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Research Paper

Filamentation Instability of Electromagnetic Beams In Nonlinear Media: A Tutorial Review

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This paper is a tutorial presentation of spatial growth of a transverse instability, associated with the propagation of an electromagnetic beam, with uniform or Gaussian irradiance along the wavefront. There are two approaches to the study of filamentation in a plasma. The results of the two approaches have been expressed in a form where they can be compared. It has been noted that the growth of the instability in the first approach is equivalent to the self-focusing of a ripple in the second approach. The dependence of the maximum growth rate and the corresponding optimum value of the wave number of the instability on the irradiance of the main beam has also been studied. Further a paraxial like approach has also been adopted to analyze the characteristics of propagation of aripple, when the dielectric function is determined by the composite (Gaussian and ripple) electric field profile of the beam. The effect of different parameters on the critical curves has been highlighted and the variation of the beam width parameter with the distance of propagation has been obtained for three typical cases viz of steady divergence, oscillatory divergence and self-focusing of the ripple.

Keywords: Filamentation Instability; Ring Ripple; Self-focusing

Introduction

There has been considerable interest in the plasma instability, associated with the propagation of a high power electromagnetic beam. A nonlinear medium is susceptible to filamentation instability, which is characterized by growing electron density and irradiance fluctuations, transverse to the direction of propagation of the beam. There are two complementary approaches to the study of the filamentation instability in a plasma, as discussed by Sodha and Sharma, 2007.

In the first usual approach (Askaryan,1962; Talanov, 1966; Hora, 1967; Palmer, 1971; Kaw *et al.*, 1973; Max *et al.*, 1974; Drake *et al.*, 1974; Mannheimer and Ott, 1974; Perkins and Valeo, 1974; Yu *et al.*, 1974; Chen, 1974; Sodha *et al.*, 1976a; Sodha *et al.*, 1976b; Bingham and Lashmore, 1976; Sodha and Tripathi, 1977; Sodha and Sharma, 2007; Gurevich, 1978; Perkins and Goldman, 1981; Kruer *et al.*, 1985; Epperlein, 1990; Berger *et al.*, 1993; Ghanshyam and Tripathi, 1993; Wilks *et al.*, 1994; Kaiser *et al.*, 1994; Vidal and Johnston, 1996; Lal *et al.*, 1997; Guzdar *et al.*, 1998; Bendib *et al.*, 2006; Keskinen and Basu, 2003; Gondarenko *et al.*, 2005), one considers an instability $E_1 \exp\left[ik_\perp x - ik_{||}z\right]$, superposed on a high power beam $E_0 \exp\left[i(\omega t - kz)\right]$; the suffixes $||$ and $\perp$ refer to the components of the wave number k of the instability parallel and perpendicular to the direction of propagation viz., z axis. The instability grows or not, as the beam propagates, depending on whether $k_{||}$ is imaginary or real. When $k_{||}$ is imaginary, the instability grows with a spatial growth rate $\left|ik_{||}\right|$. Apart from the scientific point of view, the results in the field of filamentation instability are relevant to ionospheric modification experiments (George, 1970; Utlaut and Cohen, 1971; Guzdar *et al.*, 1998; Keskinen and Basu, 2003; Gondarenko *et al.*, 2005; Perkins and Goldman, 1981; Gurevich, 1978; Brown, 1973), beams from proposed satellite power stations (Gurevich, 1978) passing through the ionosphere and

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the field of laser-induced fusion (Kasperczuk *et al.*, 2006; Chen and Wilks, 2005; Badziak *et al.*, 2005; Hora *et al.*, 2005; Hora, 2005).

Another approach for the investigation of this instability is based on the indirect (Loy and Shen, 1969)and direct (Chiligaryan, 1968; Abbi and Mahr, 1971)evidence that filamentation in a nonlinear medium is caused by the presence of irradiance spikes in the beam, normal to the direction of propagation. Following this lead, the growth of a Gaussian ripple on a plane uniform beam in plasma has been investigated (Sodha *et al.*, 1979a; Sodha *et al.*, 2006; Sodha *et Al.*, 2007; Sharma *et al.*, 2004) to a significant extent; this approach is based on the paraxial theory of self focusing of electromagnetic beams as formulated by Akhmanov *et al.*, 1968 and developed by Sodha *et al.*, 1976 (a,b) and his associates (Sodha *et al.*, 1979b; Sodha *et al.*, 1981; Sodha *et al.*, 1992; Sodha *et al.*, 2004; Asthana *et al.*, 1999; Pandey and Tripathi, 1990). The growth of a ring ripple on a Gaussian beam has also been investigatedin a paraxial-like approximation (Sodha *et al.*, 2009; Misra and Mishra, 2008; Misra and Mishra, 2009).

Media with self-focusing nonlinearity are known to be susceptible to filamentation instability; hence, a ring perturbation over a Gaussian beam may also grow to a large level in the course of propagation. This is due to the fact that the ring region, with higher intensity, would have higher index of refraction and would attract energy from the neighborhood and grow. Many researchers (Leemans *et al.*, 1992; Chessa *et al.*, 1999; Liu and V K Tripathi, 2000) have reported ring formation when nonlinear refraction causes self focusing.

Several authors have applied the conditions, derived for the first case to the second case, possibly because the first case is more well known and easier to analyze. In both approaches, one looks for the conditions and dynamics of growth of the maximum irradiance as the beam propagates. The condition for the growth of the maximum irradiance of the perturbation in the first approach corresponds to the condition for onset of self-focusing in the other approach. This is also expected intuitively because the change in the irradiance (which determines the magnitude of the nonlinearity) and the width of the beam (which determines the magnitude of diffraction) are the main parameters affecting the instability/self-focusing of the perturbation. However, there is an important difference in the results of the two approaches. In thefirst approach, the condition for the onset of the instability is independent of the irradiance of the perturbation and depends on the irradiance of the main beam, while in the second approach, the condition depends on the irradiance of the main beam as well asthe perturbation (ripple) and the phase difference between the two.

