ON THE ELECTROMAGNETIC FIELD AND THE SELF-ENERGY OF MESON.

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ABSTRACT.

Assuming Duffin-Kemmer's formulation of meson theory the charge distribution, the electromagnetic field and the self-energy of mesons with spins one and zero are discussed. The charge distribution is found to have a high order of singularity and is extended over a finite region of the order of $\frac{\hbar}{mc}$. The self-energy is calculated by an approximate method which is equivalent to the second order perturbation theory. It is found that the self-energy consists of four different parts which are due to the static field, the transverse electric field, the magnetic field and the forced vibrations under the zero-point fluctuations of the radiation field. The second and the third parts arise from the spin, and so are zero for spinless meson. In terms of the 'cut off momentum' P, for a meson with spin one these two parts diverge separately as P^4, but cancel one another to this order resulting a quadratic divergence. The first and the last parts diverge quadratically, and so also the total self-energy for both the particles. The transverse part of the self-energy is also calculated by second order perturbation calculation. The results are compared, in every stage, with the corresponding ones for an electron in the positron theory.

INTRODUCTION.

The self-energy of the electron in Dirac's positron theory was discussed by Weisskopf (1934) by a second order perturbation calculation. This self-energy, as shown by him, diverges logarithmically with infinitely small radius. Later on by an approximate method, which is equivalent to the second order perturbation calculation and is open to the same objection as the perturbation theory, he (Weisskopf, 1939) separated this total result into different parts whose physical causes are more clear than the total one. According to his treatment the different parts of the self-energy of an electron are—(i) the energy due to the longitudinal electric field (Coulomb field), (ii) the energy due to the transverse electric field arising from the oscillatory motion which produces the spin, (iii) the energy due to the magnetic field (also from the spin), and (iv) the energy of the forced vibrations due to the influence of the zero point fluctuations of the radiation field. In terms of the 'cut off momentum' P, the first part diverges logarithmically; the second, the third and the last parts separately diverge quadratically, but compensate one another to a logarithmic term. In the same paper Weisskopf applied a similar treatment to the Bose particle with spin zero, as has been developed by Pauli-Weisskopf

(1934). In this case the second and the third parts of the self-energy, mentioned above, do not appear at all as the particle has no spin; and the first and last parts diverge quadratically. Following the method of Weisskopf, Richtmyer (1940) worked out the same problem for a particle with spin one assuming the field theory as developed by Proca, Yukawa and others (Proca, 1936; Kemmer, 1938; Frölich, Heitler and Kemmer, 1938; Bhabha, 1938; Yukawa, Sakata and Taketani, 1938; Kobayasi and Okayama, 1939). In this particle the first and the last parts also diverge quadratically, and the second and the third parts separately as P^4; but the whole self-energy diverges quadratically.

Recently Duffin (1938) and Kemmer (1939) developed the theories of particles with spins one and zero from a single scheme. They derived the equations of these particles in a form which is analogous to Dirac's equation of electron. This formulation of Duffin and Kemmer, as shown by various authors (Booth and Wilson, 1940; Wilson, 1940; Christy and Kusaka, 1941) whose procedure of calculations is similar to that as applied to Dirac's electron, can be used with more advantage in the problems of the interaction between meson and the electromagnetic field than the usual field theory. It may, therefore, be of some interest to work out the problem of the self-energy of meson from the Duffin-Kemmer's formalism following the method of Weisskopf. The advantage of using this formalism is that the results for particles with spins one and zero are obtained from a single scheme; moreover, as the method of calculations is somewhat similar to that as applied to Dirac's electron, these results can be compared in every stage with the corresponding results as obtained by Weisskopf for an electron in the positron theory.

In the first two articles all the necessary equations are given and the method of approximation, which is applied in the Duffin-Kemmer's formalism, is developed. The charge distribution and the static field energy are calculated in the third article; and the electromagnetic field produced by meson and the contribution to the self-energy by the spin are worked out in the fourth one. The part of the self-energy produced by the zero point fluctuations of the radiation field is obtained in the fifth article. By way of check the transverse self-energy is calculated in the last article directly by a second order perturbation calculation.

1. Equations of Meson Theory.

Kemmer's wave equation for meson in the presence of the electromagnetic field is

$$\left(\frac{\partial}{\partial x_\mu} - \frac{ie}{\hbar c}\,\phi_\mu\right)\beta_\mu\psi + \kappa\psi = 0 \qquad \ldots \quad \ldots \quad \ldots \quad (1)$$

where $\kappa = \frac{mc}{\hbar}$, $x_4 = ict$, and the operators β_μ are hermitian matrices obeying the Duffin's commutation rules

$$\beta_\mu\beta_\nu\beta_\rho + \beta_\rho\beta_\nu\beta_\mu = \beta_\mu\delta_{\nu\rho} + \beta_\rho\delta_{\nu\mu}. \qquad \ldots \quad \ldots \quad \ldots \quad (2)$$

For meson with spin one β's are ten-row square matrices and for spinless meson they are five-row ones. The current and charge densities are given by

$$\left.\begin{aligned} i_k &= e\psi^\dagger\beta_k\psi = ie\psi^*\eta_4\beta_k\psi, \\ \rho &= -ie\psi^\dagger\beta_4\psi = e\psi^*\beta_4\psi, \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

where

$$\psi^\dagger = i\psi^*\eta_4,\ \eta_4 = 2\beta_4^2-1. \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

The equation of supplementary condition is given by

$$\left(\frac{\partial}{\partial x_k} - \frac{ie}{\hbar c}\phi_k\right)\beta_k\beta_4^2\psi + \kappa\left(1-\beta_4^2\right)\psi = 0 \quad \ldots \quad \ldots \quad (5)$$

with the help of which we can eliminate those components of ψ for which $\beta_4\psi$ is zero and which are not directly quantised. It is to be understood that repeated suffixes mean summation; Greek suffixes run from 1 to 4, and Latin from 1 to 3.

The Hamiltonian function (B & W, eqn. 55)[1] is given by

$$H = \int\Big[-i\hbar c\psi^\dagger\beta_k\frac{\partial\psi}{\partial x_k} - e\psi^\dagger\beta_\mu\phi_\mu\psi - imc^2\psi^\dagger\psi - \frac{1}{4\pi}\phi\,\mathrm{div}\,\mathbf{E}$$
$$+\frac{1}{8\pi}(\mathbf{E}^2+\mathbf{H}^2)\Big]d\mathbf{r}. \quad \ldots \quad (6)$$

This Hamiltonian is to be considered subject to the restrictions (5) and

$$\frac{1}{c}\frac{\partial\phi}{\partial t} + \mathrm{div}\,\mathbf{A} = 0.$$

In the ten-components theory, for a given $\mathbf{p}$, the equation of the free meson has six linearly independent solutions, viz.

$$\phi_k^+(\mathbf{p}) = \frac{1}{\sqrt{V}}u_k^+(\mathbf{p}).\exp\left\{-\frac{i}{\hbar}E(\mathbf{p})t + \frac{i}{\hbar}(\mathbf{pr})\right\} \quad \ldots \quad \ldots \quad (7)$$

for positively charged states with momentum $\mathbf{p}$, and

$$\phi_k^-(\mathbf{p}) = \frac{1}{\sqrt{V}}u_k^-(\mathbf{p}).\exp\left\{\frac{i}{\hbar}E(\mathbf{p})t + \frac{i}{\hbar}(\mathbf{pr})\right\} \quad \ldots \quad \ldots \quad (8)$$

for negatively charged states with momentum $-\mathbf{p}$, where $k = 1, 2, 3$ correspond to the three states of polarisation which are taken to be transverse and longitudinal to the direction of motion; while the five-components theory has two linearly independent solutions given by (7) and (8), and the polarisation is longitudinal. The equations satisfied by $u^+(\mathbf{p})$ and $u^-(\mathbf{p})$ are

$$\{\mp E(\mathbf{p})\beta_4 + ic(\mathbf{p}\beta) + mc^2\}u^\pm(\mathbf{p}) = 0 \quad \ldots \quad \ldots \quad (9)$$

with the equations of condition

$$u^\pm(\mathbf{p}) = \left\{1 - \frac{i}{mc}(\mathbf{p}\beta)\right\}\beta_4^2 u^\pm(\mathbf{p}), \quad \ldots \quad \ldots \quad \ldots \quad (10)$$

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[1] Henceforth B & W refers to the paper of Booth and Wilson.

and u's satisfy the following normalising and orthogonality relations:

$$\left.\begin{aligned} mc^2 u_k^* u_l &= E\delta_{kl}, \quad u_k^{+*}\beta_4 u_l^+ = \delta_{kl}, \\ u_k^{-*}\beta_4 u_l^- &= -\delta_{kl}, \quad u_k^{+*}\beta_4 u_l^- = 0 \end{aligned}\right\} \qquad (11)$$

where $E = c(p^2+m^2c^2)^{\frac{1}{2}}$.

