SOME NEW RELATIVISTIC DISTRIBUTIONS OF RADIAL SYMMETRY.

By V. V. NARLIKAR, G. K. PATWARDHAN *and* P. C. VAIDYA,
Benares Hindu University.

(*Received, October 27, 1942.*)

ABSTRACT.

Several new solutions, which are static and isotropic, of Einstein's field equations are obtained. The inadequacy of the usual boundary conditions is brought to light. Following Tolman's treatment of the subject it is shown that certain expressions are tacitly assumed to be continuous on the boundary. For the particular case of static isotropic solutions of radial symmetry the new condition is the continuity of μ'. The examples given have possible astronomical interest and they bring out the efficacy of the new condition.

INTRODUCTION.

Following some recent work by Tolman (1939) we have discussed in a series of communications * the solutions of Einstein's field equations for spheres of fluid of the form,

$$ds^2 = -e^{\lambda}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) + e^{\nu}dt^2 \quad \ldots \quad (1)$$

$$\lambda = \lambda(r), \; \nu = \nu(r).$$

We have also gone into the question of boundary conditions and brought to light a new necessary condition. In this paper we consider solutions of the form,

$$ds^2 = -e^{\mu}(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2) + e^{\nu}dt^2 \quad \ldots \quad (2)$$

$$\mu = \mu(r), \; \nu = \nu(r).$$

Obviously, (1) and (2) are transformable into each other. The physical content of (2) cannot, therefore, be different from that of (1). The object in choosing the other co-ordinate system is this. The form of the differential equations in μ and ν suggests some new particular solutions which are not quite obvious from the differential equations arising from (1). The results of the next section will bear this out.

For the line-element (2) the non-zero components of the energy-momentum tensor are

$$8\pi T_1^1 = -8\pi p = -e^{-\mu}\left(\frac{\mu'^2}{4} + \frac{\mu'\nu'}{2} + \frac{\mu' + \nu'}{r}\right) - \Lambda, \quad \ldots \quad (3)$$

* Two letters are recently published in *Current Science* and a paper is in the press with the *Journal of the Bombay University*.—V. V. N. (7–7–1943).

$$8\pi\, T_2^2 = 8\pi\, T_3^3 = -8\pi q = -e^{-\mu}\left(\frac{\mu''}{2}+\frac{\nu''}{2}+\frac{\nu'^2}{4}+\frac{\mu'+\nu'}{2r}\right)-\Lambda, \quad .. \quad (4)$$

$$8\pi\, T_4^4 = 8\pi\rho = -e^{-\mu}\left(\mu''+\frac{\mu'^2}{4}+\frac{2\mu'}{r}\right)-\Lambda. \quad .. \quad .. \quad (5)$$

Here and in what follows a dash denotes a differentiation with regard to r as in μ'. For fluid spheres of stellar dimensions the cosmological constant Λ can be safely neglected. For the pressure to be isotropic it is necessary that $p = q$ or

$$\mu''+\nu''+\frac{\nu'^2-\mu'^2}{2}-\mu'\nu'-\frac{\mu'+\nu'}{r} = 0. \quad .. \quad .. \quad (6)$$

Let us now consider the particular solutions arising out of the last equation.

NEW PARTICULAR SOLUTIONS.

The equation of isotropy (6) can be expressed as

$$z''-z'\left(\mu'+\frac{1}{r}\right)+z\left(\frac{\mu''}{2}-\frac{\mu'^2}{4}-\frac{\mu'}{2r}\right) = 0, \quad .. \quad .. \quad (7)$$

where

$$z = e^{\frac{\nu}{2}}. \quad .. \quad .. \quad .. \quad .. \quad (8)$$

In our search of particular solutions we may start with the simplifying assumption,

$$\frac{\mu''}{2}-\frac{\mu'^2}{4}-\frac{\mu'}{2r} = \left(1-\frac{n^2}{4}\right)\frac{1}{r^2}. \quad .. \quad .. \quad .. \quad (9)$$