In this paper expressions for the growth rate of instability and the condition for instability to occur have been obtained and the maximum value of the growth rate and the corresponding value of $q_{\perp}$ have been specifically investigated. Following the other approach a critical curve between the initial radius and power of the spike was obtained, such that for all points on the curve the ripple propagates without change of width and for points above the curve the ripple width varied between the initial width and a minimum (in other words it displayed self-focusing). For points below the curve, the ripple has either steady divergence or oscillatory divergence (the beam width varying between the original width and a maximum). The variation of the ripple width with distance of propagation for typical points in the three regions has also been evaluated and illustrated graphically for different kinds of nonlinearities (collisional (Sodha *et al.*, 1976a), ponderomotive (Hora, 1970) and relativistic (Esarey *et al.*, 1997)) in laser-plasma interaction. It is seen that the propagation characteristics of the ripple strongly depend on the initial ripple width and initial power of the beam (including the ripple), which can be expressed as a function of the amplitude ratio of the ripple and the main beam and the phase difference between the two.

Several later references with similar logic and directly or indirectly concerned with instability have been published (Hao *et al.*, 2013; Yang *et al.*, 2016; Bawaaneh *et al.*, 2010; Lin *et al.*, 2014; Hasanbeigi *et al.*, 2013; Fox *et al.*, 2013; Sharma *et al.*, 2016; Silantyev *et al.*, 2017; Pathak *et al.*, 2015; Grassi *et al.*, 2017; Alimohamadi and Hajisharifi, 2017). Despite different approaches, the status of our understanding of instabilities has not changed appreciably over the years. The theory is far ahead of available experiments.

Analysis for Beams with Uniform Illumination Along the Wave Front

Expression for Spatial Growth Rate of Instability (First Approach)

Let the electric field of a beam of uniform illumination and that of a small perturbation (filament) superimposed on the beam be represented by $\boldsymbol{J}E_0$ $expi(\omega t - kz)$ and $\boldsymbol{J}E_1$ $expi(\omega t - kz)$ respectively. The total field E propagating in the Z direction through a plasma can be expressed as

$$\boldsymbol{E} = \boldsymbol{J}(E_0 + E_1)\, expi(\omega t - kz) \tag{1}$$

where E_0, without loss of generality, is a real positive constant and E_1 ($|E_1| << E_0$) is a complex parameter, J is a unit vector along y axis, k is the wave number defined later and w is the wave frequency. Neglecting the small contribution $E_1E_1^*$ as compared to other terms, one can write

$$\boldsymbol{E}\cdot\boldsymbol{E}^* = E_0^2 + E_0\left(E_1 + E_1^*\right) \tag{2}$$

The effective dielectric function of the plasma depends on EE^* and hence can be expressed as

$$\varepsilon\left(EE^*\right) = \varepsilon_0\left(E_0^2\right) + \varepsilon_2\left(E_0^2\right)E_0\left(E_1 + E_1^*\right) \tag{3}$$

where

$$\varepsilon_2 = \left[\frac{\partial\varepsilon}{\partial\left(E\,E^*\right)}\right]_{EE^*=E_0^2}$$

The effective electric field vector $\boldsymbol{E}$ satisfies the wave equation,

$$\nabla^2\boldsymbol{E} - \nabla(\nabla\cdot\boldsymbol{E}) + (\omega^2/c^2)\varepsilon(r,z)\boldsymbol{E} = 0 \tag{4}$$

Where ε is the effective dielectric function of the plasma and C is the speed of light in free space.

In the JWKB approximation *i.e.* $k^{-2}\nabla^2(\ln\varepsilon) << 1$, the second term of Eq.(3) may be neglected, where k is the wave number of propagation. One can thus write the wave equation, as

$$\nabla^2\boldsymbol{E} + (\omega^2/c^2)\,\varepsilon(r,z)\,\boldsymbol{E} = 0 \tag{5}$$

The wave equation for the total field can be separated for E_0 and E_1. On choosing $k = \frac{\omega}{c}\sqrt{\varepsilon_0}$ the wave equation for E_0 yields a solution

$E_0 = A_0$ (constant)

The wave equation for E_1 (r, z) on neglecting the term $\frac{\partial^2 E_1}{\partial z^2}$ (assuming E_1 (r, z) to be a slowly varying function of z) and $E_1E_1^*$, reduces to

$$2ik\frac{\partial E_1}{\partial z} + iE_1\frac{\partial k}{\partial z} = \nabla_\perp^2 E_1 + \frac{\omega^2}{c^2}\varepsilon_2 E_0^2\left(E_1 + E_1^*\right) \tag{6}$$

One can express the complex amplitude E_1 of the perturbation as

$$E_1 = E_{1r} + iE_{1i}, \tag{7}$$

where E_{1r} and E_{1i} are real and $\nabla_\perp^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}$.