The wave function ψ of any system can be expanded in wave functions of free meson as

$$\psi = \sum_p \sum_k \left\{a_k(\mathbf{p})\phi_k^+(\mathbf{p})+b_k^*(\mathbf{p})\phi_k^-(\mathbf{p})\right\} \qquad (12)$$

where a's and b's are q-numbers and u's are column matrices with c-number components. As in the most of the problems we shall have to take either the sum or the average over different polarisations of meson, we shall for convenience drop the suffix k and the summation over it. By the method of second quantisation it is well known that the operators a's and b's satisfy the relations

$$\left.\begin{aligned} a^*(\mathbf{p})a(\mathbf{p}) &= N^+(\mathbf{p}), \quad a(\mathbf{p})a^*(\mathbf{p}) = N^+(\mathbf{p})+1, \\ b^*(\mathbf{p})b(\mathbf{p}) &= N^-(\mathbf{p}), \quad b(\mathbf{p})b^*(\mathbf{p}) = N^-(\mathbf{p})+1, \end{aligned}\right\} \qquad (13)$$

where $N^+(\mathbf{p})$ and $N^-(\mathbf{p})$ give the number of positive and negative mesons in the state $\mathbf{p}$.

2. The Method of Approximation.

The method, used here, is somewhat similar to that as used by Weisskopf. If the components of ψ for which $\beta_4\psi$ is zero are eliminated from (6) by (5), then after partial integration the Hamiltonian can be written in the form

$$H = H_0+eH'+e^2H'', \qquad (14)$$

where

$$H_0 = \int\left[mc^2\psi^*\beta_4^2\psi+\frac{\hbar^2}{m}\frac{\partial\psi^*}{\partial x_k}\beta_4^2\beta_k\beta_l\beta_4^2\frac{\partial\psi}{\partial x_l}-\frac{1}{4\pi}\phi\,\mathrm{div}\,\mathbf{E}+\frac{1}{8\pi}(\mathbf{E}^2+\mathbf{H}^2)\right]d\mathbf{r},$$

$$H' = -\frac{i\hbar}{mc}\int\left[\frac{\partial\psi^*}{\partial x_k}\beta_4^2\beta_k(\boldsymbol{\beta}\mathbf{A})\beta_4^2\psi-\psi^*\beta_4^2(\boldsymbol{\beta}\mathbf{A})\beta_k\beta_4^2\frac{\partial\psi}{\partial x_k}\right]d\mathbf{r}+\int\psi^*\beta_4\psi.\phi d\mathbf{r}, \qquad (15)$$

$$H'' = \frac{1}{mc^2}\int\psi^*\beta_4^2(\boldsymbol{\beta}\mathbf{A})^2\beta_4^2\psi d\mathbf{r}. \qquad (16)$$

The terms eH' and e^2H'' give the interaction energy between the matter and the field. The operator H depends explicitly on e as shown in (14), where H_0, H' and H'' are explicitly independent of e. If e is increased by de, the Hamiltonian acquires the additional term $H'de+2H''ede$, and according to the perturbation theory the energy of the stationary state s of the Hamiltonian is increased by

$$dW' = (H'(e))_{AV}de+2(H''(e))_{AV}ede,$$

where $(H'(e))_{AV}$ and $(H''(e))_{AV}$ are the time average of H' and H'' respectively in the state s with the initial charge e. Hence the total increase of energy when the charge is increased from zero to e is given by

$$W' = \int_0^e \left[(H'(e))_{AV} + 2(H''(e))_{AV}\, e\right] de \qquad \ldots \qquad \ldots \quad (17)$$

which would, of course, give the self-energy of meson. For approximation we expand $H'(e)$ and $H''(e)$ as

$$H'(e) = b_0 + b_1 e + b_2 e^2 + \ldots\ldots,$$

$$H''(e) = c_0 + c_1 e + c_2 e^2 + \ldots\ldots,$$

and from (17) we get

$$W' = e(b_0)_{AV} + \frac{e^2}{2}(b_1)_{AV} + \frac{e^3}{3}(b_2)_{AV} + \ldots\ldots$$

$$+ e^2(c_0)_{AV} + \frac{2e^3}{3}(c_1)_{AV} + \ldots\ldots$$

Since the self-energy cannot depend on the sign of e, the coefficients of all the odd powers of e must be zero, and we have approximately

$$W' = \frac{e^2}{2}(b_1)_{AV} + e^2(c_0)_{AV} \sim \frac{1}{2}\{eH'(e) + 2e^2 H''(e)\}_{AV}. \qquad \ldots \quad (18)$$

Again if the unwanted components of ψ for which $\beta_4\psi = 0$ are eliminated by (5) from the expression of current as given by (3), we get

$$\mathbf{i} = \mathbf{i}_1 + \mathbf{i}_2 \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots \quad (19)$$

where

$$\mathbf{i}_1 = \frac{ie\hbar}{mc}\left\{\frac{\partial\psi^*}{\partial x_k}\beta_4^2\beta_k\boldsymbol{\beta}\beta_4^2\psi - \psi^*\beta_4^2\boldsymbol{\beta}\beta_k\beta_4^2\frac{\partial\psi}{\partial x_k}\right\}, \qquad \ldots \qquad \ldots \quad (20)$$

$$\mathbf{i}_2 = -\frac{e^2}{mc^2}\psi^*\beta_4^2\{\boldsymbol{\beta}(\boldsymbol{\beta}\mathbf{A}) + (\boldsymbol{\beta}\mathbf{A})\boldsymbol{\beta}\}\beta_4^2\psi. \quad \ldots \qquad \ldots \qquad \ldots \quad (21)$$

Hence considering (15) and (16), (18) reduces to

$$W' = \frac{1}{2}\int \{(\rho\phi)_{AV} - (\mathbf{iA})_{AV}\}d\mathbf{r}. \quad \ldots \qquad \ldots \qquad \ldots \quad (22)$$

In this expression we write

$$\mathbf{A} = \mathbf{A}_0 + \mathbf{A}', \quad \phi = \phi' \qquad \ldots \qquad \ldots \qquad \ldots \quad (23)$$

where $\mathbf{A}'$ and ϕ' is the field generated by the meson, and $\mathbf{A}_0$ is the potential for $e = 0$, that is of the zero point field. Using Maxwell's field equations through which $\mathbf{i}$ and ρ are connected with $\mathbf{A}'$ and ϕ', we partially eliminate $\mathbf{i}$ and ρ from (22). Then on partial integrations with respect to space coordinates as well as with respect to time (which is justified since the diagonal elements are the time average) we obtain

$$W' = \frac{1}{8\pi}\int\{(\mathbf{E}'^2)_{AV} - (\mathbf{H}'^2)_{AV}\}d\mathbf{r} - \frac{1}{2}\int(\mathbf{iA}_0)_{AV}\,d\mathbf{r}. \quad \text{.. (24)}$$

We split the electric field $\mathbf{E}'$ into longitudinal part $\mathbf{E}'_{\text{long}}$ and transverse part $\mathbf{E}'_{\text{tr}}$. The former gives the static field energy

$$W_{\text{st}} = \frac{1}{8\pi}\int(\mathbf{E}'^2_{\text{long}})_{AV}\,d\mathbf{r} = \frac{1}{2}\int\int\frac{\rho(\mathbf{r}_1)\rho(\mathbf{r}_2)}{|\mathbf{r}_1-\mathbf{r}_2|}\,d\mathbf{r}_1 d\mathbf{r}_2. \quad \text{.. (25)}$$

Then

$$W' = W_{\text{st}} + W_{\text{tr}}$$

where W_{tr} is the transverse self-energy and is given by

$$W_{\text{tr}} = \frac{1}{8\pi}\int(\mathbf{E}'^2_{\text{tr}})_{AV}\,d\mathbf{r} - \frac{1}{8\pi}\int(\mathbf{H}'^2)_{AV}\,d\mathbf{r} - \frac{1}{2}\int(\mathbf{iA}_0)_{AV}\,d\mathbf{r} \quad \text{.. (26)}$$

$$= U_{\text{el}} - U_{\text{mag}} + W_{\text{fluct}}$$

in the notation of Weisskopf. W_{fluct} is the energy due to the zero point fluctuations of the radiation field.

3. The Charge Distribution and Electrostatic Field Energy of Meson.

The commutation rule (B & W, eqn. 32) for components of ψ for which $\beta_4\psi$ is not zero is given by

$$\{\beta_4\psi(\mathbf{r}')\}_\sigma\{\psi^*(\mathbf{r})\beta_4^2\}_\rho - \{\psi^*(\mathbf{r})\beta_4^2\}_\rho\{\beta_4\psi(\mathbf{r}')\}_\sigma = \delta_{\rho\sigma}\delta(\mathbf{r}-\mathbf{r}'). \quad \text{.. (27)}$$

Then following Heisenberg's rule (1934), we replace the charge density $e\psi^*\beta_4\psi$ by

$$\rho = \frac{1}{2}e\,\{\psi^*\beta_4\psi + \beta_4\psi\,.\,\psi^*\beta_4^2\} \quad \text{.. .. .. (28)}$$

with this the total charge in a volume V is given by

$$\int\rho(\mathbf{r})d\mathbf{r} = e\sum_p[N^+(\mathbf{p}) - N^-(\mathbf{p})]. \quad \text{.. .. (29)}$$

Thus the choice of (28) avoids the total zero point charge in the vacuum of the meson theory, and this result is consistent with that given by Pauli (1941). (28) may then be considered as the effective charge density of a system of particles.

