediment particles. The threshold boundary shear stress τ_c for the hydraulically rough flow regime follows: $\tau_c \sim (\rho_s - \rho_f)gd$. Thus, substituting Eq. (15) and $\tau_c \sim (\rho_s - \rho_f)gd$ into $\tau_t \sim \tau_c$ yields

$$D \sim q^{2/3} h_0^{2/3} g^{-1/3} d^{-2/3} \left(\frac{\rho_f}{\rho_s - \rho_f} \right) \quad (16)$$

Equation (16) represents the complete theoretical expression for the depth of the pothole as $d_s = D - h$. Comparing Eqs. (16) and (13) reveals $\alpha_q = \alpha_{h_0} = 2/3$, $\alpha_g = -1/3$, $\alpha_d = -2/3$ with the similarity exponent $\beta = 1$.

Manes and Brocchini (2015) also applied the phenomenology of turbulence to study the local scour around a solid cylinder embedded in a sediment bed. Applying the same concept as discussed, they determined the scaling law of the equilibrium scour depth as a function of flow properties, sediment size

and cylinder diameter. From dimensional analysis, the mass of the largest eddy is expressed as $M \sim \rho_f L^3$. The TKE dissipation rate ε is obtained as the power P associated with the largest eddy divided by the fluid mass M contained within the characteristic volume. The power P is obtained as the work done by a drag force F_D on the cylinder against the mean flow. Hence, the power $P = F_D U_m$. The drag force F_D is computed as $F_D = 0.5 \rho_f C_D a L U_m^2$, where C_D is the drag coefficient, a is the diameter of cylinder and the product aL represents the frontal area of the cylinder exposed to scouring. Thus

$$\varepsilon = \frac{P}{M} \sim \frac{C_D a U_m^3}{L^2} \sim \frac{U_L^3}{L} \Rightarrow U_L \sim U_m \left(\frac{C_D a}{L} \right)^{1/3} \quad (17)$$

Manes and Brocchini (2015) considered that the length scale L of the largest eddy can be scaled with the equilibrium scour depth d_s ($L \sim d_s$). Therefore, the Reynolds shear stress τ_t is obtained as

$$\tau_t \sim \rho_f U_L^2 \left(\frac{d}{L} \right)^{1/3} \sim U_m^2 \left(\frac{C_D a}{d_s} \right)^{2/3} d_s^{1/3} \quad (18)$$

At an equilibrium condition, the Reynolds shear stress τ_t reaches the threshold boundary shear stress τ_c [$\sim (\rho_s - \rho_f)gd$]. Thus, substituting Eq. (18) and $\tau_c \sim (\rho_s - \rho_f)gd$ into $\tau_t \sim \tau_c$ yields

$$\frac{d_s g}{U_m^2} \sim \left(\frac{\rho_f}{\rho_s - \rho_f} \right) C_D^{2/3} \left(\frac{a}{d} \right)^{2/3} \quad (19)$$

New Application of Turbulence Phenomenology

The Logarithmic Law of the Wall

The logarithmic law of the wall, commonly called the log-law, is widely used to describe the velocity distribution for the turbulent wall bounded flows and believed to be one of the marvelous contributions toward modern fluid dynamics (Bradshaw and Huang, 1995). The genesis of log-law was due to von Kármán (Schlichting, 1979), who invented this law from the idea of constant Reynolds shear stress in the turbulent wall shear layer. The log-law states that the local time-averaged flow velocity \bar{u} (scaled by the friction

velocity u_*) at a specific point is proportional to the logarithm of the vertical distance z (scaled by the zero velocity level z_0) of that point from the wall. In mathematical form, it is expressed as $u^+ \sim \ln z^+$, where u^+ is \bar{u}/u_* and z^+ is z/z_0 . The above phenomenological constant is denoted by κ^{-1} , where κ is the von Kármán constant, which is believed to be a universal constant in flow over a solid wall (Dey, 2014). However, the non-universal behaviour of log-law due to the deviation of κ from its universal value was also reported in literature (Bradshaw and Huang, 1995; Panton, 2005; George, 2007; Dey, 2014). Here, we show that the existence of log-law in the turbulent wall shear layer from the phenomenology of near-wall turbulence.

The characteristics of the localized eddy in close proximity of the wall are governed by two key parameters: The local stress s_l ($\sim d\bar{u}/dz$) and its gradient \dot{s}_l , where dot denotes the derivative with respect to z . We also consider that the tangential and normal velocity fluctuations are of the same order ($v_t \sim v_n$) within the turbulent wall shear layer (Panton, 2005). Thus, in functional forms, one can write $l \sim F(s_l, \dot{s}_l)$ and v_t ($\sim v_n$) $\sim G(s_l, \dot{s}_l)$. Simple dimensional analysis shows that $l \sim s_l \dot{s}_l^{-1}$ and u_t ($\sim u_n$) $\sim s_l^2 \dot{s}_l^{-1}$. Therefore, the Reynolds shear stress τ_t developed within the turbulent wall shear layer is $\tau_t = \rho_f \overline{v_t v_n} \sim \rho_f s_l^4 \dot{s}_l^{-2}$. The TKE production rate t_p is expressed as $t_p = (\tau/\rho_f) \times s_l \sim s_l^5 \dot{s}_l^{-2}$, while the TKE dissipation rate ε in the turbulent wall shear layer is $\varepsilon \sim u_*^3/l \sim u_*^3 s_l^{-1} \dot{s}_l$. In the turbulent wall shear layer, the TKE production rate approximately equals the TKE dissipation rate ($t_p \approx \varepsilon$), yielding $s_l^5 \dot{s}_l^{-2} \sim u_*^3 s_l^{-1} \dot{s}_l$. Therefore, one obtains $s_l^2 \sim u_* \dot{s}_l$, and the integration of which results in $s_l \sim u_*/z$ (neglecting the sign of s_l). Integrating again, one gets $u^+ \sim \ln z^+$, which is the form of *the log-law of the wall*.