However in most of analyses E_{1r} and E_{1i} which are real have been assumed to be proportional to the complex quantity $exp\left\{i\left(q_\perp x - q_{||}z\right)\right\}$ which is not consistent. However the results so obtained are the same as the ones based on the following considerations, free of any objection. Assuming E_1 to be independent of y and proportional to $cos\left(iq_\perp x - iq_{||}z\right)$, one has $\nabla_\perp^2 E_1 = -q_\perp^2 E_1$. With this assumption and using Eq. (5), one obtains two homogeneous equations in E_{1r} and E_{1i} (after equating the real and imaginary parts). Thus

$$2k\frac{\partial E_{1i}}{\partial z} - \Lambda^2 E_{1r} = 0 \tag{8a}$$

and

$$2k\frac{\partial E_{1r}}{\partial z} + q_\perp^2 E_{1i} = 0 \tag{8b}$$

where $\Lambda^2 = \frac{2\omega^2}{c^2}\varepsilon_2 A_0^2 - q_\perp^2$.

Differentiating Eqs. (8a) and (8b) with respect to z and substituting for $\frac{\partial E_{1r}}{\partial z}$ and $\frac{\partial E_{1i}}{\partial z}$ from Eqs.(8b) and (8a) respectively one gets

$$\frac{\partial^2 E_{1i}}{\partial z^2} = \frac{\Lambda^2 q_\perp^2}{4k^2} E_{1i} \tag{9A}$$

and

$$\frac{\partial^2 E_{1r}}{\partial z^2} = \frac{\Lambda^2 q_\perp^2}{4k^2} E_{1r}. \tag{9B}$$

Hence E_1 grows exponentially with z, with a growth rate

$$\Gamma = iq_{||} = \frac{\Lambda q_\perp}{2k} = \frac{q_\perp}{2k}\left\{\frac{2\omega^2}{c^2}\varepsilon_2 A_0^2 - q_\perp^2\right\}^{1/2}. \tag{10}$$

From the above equation one obtains the condition for the growth of the instability (Γ being real) as the beam propagates viz.

$$\frac{2\omega^2}{c^2}\varepsilon_2 A_0^2 > q_\perp^2. \tag{11}$$

Growth of a Gaussian Ring Ripple on a Uniform Plane Wave Front (Second Approach), Following Sharma et al. (2004)

Consider the propagation of a linearly polarized electromagnetic beam with uniform intensity along its wave front on which a Gaussian ripple is superposed. Let the electric fields of the two components be expressed, respectively, as

$$\boldsymbol{E}_0 = \boldsymbol{j}E_0 \exp(i\omega t) \tag{12A}$$

and

$$\boldsymbol{E}_1 = \boldsymbol{j}E_1 \exp i(\omega t - \phi_p) \tag{12B}$$

Where E_0 and E_1 are the amplitudes, w is the common angular frequency, and f_p is the phase difference between the main beam and the ripple. The symmetry of the ripple allows the choice of a cylindrical system of coordinates with the z axis perpendicular to the wave front and passing through the point of intensity maximum of the ripple. In the present case, one considers the perturbation to be a Gaussian ripple at $z = 0$; hence,

$$\boldsymbol{E}_1 \cdot \boldsymbol{E}_1^* = E_{10}^2 \exp\left(-\frac{r^2}{r_{10}^2}\right), \tag{13}$$

where E_{10} is the field intensity of the ripple at $z = 0$ and $r = 0$ and r_{10} is the initial ($z = 0$) width of the ripple. The propagation of the resultant electric vector $\boldsymbol{E} = \boldsymbol{E}_0 + \boldsymbol{E}_1$ is governed by the scalar wave equation (5).

Assuming the ripple to be a small perturbation ($E_{10} << E_0$), one expects Eq. (5) to hold for both E_0 and E_1 separately. As discussed in the analysis, corresponding to the first approach E_0 is constant. For the study of self focusing of the ripple it is convenient to express the dielectric constant as

$$\varepsilon(r,z) = \varepsilon_0(z) + \varepsilon_1(r,z), \tag{14}$$

where $\varepsilon_0(z)$ refers to the dielectric function of the plasma at points of maximum irradiance and $\varepsilon_1(r,z)$ represents the $\boldsymbol{r}$ dependent remainder.

Following Akhmanov *et al.* [1968] and Sodha *et al.* [1976b] the solution of Eq.(5) for E_1 may be written in the cylindrical coordinate system as

$$\boldsymbol{E}_1 = A_{10}(r,z)\exp[-ik(z+S)], \tag{15}$$

where

$$S = \phi(z) + \frac{r^2}{2}\beta(z) \tag{16}$$

is the eikonal associated with the electromagnetic ripple beam and $\beta(z)$ is the inverse of the radius of curvature of the wavefront.

Substituting for E_1 from Eq. (15) in Eq. (5) and equating the real and imaginary parts on both sides of the resulting equation, one obtains

$$\frac{2S}{k}\frac{\partial k}{\partial z}+2\frac{\partial S}{\partial z}+\left(\frac{\partial S}{\partial r}\right)^2=\frac{\varepsilon_1(r,z)}{\varepsilon_0(z)}+\frac{1}{k^2E_{10}}\left(\frac{\partial^2 E_{10}}{\partial r^2}+\frac{1}{r}\frac{\partial E_{10}}{\partial r}\right) \quad (17)$$

and

$$\frac{\partial A_{10}^2}{\partial z}+\frac{\partial S}{\partial r}.\frac{\partial A_{10}^2}{\partial r}+A_{10}^2\left(\frac{\partial^2 S}{\partial r^2}+\frac{1}{r}\frac{\partial S}{\partial r}\right)+\frac{A_{10}^2}{k}\frac{\partial k}{\partial z}=0 \quad (18)$$