We now construct a function

$$G(\xi) = \int\rho(\mathbf{r}_2)\rho(\mathbf{r}_1)\,d\mathbf{r} \quad \text{.. .. .. (30)}$$

where $\rho(\mathbf{r})$ is given by (28), and $\mathbf{r}_1 = \mathbf{r}-\xi/2$, $\mathbf{r}_2 = \mathbf{r}+\xi/2$. This function gives us information about the charge distribution in the neighbourhood of meson. In the language of the electron theory, the probability [1] of finding the charge simultaneously at two points in a distance ξ is given by $G(\xi)$. We substitute (12) in (30) and retain those terms which have the time average, that is, those terms which contribute to the expectation value $\overline{G}(\xi)$. Then by (11) and (13) we obtain

$$\overline{G}(\xi) = \frac{e^2}{V}\sum_p\sum_{p'}[N^+(\mathbf{p})N^+(\mathbf{p}')-2N^+(\mathbf{p})N^-(\mathbf{p}')+N^-(\mathbf{p})N^-(\mathbf{p}')]$$
$$+e^2\sum_p\sum_{p'}\int[N^+(\mathbf{p})(N^+(\mathbf{p}')+1)\{\phi^+(\mathbf{p},\mathbf{r}_2)^*\beta_4\phi^+(\mathbf{p}',\mathbf{r}_2)\}\{\phi^+(\mathbf{p}',\mathbf{r}_1)^*\beta_4\phi^+(\mathbf{p},\mathbf{r}_1)\}$$
$$+(N^-(\mathbf{p})+1)N^-(\mathbf{p}')\{\phi^-(\mathbf{p},\mathbf{r}_2)^*\beta_4\phi^-(\mathbf{p}',\mathbf{r}_2)\}\{\phi^-(\mathbf{p}',\mathbf{r}_1)^*\beta_4\phi^-(\mathbf{p},\mathbf{r}_1)\}$$
$$+N^+(\mathbf{p})N^-(\mathbf{p}')\{\phi^+(\mathbf{p},\mathbf{r}_2)^*\beta_4\phi^-(\mathbf{p}',\mathbf{r}_2)\}\{\phi^-(\mathbf{p}',\mathbf{r}_1)^*\beta_4\phi^+(\mathbf{p},\mathbf{r}_1)\}$$
$$+(N^-(\mathbf{p})+1)(N^+(\mathbf{p}')+1)\{\phi^-(\mathbf{p},\mathbf{r}_2)^*\beta_4\phi^+(\mathbf{p}',\mathbf{r}_2)\}\{\phi^+(\mathbf{p}',\mathbf{r}_1)^*\beta_4\phi^-(\mathbf{p},\mathbf{r}_1)\}]d\mathbf{r} \quad \ldots (31)$$

where $\phi^+(\mathbf{p})$ and $\phi^-(\mathbf{p})$ are given by (7) and (8) respectively. This expression does not vanish for the vacuum where $N^+(\mathbf{p}) = N^-(\mathbf{p}) = 0$ for all values of $\mathbf{p}$, and we have

$$\overline{G}(\xi)_{\text{vac}} = e^2\sum_p\sum_{p'}\int[\{\phi^-(\mathbf{p},\mathbf{r}_2)^*\beta_4\phi^+(\mathbf{p}',\mathbf{r}_2)\}\{\phi^+(\mathbf{p}',\mathbf{r}_1)^*\beta_4\phi^-(\mathbf{p},\mathbf{r}_1)\}]\,d\mathbf{r}$$

which is infinite. As we are interested in the behaviour of a single meson only, we shall not make further discussion on the nature of this function in the present paper. For definiteness we shall consider the case of positive meson in our subsequent discussion, but our final results will also be valid for negative meson. We consider the single positive meson at rest in the state $\mathbf{p} = 0$. If we now put $N^+(0) = 1$ and all other $N^+(\mathbf{p}\neq 0) = N^-(\mathbf{p}) = 0$ in (31), the value of $\overline{G}(\xi)_{\text{vac}+1}$ for a single positive meson at rest in the vacuum is obtained, and by subtracting from it the contribution of the vacuum $\overline{G}(\xi)_{\text{vac}}$ we get for a single positive meson only

$$\overline{G}(\xi) = \overline{G}(\xi)_{\text{vac}+1}-\overline{G}(\xi)_{\text{vac}}$$
$$=\frac{2e^2}{V}+e^2\sum_p\int[\{\phi^+(0,\mathbf{r}_2)^*\beta_4\phi^+(\mathbf{p},\mathbf{r}_2)\}\{\phi^+(\mathbf{p},\mathbf{r}_1)^*\beta_4\phi^+(0,\mathbf{r}_1)\}$$
$$+\{\phi^+(0,\mathbf{r}_2)^*\beta_4\phi^-(\mathbf{p},\mathbf{r}_2)\}\{\phi^-(\mathbf{p},\mathbf{r}_1)^*\beta_4\phi^+(0,\mathbf{r}_1)\}]\,d\mathbf{r}.$$

[1] The charge distribution is essentially a positive quantity in the non-relativistic Schrödinger theory as well as in the Dirac's relativistic electron theory and it can be defined as the probability distribution. In the relativistic theory of Bose particles the charge densities are not positive definite; and serious difficulties arise in this interpretation for then the concept of negative probability comes in. Recently Dirac (Bakerian Lecture, 1942) has suggested some ways of modifying this interpretation.

The first term in this expression is to be omitted, because this term becomes zero for $V \to \infty$. Hence by (7) and (8)

$$\bar{G}(\xi) = \frac{e^2}{V} \sum_p \left[\{ u^+(0)^* \beta_4 u^+(\mathbf{p}) \} \{ u^+(\mathbf{p})^* \beta_4 u^+(0) \} + \{ u^+(0)^* \beta_4 u^-(\mathbf{p}) \} \{ u^-(\mathbf{p})^* \beta_4 u^+(0) \} \right] e^{\frac{i}{\hbar}(\mathbf{p}\xi)}. \quad \ldots \quad (32)$$

For a meson with spin one (10×10 matrices) there are three directions of polarisation over which the summation is to be taken in the state $\mathbf{p}$. This summation is performed by using the relations

$$\left.\begin{aligned} \sum_{k=1}^{3} \beta_4^2 \{ u_k^+(\mathbf{p}) u_k^+(\mathbf{p})^* - u_k^-(\mathbf{p}) u_k^-(\mathbf{p})^* \} \beta_4^2 &= \beta_4, \\ \sum_{k=1}^{3} \beta_4^2 \{ u_k^+(\mathbf{p}) u_k^+(\mathbf{p})^* + u_k^-(\mathbf{p}) u_k^-(\mathbf{p})^* \} \beta_4^2 &= \frac{mc^2}{E(\mathbf{p})} \beta_4 \left[1 + \frac{1}{m^2c^2} (\mathbf{p}\boldsymbol{\beta})^2 \right] \beta_4. \end{aligned}\right\} \ldots \quad (33)$$

Again we shall have to average over the directions of polarisation in the state $\mathbf{p} = 0$. For this purpose we sum over the three directions of polarisation and divide by 3. This summation is to be performed by introducing the annihilation operator (B & W, eqn. 107), which is obtained from (9) and (10),

$$\left.\begin{aligned} \frac{1}{2E(\mathbf{p})} \left[E(\mathbf{p})\beta_4 + mc^2 + \frac{1}{m} (\mathbf{p}\boldsymbol{\beta})^2 \right] \beta_4^2 \, u^+(\mathbf{p}) &= \beta_4 u^+(\mathbf{p}), \\ \frac{1}{2E(\mathbf{p})} \left[E(\mathbf{p})\beta_4 + mc^2 + \frac{1}{m} (\mathbf{p}\boldsymbol{\beta})^2 \right] \beta_4^2 \, u^-(\mathbf{p}) &= 0. \end{aligned}\right\} \quad \ldots \quad (34)$$

On performing the above operations, (32) assumes the form

$$\bar{G}(\xi) = \frac{e^2 mc^2}{6V} \sum_p \frac{1}{E(\mathbf{p})} \cdot e^{\frac{i}{\hbar}(\mathbf{p}\xi)} \cdot \text{spur } \beta_4 \left[1 + \frac{1}{m^2c^2} (\mathbf{p}\boldsymbol{\beta})^2 \right] (\beta_4^2 + \beta_4). \quad \ldots \quad (35)$$

For meson with spin zero (5×5 matrices) calculations are the same; the only difference from the particle with spin one is that the factor $\frac{1}{3}$ must be omitted. The factor $\frac{1}{3}$ arises because of the averaging over the three polarisations; for spinless particles averaging is unnecessary since there is only one polarisation. Spur calculations, the detail method of which is given by Booth and Wilson, are also somewhat different.