It leads to

$$e^{-\frac{\mu}{2}} = A_1 r^{1+\frac{n}{2}}+A_2 r^{1-\frac{n}{2}}, \quad .. \quad .. \quad .. \quad (10.1)$$

$$e^{\frac{\nu}{2}} = \left(B_1 r^{1+\frac{x}{2}}+B_2 r^{1-\frac{x}{2}}\right)\left(A_1 r^{1+\frac{n}{2}}+A_2 r^{1-\frac{n}{1}}\right)^{-1}, \quad .. \quad (10.2)$$

where

$$x = (2n^2-4)^{\frac{1}{2}}. \quad .. \quad .. \quad .. \quad .. \quad (11)$$

The pressure and density are given by

$$8\pi p - \frac{\left(B_1 r^{1+\frac{x}{2}}+B_2 r^{1-\frac{x}{2}}\right)^{-1}}{4}\left[(n-x)^2\alpha+(n+x)^2\beta-2(3n^2+4)\gamma\right], \quad (10.3)$$

$$8\pi\rho = \left(1-\frac{n^2}{4}\right)A_1^2 r^n+2A_1A_2\left(1+\frac{5}{4}n^2\right)+A_2^2\left(1-\frac{n^2}{4}\right)r^{-n}, \quad (10.4)$$

where
$$\alpha = A_1^2 B_1 r^{1+n+\frac{2}{x}} + A_2^2 B_2 r^{1-n-\frac{x}{2}}, \quad \ldots \quad \ldots \quad (12.1)$$

$$\beta = A_1^2 B_2 r^{1+n-\frac{x}{2}} + A_2^2 B_1 r^{1-n+\frac{x}{2}}, \quad \ldots \quad \ldots \quad \ldots \quad (12.2)$$

$$\gamma = B_1 r^{1+\frac{x}{2}} + B_2 r^{1-\frac{x}{2}}. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (12.3)$$

The constants A_1, A_2, B_1, B_2 are to be determined (Tolman, 1934*b*) as usual by using the conditions (i) $p = 0$ at $r = a$ and that (ii) $g_{\mu\nu}$ are continuous at $r = a$. The external (Eddington, 1924) solution, valid for $r \geqslant a$ is

$$ds^2 = -\left(1+\frac{m}{2r}\right)^4 dr^2 - r^2 (d\theta^2 + \sin^2\theta d\phi^2) + \left(\frac{1-\frac{m}{2r}}{1+\frac{m}{2r}}\right)^2 dt^2. \quad \ldots \quad (13)$$

Thus (i) and (ii) provide only three conditions. We are showing in the next section that all the constants become known when we use the additional condition that μ' is continuous at $r = a$. In all this a is taken as the radius of the material distribution. We have considered the fourth condition in a more detailed way elsewhere as it was never hinted at or used in such problems before. The significance of the new condition is this: it ensures that the total energy of the distribution (2) is the same as that of a particle of mass m so far as the field at great distances is concerned. Using the conditions we get

$$A_1 = -a^{-1-\frac{n}{2}} \frac{(2-n)-(2+n)\frac{m}{2a}}{2n\left(1+\frac{m}{2a}\right)^3}, \quad \ldots \quad \ldots \quad (14.1)$$

$$A_2 = a^{-1+\frac{n}{2}} \frac{(2+n)-(2-n)\frac{m}{2a}}{2n\left(1+\frac{m}{2a}\right)^3}, \quad \ldots \quad \ldots \quad \ldots \quad (14.2)$$

$$B_1 = \frac{\frac{4m}{a} - (2-x) - (2+x)\frac{m^2}{4a^2}}{2x\left(1+\frac{m}{2a}\right)^4} a^{-1-\frac{x}{2}}, \quad \ldots \quad \ldots \quad (14.3)$$

$$B_2 = -\frac{\frac{4m}{a} - (2+x) - (2-x)\frac{m^2}{4a^2}}{2x\left(1+\frac{m}{2a}\right)^4} a^{-1+\frac{x}{2}}. \quad \ldots \quad (14.4)$$