Substituting for S from Eq. (16) in Eq. (17) one obtains for an initially Gaussian ripple (Sodha *et al.*, 1976b)

$$A_{10}^2=\boldsymbol{E}_1\cdot\boldsymbol{E}_1^*=\frac{E_{10}^2}{f^2}\exp\left(-\frac{r^2}{r_{10}^2f^2}\right) \ , \quad (19)$$

where f is a function of z and is defined by

$$\beta=\frac{1}{f}\frac{df}{dz} \ .$$

Using Eqs. (12A), (15) and (19), one can write the resultant irradiance as

$$\boldsymbol{EE}^*=(\boldsymbol{E}_0+\boldsymbol{E}_1).(\boldsymbol{E}_0+\boldsymbol{E}_1)^*=\boldsymbol{E}_0^2+\boldsymbol{E}_0(\boldsymbol{E}_1+\boldsymbol{E}_1^*)$$

$$=\left[E_0^2+2\frac{E_0E_{10}}{f}\cos\phi_p\, exp\left(\frac{-r^2}{2r_{10}^2f^2}\right)+\frac{E_{10}^2}{f^2}exp\left(\frac{-r^2}{r_{10}^2f^2}\right)\right] , \quad (20)$$

which in the paraxial approximation can be expressed as

$$\boldsymbol{EE}^*=F_1(z)-r^2F_2(z) \ , \quad (21)$$

where

$$F_1(z)=E_0^2+2\frac{E_0E_{10}}{f}\cos\phi_p+\frac{E_{10}^2}{f^2} \quad (21A)$$

and

$$F_2(z)=\frac{E_0E_{10}}{r_{10}^2f^3}\cos\phi_p+\frac{E_{10}^2}{r_{10}^2f^4} \quad (21B)$$

The dependence of nonlinear dielectric constant on the intensity of the beam may be expressed as

$$\varepsilon=\varepsilon_0+\phi(EE^*)$$

where $\varepsilon_0=1-\left(\frac{\omega_p^2}{\omega^2}\right)$ (in a plasma)

From Eqs.(20) and (21) one can write

$$\varepsilon=\varepsilon_0+\phi\left[EE^*=F_1(z)\right]-\left(\frac{d\phi}{dEE^*}\right)_{EE^*=F_1(z)}.r^2F_2(z) \quad (22)$$

The first two terms of this equation together represent $\varepsilon_0(z)$ in Eq. (14), while the third term is equal to $\varepsilon_1(r,z)$. Substituting for S and A_{10} from Eqs. (16) and (19) in Eq. (14) and equating the coefficient of r^2 to zero one gets

$$\frac{1}{f}\frac{d^2f}{dz^2}=\frac{c^2}{\omega^2\varepsilon_0(z)r_{10}^4f^4}-\frac{\varepsilon_1(r,z)}{r^2\varepsilon_0(z)} \quad (23)$$

The boundary conditions on Eq. (23) are $f = 1$ and $df/dz = 0$ (plane wave front) at $z = 0$. Equation (23) can be reduced to a simpler form by transforming the coordinate z and the initial beam width r_{10} to dimensionless form viz.,

$$\xi=cz/\omega r_{10}^2 \text{ and } \rho_0=r_{10}\omega/c.$$

Thus, one obtains

$$\varepsilon_0(f)\frac{d^2f}{dz^2}=\frac{1}{f^3}\left[1-\frac{\rho_0^2r_{10}^2\varepsilon_1(r,f)}{r^2}\right]. \quad (24)$$

Hence the condition for self focusing (f decreasing with increasing z) of the ripple is $\left(d^2f/dz^2\right)_{z=0}<0$. Thus, using Eqs. (21), (21A) and (21B) the condition comes out to be

$$\phi'\left[EE^* = F_1(z=0)\right]E_0E_{10}\cos\phi_p > \left(1/\rho^2\right), \tag{25}$$

where ϕ' is the differential coefficient of ϕ with respect to the argument.

It is interesting to compare the condition [Eq (8A)] for the growth of the perturbation in the first approach to the condition for self-focusing [(Eq.25)] of the superposed ripple in thesecond approach, because in both cases the maximum irradiance of the instability/ripple increases. The right-hand sides of the two equations are the same when $r_0 = \sqrt{2}/k_\perp$, which is very nearly equal to r_0', the width of the sinusoidal instability. However, the left-hand sides differ considerably in having the terms E_0^2 for the first approach and $E_0E_{10}\cos\phi_p$ in the second approach. Thus, the condition for the growth of instability depends only on their irradiance of the main beam in the first approach, while in the second approach the self-focusing depends on the irradiance of the main beam and that of the rippleas well as the phasedifference between the two.

Propagation of a Ring Ripple on a Gaussian Electromagnetic Beam (Paraxial Like Approach)

Consider the propagation along the z-axis of a linearly polarized Gaussian electromagnetic beam with a small coaxial perturbation (the ring ripple) having its electric vector along the y-axis, in a homogeneous plasma. The effective electric field vector $\boldsymbol{E}$ of the Gaussian electromagnetic beam with the coaxial ripple can be expressed as $z = 0$.

$$\boldsymbol{E} = \boldsymbol{j}F_0\exp(i\omega t) \tag{26}$$

where

$$(F_0)_{z=0} = E_{00}\exp\left(-\frac{r^2}{2r_0^2}\right) + E_{10}\left(\frac{r^2}{r_1^2} - \delta\right)^{n/2}$$

$$\exp\left(-\frac{r^2}{2r_1^2}\right)\ \exp(i\phi_p); \tag{27}$$

F_0 refers to the complex amplitude of the electromagnetic beam, E_{00} and E_{10} correspond to the initial amplitude of the Gaussian beam (with initial beam width r_0) and the ring ripple (with initial beam width r_1) components respectively, $\boldsymbol{n}$ and δ are positive numbers, characterizing the position of the ring ripple on the wave front of the electromagnetic beam. The first term on the right hand side of Eq. (27) corresponds to the Gaussian profile while the second term represents the radial distribution of the coaxial perturbation in the form of the ring ripple, having its maximum at $r = r_{max} = r_1\sqrt{(n+\delta)}$.