On evaluating the spurs we obtain for both the particles (mesons with spins one and zero) at rest after replacing the sum over the states by an integral over the momenta of the states

$$\bar{G}(\xi) = \frac{e^2 mc^2}{2(2\pi\hbar)^3} \int \frac{d\mathbf{p}}{E(\mathbf{p})} \left(2 + \frac{p^2}{m^2c^2} \right) e^{\frac{i}{\hbar}(\mathbf{p}\xi)} \quad \ldots \quad \ldots \quad (36)$$

which gives

$$\bar{G}(\xi) = e^2 \frac{mc}{\hbar} \cdot \frac{1}{8\pi i} \frac{1}{\xi} \cdot \frac{\partial}{\partial \xi}\left(2 - \frac{\hbar^2}{m^2c^2} \frac{\partial^2}{\partial \xi^2}\right) H_0^1\left(i \frac{mc}{\hbar} \xi\right) \qquad .. \quad (37)$$

where $H_0^1(z)$ is the Hankel function of the first kind which has a logarithmic singularity for $|z| = 0$ and decreases exponentially for $|z| \gg 1$. The expression (37) varies as $\frac{1}{\xi^4}$ for $\xi \ll \frac{\hbar}{mc}$ and falls off exponentially for $\xi \gg \frac{\hbar}{mc}$, and so shows a spread of charge over a finite region of the order of $\frac{\hbar}{mc}$. In the case of the electron in the positron theory, as shown by Weisskopf, $\bar{G}(\xi)$ varies as $\frac{1}{\xi^2}$ for $\xi \ll \frac{\hbar}{mc}$. Hence in the meson theory the charge distribution has a higher order singularity than that in the positron theory. This result is quite consistent with what is to be expected for in the case of particles obeying Bose statistics the probability of two like particles being closer than their wave-lengths is larger than that at longer distances. This higher order singularity in the charge distribution is much reflected on the electrostatic self-energy which will be shown presently.

The electrostatic energy W_{st} is given by (25) and can be written as

$$W_{st} = \frac{1}{2}\int\int \frac{\rho(\mathbf{r}+\boldsymbol{\xi}/2)\,\rho(\mathbf{r}-\boldsymbol{\xi}/2)}{|\xi|} d\mathbf{r}\, d\xi = \frac{1}{2}\int \frac{\bar{G}(\xi)}{|\xi|} d\xi$$

where $\bar{G}(\xi)$ is given by (36). On evaluating the integration over ξ first, we obtain, in terms of the 'cut off momentum' P which is the greatest value we allow for p,

$$\begin{aligned} W_{st} &= \frac{e^2}{8\pi^2 \hbar mc} \int d\mathbf{p} \cdot \frac{p^2 + 2m^2c^2}{p^2\sqrt{p^2+m^2c^2}} \\ &= \frac{e^2}{4\pi \hbar mc} \lim_{P\to\infty} \left[P\sqrt{P^2+m^2c^2} + 3m^2c^2 \log \frac{P+\sqrt{P^2+m^2c^2}}{mc} \right]. \qquad .. \quad (38) \end{aligned}$$

Putting $\frac{\hbar}{P} = a$, where a is the 'cut off radius', we have

$$\begin{aligned} W_{st} &= \frac{e^2\kappa}{12\pi} \lim_{P\to\infty} \left[3\left(\frac{P}{mc}\right)^2 + 9 \log\left(\frac{P}{mc}\right) \right] + \text{finite terms} \\ &= \frac{e^2\kappa}{12\pi} \lim_{a\to 0} \left[3\left(\frac{\hbar}{mca}\right)^2 + 9 \log\left(\frac{\hbar}{mca}\right) \right] + \text{finite terms}. \qquad .. \quad (39) \end{aligned}$$

Thus the singularity of the order of $\frac{1}{\xi^4}$ in $\bar{G}(\xi)$ causes the quadratic divergence of W_{st}, whereas for an electron in the positron theory this divergence is logarithmic produced by the quadratic singularity of $\bar{G}(\xi)$.

4. The Electromagnetic Field of Meson.

The vector potential $\mathbf{A}'$ of the electromagnetic field produced by meson is given by

$$\mathbf{A}'(\mathbf{r}, t) = \int \frac{\mathbf{i}(\mathbf{r}', t')}{|\mathbf{r}-\mathbf{r}'|} d\mathbf{r}'$$

where $t' = t - \frac{|\mathbf{r}-\mathbf{r}'|}{c}$, and the current density $\mathbf{i}$ is given by (3). In the order of approximation we are interested we consider ψ to be the solutions of the equations of free meson. Then expanding ψ according to (12) and considering (7) and (8) we obtain after some calculations

$$\mathbf{A}'(\mathbf{r}, t) = \frac{2\pi ie\hbar^2c^2}{V} \sum_p \sum_{p'} \Big[\{E(\mathbf{p})E(\mathbf{p}') - m^2c^4 - c^2(\mathbf{pp}')\}^{-1}$$
$$\times \Big\{a^*(\mathbf{p})a(\mathbf{p}')u^+(\mathbf{p})^*\eta_4\boldsymbol{\beta}u^+(\mathbf{p}')\exp. -\frac{i}{\hbar}(E(\mathbf{p}')-E(\mathbf{p}))t$$
$$+b(\mathbf{p})b^*(\mathbf{p}')u^-(\mathbf{p})^*\eta_4\boldsymbol{\beta}u^-(\mathbf{p}')\exp. \frac{i}{\hbar}(E(\mathbf{p}')-E(\mathbf{p}))t\Big\} - \{E(\mathbf{p})E(\mathbf{p}')+m^2c^4$$
$$+c^2(\mathbf{pp}')\}^{-1}\Big\{a^*(\mathbf{p})b^*(\mathbf{p}')u^+(\mathbf{p})^*\eta_4\boldsymbol{\beta}u^-(\mathbf{p}')\exp. \frac{i}{\hbar}(E(\mathbf{p}')+E(\mathbf{p}))t$$
$$+b(\mathbf{p})a(\mathbf{p}')u^-(\mathbf{p})^*\eta_4\boldsymbol{\beta}u^+(\mathbf{p}')\exp. -\frac{i}{\hbar}(E(\mathbf{p}')+E(\mathbf{p}))t\Big\}\Big] \exp. \frac{i}{\hbar}(\mathbf{p}'-\mathbf{p}, \mathbf{r}) \quad \ldots \quad (40)$$

Hence the magnetic field $\mathbf{H}'$ is given by

$$\mathbf{H}' = \text{rot } \mathbf{A}' = -\frac{2\pi e\hbar c^2}{V} \sum_p \sum_{p'} \Big[\{E(\mathbf{p})E(\mathbf{p}') - m^2c^4 - c^2(\mathbf{pp}')\}^{-1}$$
$$\times \Big\{a^*(\mathbf{p})a(\mathbf{p}')u^+(\mathbf{p})^*\eta_4\boldsymbol{\beta}_s u^+(\mathbf{p}')\exp. -\frac{i}{\hbar}(E(\mathbf{p}')-E(\mathbf{p}))t$$
$$+b(\mathbf{p})b^*(\mathbf{p}')u^-(\mathbf{p})^*\eta_4\boldsymbol{\beta}_s u^-(\mathbf{p}')\exp. \frac{i}{\hbar}(E(\mathbf{p}')-E(\mathbf{p}))t\Big\} - \{E(\mathbf{p})E(\mathbf{p}')+m^2c^4$$
$$+c^2(\mathbf{pp}')\}^{-1}\Big\{a^*(\mathbf{p})b^*(\mathbf{p}')u^+(\mathbf{p})^*\eta_4\boldsymbol{\beta}_s u^-(\mathbf{p}')\exp. \frac{i}{\hbar}(E(\mathbf{p}')+E(\mathbf{p}))t$$
$$+b(\mathbf{p})a(\mathbf{p}')u^-(\mathbf{p})^*\eta_4\boldsymbol{\beta}_s u^+(\mathbf{p}')\exp. -\frac{i}{\hbar}(E(\mathbf{p}')+E(\mathbf{p}))t\Big\}\Big]$$
$$\times |\mathbf{p}-\mathbf{p}'| \exp. \frac{i}{\hbar}(\mathbf{p}'-\mathbf{p}, \mathbf{r}) \quad \ldots \quad \ldots \quad \ldots \quad (41)$$

where $\boldsymbol{\beta}_s |\mathbf{p}-\mathbf{p}'| = [\boldsymbol{\beta}, \mathbf{p}-\mathbf{p}']$, that is, $\boldsymbol{\beta}_s$ is the projection of $\boldsymbol{\beta}$ perpendicular to $\mathbf{p}-\mathbf{p}'$. Again the transverse part of the electric field is given by

$$\mathbf{E}'_{\text{tr}} = -\frac{1}{c}\frac{\partial \mathbf{A}'_{\text{tr}}}{\partial t}$$

where $\mathbf{A}'_{\rm tr}$ is the transverse part of $\mathbf{A}'$. Then by (40)