The Power Law of the Wall

The power law of the wall, commonly called the power-law, is also widely used to describe the velocity

distribution for the turbulent wall bounded flows and is regarded as a good approximation of the log-law over the entire boundary layer thickness (Schlichting, 1979). The power-law of velocity distribution states that the local time-averaged flow velocity \bar{u} (scaled by the maximum flow velocity U_{max} or flow velocity at the edge of the boundary layer) is proportional to the $1/m$ -th power of the vertical distance z (scaled by the boundary layer thickness δ). In mathematical form, it is expressed as $\hat{u} = \hat{z}^{1/m}$, where \hat{u} is \bar{u}/U_{max} , \hat{z} is z/δ and m is an exponent of the power-law. To seek the origin of the power-law, a turbulent wall bounded flow having a mean flow velocity U_m is considered. The length scale L and velocity scale U_L of the largest eddy can be scaled with δ and U_m , respectively. At the extremity of the inertial subrange ($l \sim L$), one gets $\varepsilon \sim U_m^3/\delta$ and $v_l \sim U_m(l/\delta)^{1/3}$. To determine v_t at any vertical distance z , it can be assumed that $v_t \sim U_L \sim U_m$ since the larger eddies have a substantial contribution to the momentum flux along the flow direction. However, conceptually $v_n \sim v_l$ since the smaller eddies have a considerable contribution to the momentum flux normal to the flow direction, while larger eddies have a weak contribution. Thus, at any z , one obtains $\tau_t \sim U_m v_l \sim U_m^2(l/\delta)^{1/3}$. Furthermore, the mean flow velocity U_m can be scaled with the maximum flow velocity U_{max} ($U_m \sim U_{max}$). Thus, one can write $\tau_t \sim U_{max}^2 (l/\delta)^{1/3} \sim s_l^4 \dot{s}_l^{-2}$ (as already obtained from the preceding section). Using $l \sim s_l \dot{s}_l^{-1}$ the above phenomenological relationship yields $\dot{s}_l \sim s_l^{11/5} \delta^{1/5} U_{max}^{-6/5}$, and integration of which produces $s_l \sim U_{max} \delta^{-1/6} z^{-5/6}$ (neglecting the sign of s_l). Integrating again, one obtains $\hat{u} \sim \hat{z}^{1/6}$. This shows the origin of *the power law* of velocity distribution with $m = 6$. Measurements showed that the value of m typically varies from 6 to 10 depending on the flow Reynolds numbers (Schlichting, 1979). The present phenomenological study predicts $m = 6$, which is the lower limit of the predicted values.

Conclusions

The scaling laws of rough turbulent flows obtained from the phenomenological theory of turbulence are

reviewed. A close agreement between these scaling laws and the empirical laws reveals that the phenomenological theory provides an easy but scientific means to find the scaling laws of rough turbulent flows. The theory is capable to capture the essential features of the physics of the flows in the context of the momentum transfer. In addition, the new scaling laws of the velocity distributions (*the logarithmic law* and *the power law of the wall*) are explored by applying the same theory.

Notation

a	diameter of cylinder	q	discharge of jet per unit thickness
b	channel width	R_b	hydraulic radius
C_D	drag coefficient	Re	Reynolds number ($= U_m R_b / \nu$)
D	sum of depth of pothole and depth of pool ($= d_s + h$)	S_0	bottom slope of channel
d	characteristic size of sediment particles	s_l	local stress
d_s	depth of pothole	\dot{s}_l	gradient of s_l
E_l	energy flux	t_l	circulation time
F_g	dynamic component of gravitational force per unit length of channel	t_p	TKE production rate
F_t	streamwise component of force per unit length of channel	U_L	velocity scale corresponding to L
f	resistance coefficient	U_m	mean velocity
g	acceleration due to gravity	U_{max}	maximum velocity
h	flow depth	u_*	friction velocity
h_0	head of jet	\bar{u}	time-averaged flow velocity
k_s	boundary roughness height	u^+	\bar{u} / u_*
L	integral length scale	\hat{u}	\bar{u} / U_{max}
l	length scale lying between L and h	v_l	velocity scale corresponding to l
M	fluid mass	v_n	velocity fluctuations normal to W
m	exponent of the power law	v_t	velocity fluctuations tangential to W
n	Manning roughness coefficient	W	wetted surface
P	power of fluid	z	vertical distance
		z_0	zero velocity level
		z^+	z / z_0
		\hat{z}	z / δ
		δ	boundary layer thickness
		ε	TKE dissipation rate
		η	Kolmogorov length scale
		κ	von Kármán constant
		ρ_f	mass density of fluid
		ρ_s	mass density of sediment particles

τ_c threshold boundary shear stress

τ_t Reynolds shear stress

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