The effective field vector $\boldsymbol{E}$ (or F_0) satisfies the wave equation (5). Taking the solution for $F_0(r, z)$ as $F_0(r,z) = A_0(r,z)\ \exp\left[i(-kz - ks)\right]$, analogous to equation (15), one obtains two equations viz., (17) and (18). The only difference is that there is amplitude of the composite beam $A_0(r, z)$ instead of that of the ripple $A_{10}(r, z)$ only.

To investigate the propagation of the ring ripple component, far from the axis $r = 0$, one may use a paraxial like approximation, which is valid around $r = r_{\max}$, the position of the maximum irradiance of the ring ripple; this is analogous to the usual paraxial approach. One can thus express Eqs. (17) and (18) in terms of z and a new variable χ, where is a parameter introduced for algebraic convenience, and defined as

$$\chi^2 = \left(\frac{r^2}{r_1^2 f^2} - \lambda\right) \tag{28}$$

where $\lambda = (n+\delta)$, $r_1 f(z)$ is the width of the ring ripple and $r_{\max}^2 = \lambda r_1^2 f^2$ indicates the position of the maximum irradiance of the ring ripple; it is shown later that Eqs.(17) and (18) lead to retention of the original profile of the beam during propagation in the paraxial like approximation i.e., when $\chi^2 << n$.

In the paraxial like approximation the relevant parameters (i.e., dielectric function $\varepsilon(r,z)$, eikonal and irradiance) may be expanded around the maximum of the ring ripple i.e., around $\chi^2 = 0$. Thus the dielectric function $\varepsilon(\chi,z)$ can be expressed around $\chi^2=0$ as

$$\varepsilon(\chi,z)=\varepsilon_0(z)-\chi^2\varepsilon_2(z) \tag{29}$$

where $\varepsilon_0(z)$ and $\varepsilon_2(z)$ are the coefficients associated with χ^0 and χ^2 in the expansion of $\varepsilon(\chi,z)$ around $\chi^2=0$. The expressions for these coefficients have been derived later.

The present paraxial like theory is valid, when $\chi^2 << n$ (Eq.28). This condition defines the range of r^2/r_1^2 (around $\lambda=(n+\delta)$) for which the theory is valid.

Such a transformation leads to

$$\frac{\partial}{\partial z}=\frac{\partial}{\partial z}-\frac{(\lambda+\chi^2)}{\chi f}\frac{df}{dz}\frac{\partial}{\partial\chi} \tag{30}$$

and

$$\frac{\partial}{\partial r}=\frac{1}{r_1 f}\frac{\partial}{\partial\chi} \tag{31}$$

Thus with the help of Eqs.(29), (30) and (31), the set of focusing equations [Eqs.(17) and (18)] reduce to

$$\frac{2S}{k}\frac{\partial k}{\partial z}+2\left(\frac{\partial S}{\partial z}-\frac{(\lambda+\chi^2)}{\chi f}\frac{df}{dz}\frac{\partial S}{\partial\chi}\right)+$$

$$\frac{\delta_m}{r_1^2 f^2}\frac{(\lambda+\chi^2)}{\chi^2}\left(\frac{\partial S}{\partial\chi}\right)^2=\frac{1}{k^2A_0r_1^2f^2}$$

$$\left[\frac{\lambda}{\chi^2}\left(\frac{\partial^2A_0}{\partial\chi^2}-\frac{1}{\chi}\frac{\partial A_0}{\partial\chi}\right)+\left(\frac{\partial^2A_0}{\partial\chi^2}+\frac{1}{\chi}\frac{\partial A_0}{\partial\chi}\right)\right]$$

$$-\chi_\pm^2\frac{\omega^2}{k_\pm^2c^2}\varepsilon_{2\pm} \tag{32}$$

and

$$\frac{A_0^2}{k}\frac{\partial k}{\partial z}+\left(\frac{\partial A_0^2}{\partial z}-\frac{(\lambda+\chi^2)}{\chi f}\frac{df}{dz}\frac{\partial A_0^2}{\partial\chi}\right)+$$

$$\frac{\delta_mA_0^2}{r_1^2f^2}\left[\frac{\lambda}{\chi^2}\left(\frac{\partial^2S}{\partial\chi^2}-\frac{1}{\chi}\frac{\partial S}{\partial\chi}\right)+\left(\frac{\partial^2S}{\partial\chi^2}+\frac{1}{\chi}\frac{\partial S}{\partial\chi}\right)\right]$$

$$+\frac{\delta_m}{r_1^2f^2}\frac{(\lambda+\chi^2)}{\chi^2}\frac{\partial A_0^2}{\partial\chi}\frac{\partial S}{\partial\chi}=0 \tag{33}$$

In the paraxial like approximation $\chi^2 << n$, the solution of Eq.(33) may be chosen as,

$$A_0^2=\frac{E_0^2}{f^2}\exp[-m(\lambda+\chi^2)]+\frac{E_1^2}{f^2}(n+\chi^2)^n$$

$$\exp[-(\lambda+\chi^2)]+\frac{E_0E_1}{f^2}(n+\chi^2)^{n/2}$$

$$\exp\left[-\frac{1}{2}(1+m)(\lambda+\chi^2)\right]Cos\phi_p \tag{34}$$

where

$$S(\chi,z)=\frac{\chi^2}{2}\beta(z)+\varphi(z), \tag{35}$$

$$\beta(z)=r_1^2f\frac{df}{dz},$$

$$E_0^2=E_{00}^2\left(\frac{k(0)}{k(z)}\right)=E_{00}^2\left(\frac{\varepsilon_0(0)}{\varepsilon_0(z)}\right)^{1/2},$$

$$E_1^2=E_{10}^2\left(\frac{k(0)}{k(z)}\right)=E_{10}^2\left(\frac{\varepsilon_0(0)}{\varepsilon_0(z)}\right)^{1/2},$$

$m = (r_1^2 / r_0^2)$, $\varphi(z)$ is an arbitrary function of z and $f(z)$ is the beam width parameter.