$$\mathbf{E}'_{\rm tr} = -\frac{2\pi e\hbar c}{V}\sum_{p}\sum_{p'}\Bigg[\frac{E(\mathbf{p}')-E(\mathbf{p})}{E(\mathbf{p})E(\mathbf{p}')-m^2c^4-c^2(\mathbf{pp}')}$$

$$\times\Big\{a^*(\mathbf{p})a(\mathbf{p}')u^+(\mathbf{p})^*\eta_4\boldsymbol{\beta}_s u^+(\mathbf{p}')\exp.-\frac{i}{\hbar}(E(\mathbf{p}')-E(\mathbf{p}))t$$

$$-b(\mathbf{p})b^*(\mathbf{p}')u^-(\mathbf{p})^*\eta_4\boldsymbol{\beta}_s u^-(\mathbf{p}')\exp.\frac{i}{\hbar}(E(\mathbf{p}')-E(\mathbf{p}))t\Big\}+\frac{E(\mathbf{p}')+E(\mathbf{p})}{E(\mathbf{p})E(\mathbf{p}')+m^2c^4+c^2(\mathbf{pp}')}$$

$$\times\Big\{a^*(\mathbf{p})b^*(\mathbf{p}')u^+(\mathbf{p})^*\eta_4\boldsymbol{\beta}_s u^-(\mathbf{p}')\exp.\frac{i}{\hbar}(E(\mathbf{p}')+E(\mathbf{p}))t$$

$$-b(\mathbf{p})a(\mathbf{p}')u^-(\mathbf{p})^*\eta_4\boldsymbol{\beta}_s u^+(\mathbf{p}')\exp.-\frac{i}{\hbar}(E(\mathbf{p}')+E(\mathbf{p}))t\Big\}\Bigg]\exp.\frac{i}{\hbar}(\mathbf{p}'-\mathbf{p},\mathbf{r}). \qquad (42)$$

It is evident that div $\mathbf{E}'_{\rm tr}=0$ is satisfied, since $(\boldsymbol{\beta}_s,\,\mathbf{p}'-\mathbf{p})=0$.

Magnetic field energy.—In order to evaluate the time average of the magnetic field energy

$$\frac{1}{8\pi}\int|\,\mathbf{H}'\,|^2\,d\mathbf{r}$$

we shall have to substitute (41), and retain only the diagonal elements. In consequence of the relations (13) we have then

$$U_{\rm mag}=\frac{\pi e^2\hbar^2c^4}{2V}\sum_{p}\sum_{p'}|\,\mathbf{p}-\mathbf{p}'\,|^2\Big[\{E(\mathbf{p})E(\mathbf{p}')-m^2c^4-c^2(\mathbf{pp}')\}^{-2}$$

$$\times\Big\{N^+(\mathbf{p})(N^+(\mathbf{p}')+1)\{u^+(\mathbf{p})^*\eta_4\boldsymbol{\beta}_s u^+(\mathbf{p}')\}\cdot\{u^+(\mathbf{p}')^*\boldsymbol{\beta}_s\eta_4u^+(\mathbf{p})\}$$

$$+(N^-(\mathbf{p})+1)N^-(\mathbf{p}')\{u^-(\mathbf{p})^*\eta_4\boldsymbol{\beta}_s u^-(\mathbf{p}')\}\cdot\{u^-(\mathbf{p}')^*\boldsymbol{\beta}_s\eta_4u^-(\mathbf{p})\}\Big\}+\{E(\mathbf{p})E(\mathbf{p}')$$

$$+m^2c^4+c^2(\mathbf{pp}')\}^{-2}\Big\{N^+(\mathbf{p})N^-(\mathbf{p}')\{u^+(\mathbf{p})^*\eta_4\boldsymbol{\beta}_s u^-(\mathbf{p}')\}\cdot\{u^-(\mathbf{p}')^*\boldsymbol{\beta}_s\eta_4u^+(\mathbf{p})\}$$

$$+(N^-(\mathbf{p})+1)(N^+(\mathbf{p}')+1)\{u^-(\mathbf{p})^*\eta_4\boldsymbol{\beta}_s u^+(\mathbf{p}')\}\cdot\{u^+(\mathbf{p}')^*\boldsymbol{\beta}_s\eta_4u^-(\mathbf{p})\}\Big\}\Big]\quad\ldots\quad(43)$$

If we now calculate $U_{\rm mag}$ for vacuum by putting $N^+(\mathbf{p})=N^-(\mathbf{p})=0$ for all $\mathbf{p}$, we get an expression which is highly divergent. This highly divergent magnetic field energy $U_{\rm mag}$ (vac) is produced by the current fluctuations of the vacuum. As we are interested only in the magnetic field energy of a single meson at rest ($\mathbf{p}=0$), we first calculate $U_{\rm mag}$ (vac+1) by putting $N^+(0)=1$, and all other $N^+(\mathbf{p}\neq 0)=N^-(\mathbf{p})=0$, and subtract the contribution of the vacuum $U_{\rm mag}$ (vac). Then for a single positive meson at rest

$$U_{\rm mag}=U_{\rm mag}\,({\rm vac}+1)-U_{\rm mag}\,({\rm vac})$$

$$=\frac{\pi e^2\hbar^2}{2Vm^2}\sum_{p}|\mathbf{p}|^2\Bigg[\frac{\{u^+(0)^*\eta_4\boldsymbol{\beta}_s u^+(\mathbf{p})\}\cdot\{u^+(\mathbf{p})^*\boldsymbol{\beta}_s\eta_4u^+(0)\}}{\{E(\mathbf{p})-mc^2\}^2}$$

$$+\frac{\{u^+(0)^*\eta_4\boldsymbol{\beta}_s u^-(\mathbf{p})\}\cdot\{u^-(\mathbf{p})^*\boldsymbol{\beta}_s\eta_4u^+(0)\}}{\{E(\mathbf{p})+mc^2\}^2}\Bigg].$$

Eliminating by (10) the components of $u^{\pm}$ for which $\beta_4 u^{\pm} = 0$, we as before sum over the directions of polarisation in the state $\mathbf{p}$ and average over that of the state $\mathbf{p} = 0$, and obtain for a meson with spin one

$$U_{\text{mag}} = \frac{\pi e^2 \hbar^2}{12 V m^3 c^2} \sum_p \frac{1}{p^2 E(\mathbf{p})} (p^2 + 2m^2c^2) \text{ spur } \beta_4 \boldsymbol{\beta}_s (\mathbf{p}\boldsymbol{\beta}) \beta_4^2 (\mathbf{p}\boldsymbol{\beta}) \boldsymbol{\beta}_s (\beta_4^2 + \beta_4),$$

and for spinless meson the factor $\frac{1}{3}$ will be absent. On evaluating the spurs in both the cases, we get for a meson with spin zero

$$U_{\text{mag}} = 0, \qquad \ldots \quad (44)$$

and for a meson with spin one

$$U_{\text{mag}} = \frac{\pi e^2 \hbar^2}{3 V m^3 c^2} \sum_p \frac{p^2 + 2m^2c^2}{E(\mathbf{p})} = \frac{\pi e^2 \hbar^2}{3 m^3 c^3} \cdot \frac{1}{(2\pi\hbar)^3} \int d\mathbf{p} \frac{p^2 + 2m^2c^2}{\sqrt{p^2 + m^2c^2}}$$

$$= \frac{c^2 \kappa}{12\pi} \lim_{P \to \infty} \cdot \left[\frac{P^3 \sqrt{P^2 + m^2c^2}}{2m^4c^4} + \frac{5}{4} \frac{P\sqrt{P^2 + m^2c^2}}{m^2c^2} - \frac{5}{4} \log \frac{P + \sqrt{P^2 + m^2c^2}}{mc} \right] \qquad \ldots \quad (45a)$$

$$= \frac{e^2 \kappa}{12\pi} \cdot \lim_{P \to \infty} \left[\frac{1}{2} \left(\frac{P}{mc} \right)^4 + \frac{3}{2} \left(\frac{P}{mc} \right)^2 - \frac{5}{4} \log \left(\frac{P}{mc} \right) \right] + \text{finite terms} \qquad \ldots \quad (45b)$$

which diverges as P^4.