For further algebraic analysis, it is convenient to expand the solution for A_0^2 as a polynomial in χ^2; thus

$$A_0^2 = g_0 + g_2\chi^2 + g_4\chi^4 + g_6\chi^6, \qquad (36)$$

where

$$g_0 = \frac{E_0^2}{f^2}\left(e^{-m\lambda} + p^2 n^n e^{-\lambda} + 2pn^{n/2} e^{-(m+1)\lambda/2} Cos\phi_p\right) \qquad (37)$$

$$g_2 = -\frac{E_0^2}{f^2}\left(me^{-m\lambda} + pmn^{n/2} e^{-(m+1)\lambda/2} Cos\phi_p\right), \qquad (38)$$

$$g_4 = \frac{E_0^2}{f^2}\left(\frac{m^2}{2}e^{-m\lambda} - \frac{1}{2n}p^2 n^n e^{-\lambda} + pn^{n/2} e^{-(m+1)\lambda/2} Cos\phi_p\left(\frac{m^2}{4} - \frac{1}{2n}\right)\right), \qquad (39)$$

$$g_6 = \frac{E_0^2}{f^2}\left(-\frac{m^3}{6}e^{-m\lambda} + \frac{1}{3n^2}p^2 n^n e^{-\lambda} + pn^{n/2} e^{-(m+1)\lambda/2} Cos\phi_p\left(\frac{m}{4n} + \frac{1}{3n^2} - \frac{m^3}{24}\right)\right), \qquad (40)$$

and $p = (E_1/E_0)$.

On substituting for A_0^2 and S from Eqs.(36) and (35) in Eq.(32) and equating the coefficients of χ^0 and χ^2 on both sides of the resulting equation, one obtains

$$f\left(\varepsilon_0 \frac{d^2 f}{d\xi^2} + \frac{1}{2}\frac{df}{d\xi}\frac{d\varepsilon_0}{d\xi}\right)g_0^2 + 2g_2 g_0 \left(\Phi\frac{d\varepsilon_0}{d\xi} + \varepsilon_0\left(2\frac{d\Phi}{d\xi} - \lambda\left(\frac{df}{d\xi}\right)^2\right)\right) \qquad (41)$$

and

$$g_0^2\left[\Phi\frac{d\varepsilon_0}{d\xi} + \varepsilon_0\left(2\frac{d\Phi}{d\xi} - \lambda\left(\frac{df}{d\xi}\right)^2\right)\right] = \frac{1}{f^2}[2g_0(g_2 + 2\lambda g_4) - \lambda g_2^2], \qquad (42)$$

where $\xi = (c/r_1^2\omega)z$ is the dimensionless distance of propagation, $\rho = (r_1\omega/c)$ is the dimensionless initial width of the ring ripple and $\mathbf{\Phi} = (\omega/c)\varphi$ $\Phi = (\omega/c)\varphi$ is the dimensionless arbitrary function associated with the eikonal. The parameter $\mathbf{\Phi}$ can be eliminated between Eqs.(41) and (42); thus

$$f\left(\varepsilon_0 \frac{d^2 f}{d\xi^2} + \frac{1}{2}\frac{df}{d\xi}\frac{d\varepsilon_0}{d\xi}\right)g_0^3 + \frac{2g_2}{f^2}[2g_0(g_2 + 2\lambda g_4) - \lambda g_2^2] = \frac{1}{f^2}[4g_0(2g_4 + 3\lambda g_6) + g_2^2]g_0 - \rho^2 g_0^3 \varepsilon_2 \qquad (43)$$

is the equation which determines the width of the ripple.

Numerical Results and Discussion

Figure 1 illustrates the dependence of the dimensionless growth rate Γ of the self focusing instability with the dimensionless wave number $q = (c/\omega)q_\perp$ in the direction transverse to propagation, for different values of the dimensionless background irradiance αA_0^2 = 1, 5 and 10. It is seen that corresponding to a certain value of the beam irradiance, there is an optimum value of wave number namely q_{opt} for which the growth rate is maximum.

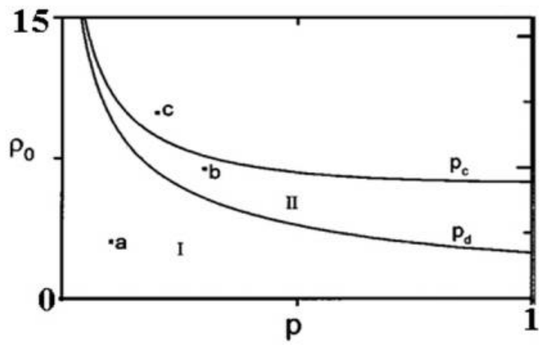

Fig. 1: Dependence of dimensionless growth rate $(c/\omega)\Gamma$ of self-focusing filamentation instability on q, the dimensionless wave number of the perturbation in the transverse direction corresponding to the dimensionless background irradiance αA_0^2 = 1(I), 5(II) and 10(III). The other parameters are $\Omega_p^2 = \omega_p^2/\omega^2 = 0.5$, $v_0^2/\omega^2 = 0.1$

Figure 2 depicts the variation of both optimum value of the wave number $q_{opt.}$ and maximum growth rate Γ_{max} on the uniform background irradiance. It is seen that Γ_{max} and $q_{opt.}$ increase monotonically. This may be readily understood as follows; as q increases the width of disturbance ($\propto$ the wavelength) decreases, diffraction becomes more important, requiring larger magnitudes of nonlinearity,

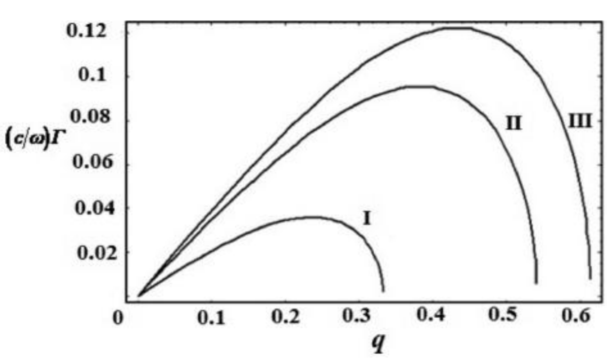

Fig. 2: Variation of optimum value of the dimensionless wavenumber $q_{opt.}$ and dimensionless maximum growth rate $(c/w)G_{max}$ with the dimensionless background irradiance αA_0^2 ; $\omega_p^2/\omega^2 = 0.5$

corresponding to larger values of the irradiance αA_0^2 of the main beam.

The variation of beam width parameter f with the dimensionless distance of propagation ξ can be obtained by numerically integrating Eq. (21.) The dielectric functions $\varepsilon_0(f)$ and $\varepsilon_1(r, f)$ depend on the nature of the nonlinearity. For relativistic nonlinearity, the function $\phi(EE^*)$ assumes the form (Esarey *et al.*, 1997).

$$\phi(EE^*) = \Omega^2\left[1-\left(1+\alpha EE^*\right)^{-1/2}\right]$$

where

$$\alpha = (e/m\omega c)^2 \text{ and } \Omega = \omega\rho/\omega$$

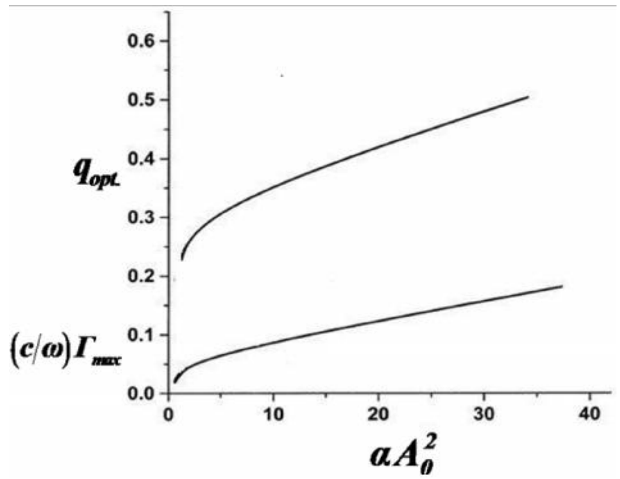

Fig. 3: Three regions for relativistic nonlinearity with different propagation characteristics in the dimensionless parameter $p = E_{10}/E_0$ and dimensionless ripple width $\rho_0 = r_{10}\omega/c$ for $\Omega^2 = 0.5$ and $\alpha E_0^2 = 5$

Substituting for EE^* from Eq. (20) in the expression of $\phi(EE^*)$ one obtains the two parts of dielectric function as

$$\varepsilon_0(f) = 1-\Omega^2\left[1+P\left\{1+\left(2pf\ \cos\varphi_p + p^2\right)/f^2\right\}\right]^{-1/2} \tag{44}$$

and

$$\varepsilon_1(r,f)=\frac{\Omega^2 P\left(pf\ \cos\varphi_p+p^2\right)}{2r_{10}^2 f^4}$$

$$\left[1+P\left\{1+\left(2pf\ \cos\varphi_p+p^2\right)/f^2\right\}\right]^{-3/2}.r^2,$$

where $P=\alpha E_0^2;\ p=E_{10}/E_0$

The expressions for $\varepsilon_0(f)$ and $\varepsilon_1(r,f)$ can be substituted in Eq. (24), to completely specify the differential equation which can be integrated keeping in view the boundary conditions given earlier. Corresponding to $d^2f/d\xi^2=0$ a relation between $\rho=\rho_0 f$ and a critical value of p, say $p_c=p/f$ can be obtained from Eq. (24). Thus,

$$\rho_0^2=\frac{2\left[1+P\left(1+2p_{0c}\cos\varphi_p+p_{0c}^2\right)\right]^{3/2}}{\Omega^2 P\left(p_{0c} f\ \cos\varphi_p+p_c^2\right)}\ ,\quad (45)$$