Transverse electric field energy.—On performing the calculations as in the previous cases, the time average of the transverse electric field energy is given by (42) as

$$U_{\text{el}} = \frac{1}{8\pi} \int (|\mathbf{E}'_{\text{tr}}|^2)_{AV} \, d\mathbf{r}$$

$$= \frac{\pi e^2 \hbar^2 c^2}{2V} \sum_p \sum_{p'} \Bigg[\frac{\{E(\mathbf{p}') - E(\mathbf{p})\}^2}{\{E(\mathbf{p})E(\mathbf{p}') - m^2c^4 - c^2(\mathbf{p}\mathbf{p}')\}^2} \Big\{ N^+(\mathbf{p}) \big(N^+(\mathbf{p}') + 1 \big)$$

$$\times \{ u^+(\mathbf{p})^* \eta_4 \boldsymbol{\beta}_s u^+(\mathbf{p}') \} \cdot \{ u^+(\mathbf{p}')^* \boldsymbol{\beta}_s \eta_4 u^+(\mathbf{p}) \}$$

$$+ \big(N^-(\mathbf{p}) + 1 \big) N^-(\mathbf{p}') \{ u^-(\mathbf{p})^* \eta_4 \boldsymbol{\beta}_s u^-(\mathbf{p}') \} \cdot \{ u^-(\mathbf{p}')^* \boldsymbol{\beta}_s \eta_4 u^-(\mathbf{p}) \} \Big\}$$

$$+ \frac{\{E(\mathbf{p}') + E(\mathbf{p})\}^2}{\{E(\mathbf{p})E(\mathbf{p}') + m^2c^4 + c^2(\mathbf{p}\mathbf{p}')\}^2}$$

$$\times \Big\{ N^+(\mathbf{p}) N^-(\mathbf{p}') \{ u^+(\mathbf{p})^* \eta_4 \boldsymbol{\beta}_s u^-(\mathbf{p}') \} \cdot \{ u^-(\mathbf{p}')^* \boldsymbol{\beta}_s \eta_4 u^+(\mathbf{p}) \}$$

$$+ \big(N^-(\mathbf{p}) + 1 \big) \big(N^+(\mathbf{p}') + 1 \big) \{ u^-(\mathbf{p})^* \eta_4 \boldsymbol{\beta}_s u^+(\mathbf{p}') \} \cdot \{ u^+(\mathbf{p}')^* \boldsymbol{\beta}_s \eta_4 u^-(\mathbf{p}) \} \Big\} \Bigg]. \qquad (46)$$

When applied to a single positive meson at rest this gives

$$U_{\text{el}} = U_{\text{el}}(\text{vac}+1) - U_{\text{el}}(\text{vac})$$

$$= \frac{\pi e^2 \hbar^2}{2Vm^2c^2} \sum_p \Big[\{u^+(0)^* \eta_4 \boldsymbol{\beta}_s u^+(\mathbf{p})\} \cdot \{u^+(\mathbf{p})^* \boldsymbol{\beta}_s \eta_4 u^+(0)\} + \{u^+(0)^* \eta_4 \boldsymbol{\beta}_s u^-(\mathbf{p})\} \cdot \{u^-(\mathbf{p})^* \boldsymbol{\beta}_s \eta_4 u^+(0)\} \Big],$$

and we then get for a spinless meson

$$U_{\text{el}} = 0, \qquad \text{.. .. .. .. (47)}$$

and for a meson with spin one

$$U_{\text{el}} = \frac{\pi e^2 \hbar^2}{3Vm^2c^2} \cdot \sum_p \frac{p^2}{E(\mathbf{p})} = \frac{\pi e^2 \hbar^2}{3m^3c^3} \cdot \frac{1}{(2\pi\hbar)^3} \int d\mathbf{p} \cdot \frac{p^2}{\sqrt{p^2+m^2c^2}}$$

$$= \frac{e^2\kappa}{12\pi} \lim_{P\to\infty} \left[\frac{P^3\sqrt{P^2+m^2c^2}}{2m^4c^4} - \frac{3}{4}\frac{P\sqrt{P^2+m^2c^2}}{m^2c^2} + \frac{3}{4}\log\frac{P+\sqrt{P^2+m^2c^2}}{mc} \right] \qquad \text{.. (48}a\text{)}$$

$$= \frac{e^2\kappa}{12\pi} \cdot \lim_{P\to\infty} \left[\frac{1}{2}\left(\frac{P}{mc}\right)^4 - \frac{1}{2}\left(\frac{P}{mc}\right)^2 + \frac{3}{4}\log\left(\frac{P}{mc}\right) \right] + \text{finite terms} \qquad \text{.. (48}b\text{)}$$

which diverges also as P^4.

Thus for a meson with spin one

$$W_{\text{sp}} = U_{\text{el}} - U_{\text{mag}}$$

$$= \frac{e^2\kappa}{6\pi} \lim_{P\to\infty} \left[-\frac{P\sqrt{P^2+m^2c^2}}{m^2c^2} + \log\frac{P+\sqrt{P^2+m^2c^2}}{mc} \right] \qquad \text{.. (49)}$$

$$= \frac{e^2\kappa}{6\pi} \lim_{P\to\infty} \left[-\left(\frac{P}{mc}\right)^2 + \log\left(\frac{P}{mc}\right) \right] + \text{finite terms},$$

but for a spinless meson this quantity is zero. Hence W_{sp} which diverges quadratically, may be considered as a contribution to the self-energy of meson at rest due to spin only. In the case of an electron in the positron theory both U_{el} and U_{mag} diverge quadratically, and so also W_{sp}.

5. Energy due to the Fluctuations of the Radiation Field.

It has already been shown that

$$\mathbf{i} = \mathbf{i}_1 + \mathbf{i}_2$$

where $\mathbf{i}_1$ and $\mathbf{i}_2$ are given by (20) and (21) respectively. Then by (26)

$$W_{\text{fluct}} = W^1_{\text{fluct}} + W^2_{\text{fluct}}$$

$$= -\frac{1}{2}\int (\mathbf{i}_1\mathbf{A}_0)_{AV}\, d\mathbf{r} - \frac{1}{2}\int (\mathbf{i}_2\mathbf{A}_0)_{AV}\, d\mathbf{r} \qquad \text{.. .. (50)}$$

where $\mathbf{A}_0$ is the vector potential of the zero-point radiation field. In (21) $\mathbf{i}_2$ is proportional to e^2, so in order to get a result to the desired degree of approximation, ψ in $\mathbf{i}_2$ can be replaced by unperturbed wave functions. Hence introducing (12) and retaining only the diagonal elements, we get

$$W^2_{\text{fluct}} = -\frac{1}{2}\int(\mathbf{i}_2\mathbf{A}_0)d\mathbf{r} = \frac{e^2}{mc^2}\int\psi^*\beta_4^2(\beta\mathbf{A}_0)^2\beta_4^2\psi d\mathbf{r}$$

$$= \frac{e^2}{mc^2}\sum_p\Big\{N^+(\mathbf{p})u^+(\mathbf{p})^*\beta_4^2(\beta\mathbf{A}_0)^2\beta_4^2u^+(\mathbf{p})+ (N^-(\mathbf{p})+1)u^-(\mathbf{p})^*\beta_4^2(\beta\mathbf{A}_0)^2\beta_4^2u^-(\mathbf{p})\Big\}.$$

If we now sum over the directions of polarisation and take the average, we find for a meson with spin one

$$W^2_{\text{fluct}} = \frac{e^2}{6}\sum_p\Big[3A_0^2+\frac{2}{m^2c^2}\big\{A_0^2p^2-(\mathbf{A}_0\mathbf{p})^2\big\}\Big]_{AV}\frac{N^+(\mathbf{p})+N^-(\mathbf{p})+1}{E(\mathbf{p})},$$

and for a spinless meson

$$W^2_{\text{fluct}} = \frac{e^2}{2}(A_0^2)_{AV}\sum_p\frac{N^+(\mathbf{p})+N^-(\mathbf{p})+1}{E(\mathbf{p})}.$$

Thus for a positive meson at rest ($\mathbf{p}=0$), we have in both the cases

$$W^2_{\text{fluct}} = W^2_{\text{fluct}}(\text{vac}+1)-W^2_{\text{fluct}}(\text{vac}) = \frac{e^2}{2mc^2}(A_0^2)_{AV}. \quad .. \quad (51)$$

We write $\mathbf{i}_1 = \mathbf{i}_1^0+\mathbf{i}_1'$, where $\mathbf{i}_1^0$ is calculated for free meson and $\mathbf{i}_1'$ is the correction obtained by first order perturbation theory. Now if

$$\psi_0 = \sum_p\{a(\mathbf{p})\phi^+(\mathbf{p})+b^*(\mathbf{p})\phi^-(\mathbf{p})\} \quad .. \quad .. \quad .. \quad (52)$$

is the wave function unperturbed by the field and ψ_1 is the correction in the first order perturbation theory:

$$\psi_1 = \sum_p\{a(\mathbf{p})\phi^+(\mathbf{p})'+b^*(\mathbf{p})\phi^-(\mathbf{p})'\}, \quad .. \quad .. \quad (53)$$

then

$$\mathbf{i}_1^0 = \frac{ie\hbar}{mc}\frac{\partial\psi_0^*}{\partial x_k}\beta_4^2\beta_k\beta\beta_4^2\psi_0+\text{conj.},$$

$$\mathbf{i}_1' = \frac{ie\hbar}{mc}\Big\{\frac{\partial\psi_0^*}{\partial x_k}\beta_4^2\beta_k\beta\beta_4^2\psi_1-\psi_0^*\beta_4^2\beta\beta_k\beta_4^2\frac{\partial\psi_1}{\partial x_k}\Big\}+\text{conj.} \quad .. \quad .. \quad (54)$$

to a first approximation. It is evident that the term $(\mathbf{i}_1^0\mathbf{A}_0)$ has no diagonal elements because there is no phase relation between $\mathbf{i}_1^0$ and $\mathbf{A}_0$. Again if we substitute (52) and (53) in (54), and retain only the diagonal elements we get

$$\mathrm{i}_1' = \frac{ie\hbar}{mc}\sum_p \Big[N^+(\mathbf{p}) \Big\{ \frac{\partial \phi^+(\mathbf{p})^*}{\partial x_k} \beta_4^2\beta_k\boldsymbol{\beta}\beta_4^2\phi^+(\mathbf{p})' - \phi^+(\mathbf{p})^*\beta_4^2\boldsymbol{\beta}\beta_k\beta_4^2 \frac{\partial \phi^+(\mathbf{p})'}{\partial x_k} \Big\}$$

$$+ (N^-(\mathbf{p})+1) \Big\{ \frac{\partial \phi^-(\mathbf{p})^*}{\partial x_k} \beta_4^2\beta_k\boldsymbol{\beta}\beta_4^2\phi^-(\mathbf{p})' - \phi^-(\mathbf{p})^*\beta_4^2\boldsymbol{\beta}\beta_k\beta_4^2 \frac{\partial \phi^-(\mathbf{p})'}{\partial x_k} \Big\} \Big] + \text{conj.}$$

Hence for a single positive meson at rest ($\mathbf{p} = 0$)

$$\mathrm{i}_1' = \mathrm{i}_1'(\text{vac}+1) - \mathrm{i}_1'(\text{vac}) = -\frac{ie\hbar}{mc}\phi^+(0)^*\beta_4^2\boldsymbol{\beta}\beta_k\beta_4^2\frac{\partial\phi^+(0)'}{\partial x_k} + \text{conj.} \quad .. \quad (55)$$

$\phi^+(\mathbf{p})'$ can be calculated by the usual first order perturbation theory expanding the vector potential $\mathbf{A}_0$ in a Fourier series

$$\mathbf{A}_0 = \sum_k \mathbf{e}_k \Big[A_k^+ \exp.\frac{i}{\hbar}\{(\mathbf{kr})+c|k|t\} + A_k^- \exp.-\frac{i}{\hbar}\{(\mathbf{kr})+c|k|t\} \Big].$$

If we take solutions for free meson in the form (7) and (8), we obtain for $\phi^+(\mathbf{p})'$

$$\phi^+(\mathbf{p})' = ie\sum_{p'} \Big[u^+(\mathbf{p}')^*\beta_4^2 \Big\{1 - \frac{i}{mc}(\mathbf{p}'\boldsymbol{\beta})\Big\}(\boldsymbol{\beta}\mathbf{e}_k)u^+(\mathbf{p})$$

$$\times \Big\{ A_k^+ \frac{\exp.\frac{i}{\hbar}\{E(\mathbf{p}')-E(\mathbf{p})+c|k|\}t}{E(\mathbf{p}')-E(\mathbf{p})+c|k|} + A_k^- \frac{\exp.\frac{i}{\hbar}\{E(\mathbf{p}')-E(\mathbf{p})-c|k|\}t}{E(\mathbf{p}')-E(\mathbf{p})-c|k|} \Big\}\phi^+(\mathbf{p}')$$

$$+ u^-(\mathbf{p}')^*\beta_4^2\Big\{1 - \frac{i}{mc}(\mathbf{p}'\boldsymbol{\beta})\Big\}(\boldsymbol{\beta}\mathbf{e}_k)u^+(\mathbf{p})\Big\{ A_k^+ \frac{\exp.-\frac{i}{\hbar}\{E(\mathbf{p}')+E(\mathbf{p})-c|k|\}t}{E(\mathbf{p}')+E(\mathbf{p})-c|k|}$$

$$+ A_k^- \frac{\exp.-\frac{i}{\hbar}\{E(\mathbf{p}')+E(\mathbf{p})+c|k|\}t}{E(\mathbf{p}')+E(\mathbf{p})+c|k|}\Big\}\phi^-(\mathbf{p}')\Big] \quad .. \quad (56)$$

where $\mathbf{k} = \pm(\mathbf{p}'-\mathbf{p})$, $+$ sign for the terms with A_k^+ and $-$ sign for the terms with A_k^-. Replacing $\phi^+(0)'$ in (55) by the corresponding expression as obtained from (56) we sum over the polarisations in the intermediate states and average over those of the state $\mathbf{p} = 0$, and obtain after some calculations $\mathrm{i}_1' = 0$ both for spins one and zero. Hence W^1_{fluct} contributes nothing to the self-energy.

Finally from (50) and (51) we get for a meson at rest with spin one or zero

$$W_{\text{fluct}} = \frac{e^2}{2mc^2}(A_0^2)_{AV}$$

which is the same as the corresponding expression for the Dirac electron as given by Weisskopf and so we can take his result

$$W_{\text{fluct}} = \frac{e^2\kappa}{\pi} \lim_{P\to\infty} \left(\frac{P}{mc}\right)^2. \quad \ldots \quad \ldots \quad \ldots \quad (57)$$

Thus we have from (26), for spinless meson [Eqns. (44), (47) and (57)]

$$W_{\text{tr}} = \frac{e^2\kappa}{\pi} \lim_{P\to\infty} \left(\frac{P}{mc}\right)^2, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (58)$$

U_{el} and U_{mag} being zero as the particle has no spin; and for meson with spin one [Eqns. (45), (48) and (57) which agree with the corresponding results as given by Richtmyer]

$$W_{\text{tr}} = \frac{e^2\kappa}{6\pi} \lim_{P\to\infty} \left[6\left(\frac{P}{mc}\right)^2 - \frac{P\sqrt{P^2+m^2c^2}}{m^2c^2} + \log\frac{P+\sqrt{P^2+m^2c^2}}{mc}\right] \quad \ldots \quad (59)$$

$$= \frac{e^2\kappa}{6\pi} \lim_{P\to\infty} \left[5\left(\frac{P}{mc}\right)^2 + \log\left(\frac{P}{mc}\right)\right] + \text{finite terms} \quad \ldots \quad \ldots \quad \ldots \quad (60)$$

where it should be noticed that portions of U_{el} and U_{mag}, which diverge as P^4, cancel one another. Hence in both the cases the transverse part as well as the statical part of the self-energy diverge quadratically; so also the total self-energy. It will be shown in the next section that these transverse self-energies can be obtained directly by a second order perturbation calculation.

6. Direct Calculations of Transverse Self-energy by Second order Perturbation.

The interaction terms in the Hamiltonian due to the transverse electromagnetic field are given by (15) and (16) which can be written in the forms

$$H^1 = -\frac{e}{mc}\int \psi^*\beta_4^2 \{(\mathbf{p}\boldsymbol{\beta})(\mathbf{A}\boldsymbol{\beta}) + (\mathbf{A}\boldsymbol{\beta})(\mathbf{p}\boldsymbol{\beta})\}\beta_4^2\psi d\mathbf{r} \quad \ldots \quad \ldots \quad (61)$$

which is linear in $\mathbf{A}$, and where the first $\mathbf{p}$ operates backward on ψ^* and the second $\mathbf{p}$ operates forward on ψ, and

$$H^2 = \frac{e^2}{mc^2}\int \psi^*\beta_4^2(\mathbf{A}\boldsymbol{\beta})^2\beta_4^2\psi d\mathbf{r} \quad \ldots \quad \ldots \quad \ldots \quad (62)$$

which is quadratic in $\mathbf{A}$. We expand $\mathbf{A}$ as a Fourier series

$$\mathbf{A} = \sum_k \sqrt{\frac{2\pi c\hbar^2}{k}}\,.\,\mathbf{e}_k\left\{C_k e^{\frac{i}{\hbar}(\mathbf{kr})} + C_k^* e^{-\frac{i}{\hbar}(\mathbf{kr})}\right\}, \quad \ldots \quad \ldots \quad (63)$$

where $\mathbf{e}_k$ is a real unit vector in the direction of polarisation of the Fourier component $\mathbf{k}\left(k=\frac{\hbar\nu}{c}\right)$, and ψ as

$$\psi=\sum_p \{a(\mathbf{p},t)u^+(\mathbf{p})+b^*(\mathbf{p},t)u^-(\mathbf{p})\}e^{\frac{i}{\hbar}(\mathbf{pr})} \qquad \text{(64)}$$