where a suffix 0 has been added to p_c to indicate its value at $\xi=0$. It is obvious from Eq. (24) and its subsequent boundary conditions that if ρ_0 and p_{0c} satisfy Eq. (45), (i.e. $d^2f/d\xi^2$ at $\xi=0$) then f will remain unchanged all along the path of propagation of the beam; in other words the beam will propagate in the uniform wave guide mode. A graph drawn between ρ_0 and p_{0c} and is usually termed the critical curve. If the initial values of the beam width ρ_0 and the amplitude ratio $p=E_{10}/E_0$ do not lie on the critical curve the initial value of $d^2f/d\xi^2$ will have a positive value if the point (p,ρ_0) falls below, i.e., on the same side of the critical curve as the origin and negative if it falls above, i.e., on the other side. If the initial point (p,ρ_0) falls above the critical curve f starts decreasing as the beam propagates through the plasma and the point (p,ρ_0) shifts towards a lower right direction meeting the critical curve at some point so that $d^2f/d\xi^2$ equals zero at that value of ξ. This implies a point of inflexion in the f vs ξ graph. Consequently the graph starts turning in the other direction so that $df/d\xi$ reaches a value of zero at some ξ and f acquires its minimum value. This represents the oscillatory self-focusing mode of propagation. However, if the initial point falls below the critical curve the point (p,ρ_0) shifts towards the left upward direction and depending on the location of the initial point the point (p,ρ_0) may reach the critical curve. In case it does, the beam will again reach a ξ value where the f vs ξ curve has a point of inflexion and therefore the beam experiences an oscillatory diverging mode of propagation. We conclude that the condition for oscillatory divergence or convergence is the vanishing of $d^2f/d\xi^2$ at some value of ξ in the path of propagation for which Eq. (45) must hold which requires

$$\left[1-\varepsilon_0(f)\right]^3=\frac{2\Omega^2}{\rho_0^2 P\left(p_c\cos\varphi_p+p_c^2\right)}\qquad(46)$$

Equation (46) should yield a real value for $\varepsilon_0(f)$ $[0<\varepsilon_0(f)<1]$ for any given values of other parameters. However, Eq.(44) does not allow a real value of f for all allowed values of $\varepsilon_0(f)$. One can easily obtain from Eq.(44):

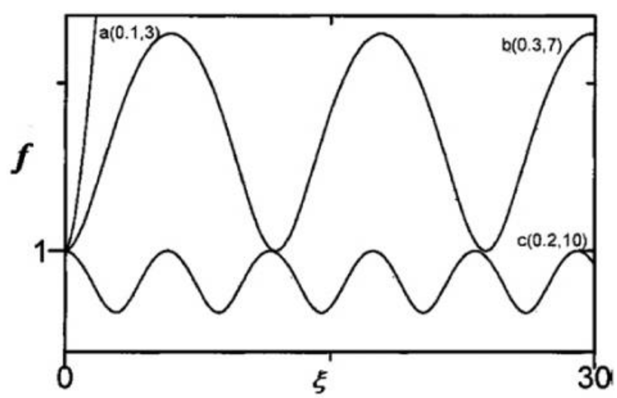

Fig. 4: Dependence of dimensionless ripple width parameter f on dimensionless distance of propagation ξ for (a) $p=0.1$, $\rho_0=0.3$; (b) $p=0.7$, $\rho_0=0.2$; (c) $p=0.2$, $\rho_0=0.10$; $\Omega^2=0.5$ and $\alpha E_0^2=5$

$$f^2=\frac{P\left(2pf\ \cos\varphi_p+p^2\right)}{\left[\Omega^2/\left\{1-\varepsilon_0(f)\right\}\right]^2-\Lambda}$$

where $\Lambda=1+P$

Obviously a real positive value of f requires the inequality

$$\frac{1}{\varepsilon_0(f)} < \frac{1}{1-\Omega^2/\sqrt{\Lambda}}$$

Substitution of this inequality in Eq. (46), yields

$$\rho_0^2 > \frac{2\Lambda^{3/2}}{\Omega^2 P\left(p_{0c} cos\varphi_p + p_c^2\right)}.$$

Replacing the inequality sign by an equality, one gets a curve satisfied by p and ρ_0 which separates the regions corresponding to steady and oscillatory divergence. This divider curve can be explicitly expressed as

$$\rho_0^2 = \frac{2\Lambda^{3/2}}{\Omega^2 P\left(p_{0c} cos\varphi_p + p_c^2\right)}$$

If the initial point (p, ρ_0) lies above this curve the propagating ripple will reach a point of inflexion and, hence, an oscillatory divergence. In case (p, ρ_0) lies below this divider curve the ripple will diverge steadily. Thus, the (p, ρ_0) plane can be divided in three regions such that if the initial point (p, ρ_0) lies:

I) below the divider curve, the ripple diverges steadily;

II) between the divider curve and the critical curve, the ripple undergoes oscillatory divergence with its width varying between the original width and some maximum; and

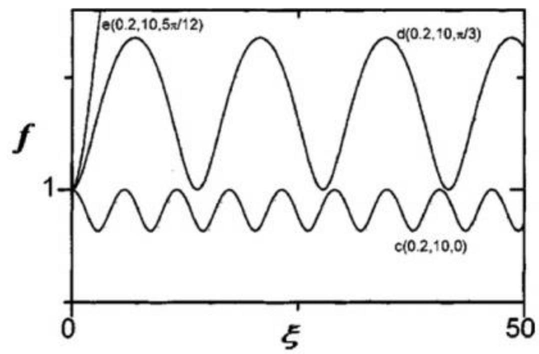

Fig. 5: Dependence of dimensionless ripple width parameter f on dimensionless distance of propagation ξ for p = 0.2, ρ_0 = 0.10; Ω^2 = 0.5 and $\alpha E_0^2 = 5$ c, d and e correspond to ψ_p = 0, $\pi/3$ and $5\pi/3$ respectively (relativistic nonlinearity)

III) above the critical curve, the ripple self-focuses, i.e., the width of the ripple varies, between the original width and some minimum

The three regions in the plane have been shown in Fig. 3. Dependence of the beam width parameter on the distance of propagation is shown in Fig. 4 for typical points in the three regions for . For the same height of the ripple, the corresponding point in the plane can shift from one region to the other. Thus, for typical parameters, the points*c*, *d*, and *e* correspond to, respectively. This type of dependence of on for the three cases has been shown in Fig. 5.

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