Now inserting these values of ψ and $\mathbf{A}$ in (61) and (62), we obtain

$$H^1=-\frac{e\hbar}{mc}\sum_{p'}\sum_p\sum_k\sqrt{\frac{2\pi c}{k}}\{a^*(\mathbf{p}')u^+(\mathbf{p}')^*+b(\mathbf{p}')u^-(\mathbf{p}')^*\}$$
$$\times\beta_4^2\{(\mathbf{p}'\boldsymbol{\beta})(\mathbf{e}_k\boldsymbol{\beta})+(\mathbf{e}_k\boldsymbol{\beta}(\mathbf{p}\boldsymbol{\beta})\}\beta_4^2\{C_k\delta(\mathbf{p}-\mathbf{p}'+\mathbf{k})+C_k^*\delta(\mathbf{p}-\mathbf{p}'-\mathbf{k})\}$$
$$\times\{a(\mathbf{p})u^+(\mathbf{p})+b^*(\mathbf{p})u^-(\mathbf{p})\}, \qquad \text{(65)}$$

and

$$H^2=\frac{2\pi e^2\hbar^2}{mc}\cdot\sum_{p'}\sum_p\sum_{k'}\sum_k\frac{1}{\sqrt{kk'}}\{a^*(\mathbf{p}')u^+(\mathbf{p}')^*+b(\mathbf{p}')u^-(\mathbf{p}')^*\}$$
$$\times\beta_4^2(\mathbf{e}_k\boldsymbol{\beta})(\mathbf{e}_{k'}\boldsymbol{\beta})\beta_4^2\Big\{C_kC_{k'}\delta(\mathbf{p}-\mathbf{p}'+\mathbf{k}+\mathbf{k}')+C_kC_{k'}^*\delta(\mathbf{p}-\mathbf{p}'+\mathbf{k}-\mathbf{k}')$$
$$+C_k^*C_{k'}\delta(\mathbf{p}-\mathbf{p}'-\mathbf{k}+\mathbf{k}')+C_k^*C_{k'}^*\delta(\mathbf{p}-\mathbf{p}'-\mathbf{k}-\mathbf{k}')\Big\}\{a(\mathbf{p})u^+(\mathbf{p})+b^*(\mathbf{p})u^-(\mathbf{p})\}. \qquad \text{(66)}$$

It is well known that the operators a^* and b^* respectively increase the number of positive and negative mesons by one, whereas a, b without asterisk decrease the corresponding number of mesons by one. The operator C^* increases the number of photons by one, whereas C decreases it by one. The interaction H^1 gives rise to various transitions involving one light quantum, but H^2 gives rise to transitions in which two light quanta are concerned.

The second approximation to the interaction energy is given by

$$W=W_1+W_2=\sum_i\frac{H^1_{ni}H^1_{in}}{E_n-E_i}+H^2_{nn} \qquad \text{(67)}$$

where the summation is to be taken over all intermediate states. We consider the case of free positive meson at rest ($\mathbf{p}=0$). We assume that in the state n for which we compute the average energy no light quanta are present. We have the following set of transitions to consider from (65) and (66).

1. The meson $u^+(0)$ is absorbed, and a light quantum $\mathbf{k}$ and the meson $u^+(-\mathbf{k})$ are emitted.

$$H^1_{\text{I}n}=\frac{e\hbar}{mc}\sqrt{\frac{2\pi c}{k}}u^+(-\mathbf{k})^*\beta_4^2(\mathbf{k}\boldsymbol{\beta})(\mathbf{e}\boldsymbol{\beta})\beta_4^2u^+(0). \qquad \text{(68)}$$

The light quantum $\mathbf{k}$ and the meson $u^{+}(-\mathbf{k})$ are then absorbed and the meson $u^{+}(0)$ is emitted.

$$H^{1}_{n\mathrm{I}} = \frac{e\hbar}{mc}\sqrt{\frac{2\pi c}{k}}\, u^{+}(0)^{*}\beta_4^2(\mathbf{e}\boldsymbol{\beta})(\mathbf{k}\boldsymbol{\beta})\beta_4^2 u^{+}(-\mathbf{k}). \quad \ldots \quad \ldots (69)$$

2. The light quantum $-\mathbf{k}$ is emitted and a pair of mesons $u^{+}(0)$ and $u^{-}(-\mathbf{k})$ are created.

$$H^{1}_{\mathrm{II}n} = \frac{e\hbar}{mc}\sqrt{\frac{2\pi c}{k}}\, u^{+}(0)^{*}\beta_4^2(\mathbf{e}\boldsymbol{\beta})(\mathbf{k}\boldsymbol{\beta})\beta_4^2 u^{-}(-\mathbf{k}). \quad \ldots \quad \ldots (70)$$

The light quantum $-\mathbf{k}$ is then absorbed and a pair of mesons $u^{+}(0)$ and $u^{-}(-\mathbf{k})$ are annihilated.

$$H^{1}_{n\mathrm{II}} = \frac{e\hbar}{mc}\sqrt{\frac{2\pi c}{k}}\, u^{-}(-\mathbf{k})^{*}\beta_4^2(\mathbf{k}\boldsymbol{\beta})(\mathbf{e}\boldsymbol{\beta})\beta_4^2 u^{+}(0). \quad \ldots \quad \ldots (71)$$

3. The meson $u^{+}(0)$ and the light quantum $\mathbf{k}$ are emitted, and simultaneously the meson $u^{+}(0)$ and the light quantum $\mathbf{k}$ are absorbed.

$$H^{2}_{nn} = \frac{4\pi e^2\hbar^2}{mc}\cdot\frac{1}{k}\, u^{+}(0)^{*}\beta_4^2(\mathbf{e}\boldsymbol{\beta})^2\beta_4^2 u^{+}(0). \quad \ldots \quad \ldots \quad \ldots (72)$$

The energy differences $E_n - E_i$ are given by

$$E_n - E_{\mathrm{I}} = E(0) - ck - E(-\mathbf{k}) = mc^2 - ck - E(-\mathbf{k}),$$

$$E_n - E_{\mathrm{II}} = E(0) - \{2E(0) + ck + E(-\mathbf{k})\} = -mc^2 - ck - E(-\mathbf{k}).$$

From equations (68–71), we then get

$$W_1 = \frac{2\pi e^2\hbar^2}{m^2 c}\cdot\frac{1}{k}\left[\frac{\{u^{+}(0)^{*}\beta_4^2(\mathbf{e}\boldsymbol{\beta})(\mathbf{k}\boldsymbol{\beta})\beta_4^2 u^{+}(-\mathbf{k})\}\{u^{+}(-\mathbf{k})^{*}\beta_4^2(\mathbf{k}\boldsymbol{\beta})(\mathbf{e}\boldsymbol{\beta})\beta_4^2 u^{+}(0)\}}{mc^2 - ck - E(-\mathbf{k})}\right.$$

$$\left. - \frac{\{u^{+}(0)^{*}\beta_4^2(\mathbf{e}\boldsymbol{\beta})(\mathbf{k}\boldsymbol{\beta})\beta_4^2 u^{-}(-\mathbf{k})\}\{u^{-}(-\mathbf{k})^{*}\beta_4^2(\mathbf{k}\boldsymbol{\beta})(\mathbf{e}\boldsymbol{\beta})\beta_4^2 u^{+}(0)\}}{mc^2 + ck + E(-\mathbf{k})}\right], \quad \ldots (73)$$

and from (72)

$$W_2 = \frac{4\pi e^2\hbar^2}{mc}\cdot\frac{1}{k}\, u^{+}(0)^{*}\beta_4^2(\mathbf{e}\boldsymbol{\beta})^2\beta_4^2 u^{+}(0)\,. \quad \ldots \quad \ldots (74)$$

We now sum over the polarisations of meson in the intermediate state $-\mathbf{k}$ and average over that of the initial state; then summing over both directions of polarisation of photon, we obtain from (73) for a spinless meson

$$W_1 = 0,$$

and for a meson with spin one

$$W_1 = -\frac{2\pi e^2\hbar^2}{3m}\cdot\frac{1}{E(-\mathbf{k})}\,.$$

Again for W_2 given by (74) if we average over the polarisations in the state $\mathbf{p} = 0$, and sum over that of the photon, we get in both the cases

$$W_2 = \frac{4\pi e^2 \hbar^2}{mc} \cdot \frac{1}{k}.$$

We shall now have to sum over all light quanta and this summation can be replaced by an integration. Hence for a spinless meson

$$W = \frac{4\pi e^2 \hbar^2}{mc} \cdot \frac{1}{(2\pi\hbar)^3} \int \frac{d\mathbf{k}}{k} = \frac{e^2 \kappa}{\pi} \lim_{P \to \infty} \left(\frac{P}{mc}\right)^2, \qquad .. (75)$$

and for a meson with spin one

$$W = \frac{4\pi e^2 \hbar^2}{mc} \cdot \frac{1}{(2\pi\hbar)^3} \int \frac{d\mathbf{k}}{k} - \frac{2\pi e^2 \hbar^2}{3mc} \cdot \frac{1}{(2\pi\hbar)^3} \int \frac{d\mathbf{k}}{\sqrt{k^2 + m^2c^2}}$$

$$= \frac{e^2 \kappa}{6\pi} \lim_{P \to \infty} \left[6\left(\frac{P}{mc}\right)^2 - \frac{P\sqrt{P^2 + m^2c^2}}{m^2c^2} + \log \frac{P + \sqrt{P^2 + m^2c^2}}{mc} \right] \qquad .. (76)$$

Thus (75) is in agreement with (58) and (76) with (59). It should be noticed that for a meson at rest the interacting term H^2 contributes only to that part of the self-energy which is produced by the zero-point fluctuations of the radiation field, while H^1 is responsible for the part which depends on the spin.

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