For ρ to be non-negative n must not exceed 2 and for x to be real n cannot be less than $\sqrt{2}$. Thus the above solution is possible only if

$$2 \geqslant n > \sqrt{2} \qquad \text{(15)}$$

and for $n = \sqrt{2}$ we have

$$e^{-\frac{\mu}{2}} = A_1 r^{1+\frac{1}{\sqrt{2}}} + A_2 r^{1-\frac{1}{\sqrt{2}}} \qquad \text{(16.1)}$$

$$e^{\frac{\nu}{2}} = \frac{B_1 + B_2 \log r}{A_1 r^{\frac{1}{\sqrt{2}}} + A_2 r^{-\frac{1}{\sqrt{2}}}}, \qquad \text{(16.2)}$$

$$8\pi p = A_1^2 \left\{ \frac{1}{2}\left(r^{\frac{1}{\sqrt{2}}} - br^{-\frac{1}{\sqrt{2}}}\right)^2 - 4b - \sqrt{2} B_2 \frac{r^{\frac{1}{\sqrt{2}}} + br^{-\frac{1}{\sqrt{2}}}}{B_1 + B_2 \log r} \right\}, \qquad \text{(16.3)}$$

$$8\pi\rho = A_1^2 \left[\frac{1}{2}\left(r^{\frac{1}{\sqrt{2}}} + br^{-\frac{1}{\sqrt{2}}}\right)^2 + 6b \right], \qquad \text{(16.4)}$$

where

$$b = \frac{A_2}{A_1}. \qquad \text{(17)}$$

There is no change in the forms of A_1 and A_2 but the values of B_1 and B_2 now become

$$B_1 = \frac{1 - \frac{m}{2a}}{a\left(1 + \frac{m}{2a}\right)^3} - \frac{\frac{m}{a} - \left(1 - \frac{m}{2a}\right)^2}{a\left(1 + \frac{m}{2a}\right)^4} \log a, \qquad \text{(18.1)}$$

$$B_2 = \frac{\frac{m}{a} - \left(1 - \frac{m}{2a}\right)^2}{a\left(1 + \frac{m}{2a}\right)^4}. \qquad \text{(18.2)}$$

It can easily be verified that $n = x = 2$ corresponds to Schwarzschild's solution in the isotropic form. For $n < \sqrt{2}$ the forms of μ and ρ are unchanged But we have

$$e^{\frac{\nu}{2}} = \left(A_1 r^{\frac{n}{2}} + A_2 r^{-\frac{n}{2}}\right) r^2 \left[B_1 \cos\left(\frac{x}{2} \log r\right) + B_2 \sin\left(\frac{x}{2} \log r\right) \right] \qquad \text{(19.1)}$$

and

$$8\pi p = A_1^2 r^n \left\{ \frac{3}{4} n^2 - 1 + \frac{nx}{2} \tan\left(\frac{x}{2} \log r - \phi\right) \right\} - 2A_1 A_2 \left(1 + \frac{3}{4} n^2\right)$$
$$+ A_2^2 r^{-n} \left\{ \frac{3}{4} n^2 - 1 - \frac{nx}{2} \tan\left(\frac{x}{2} \log r - \phi\right) \right\}, \qquad \text{(19.2)}$$

where
$$x = (4-2n^2)^{\frac{1}{2}}, \qquad (20)$$
and
$$\tan\phi = B_2/B_1. \qquad (21)$$

Another solution, which is new and important and which brings out the inadequacy of the usual boundary conditions follows from the assumption

$$\mu' = \frac{k}{r}, \qquad (22)$$

where k is a constant. From the assumption and the usual boundary conditions one gets

$$e^{\mu} = \left(1+\frac{m}{2a}\right)^4 \left(\frac{r}{a}\right)^k, \qquad (23.1)$$

$$e^{\frac{\nu}{2}} = \frac{1-\frac{m}{2a}}{1+\frac{m}{2a}} \cdot \frac{1}{2d} \cdot \left(\frac{r}{a}\right)^{1-n+\frac{k}{2}} \left[(c+d)-(c-d)\left(\frac{r}{a}\right)^{2n}\right], \qquad (23.2)$$

where
$$c = 2+3k+\frac{3}{4}k^2, \qquad (24.1)$$
$$d = n(k+2), \qquad (24.2)$$
$$n = \left(1+2k+\frac{1}{2}k^2\right)^{\frac{1}{2}}. \qquad (24.3)$$

The density and pressure are given by the expressions

$$8\pi\rho = -\frac{a^k r^{-k-2}}{\left(1+\frac{m}{2a}\right)^4}\left(k+\frac{k^2}{4}\right), \qquad (23.3)$$

$$8\pi p = \frac{a^k(c^2-d^2)}{\left(1+\frac{m}{2a}\right)^4 r^{k+2}}\left[\frac{a^{2n}-r^{2n}}{(c+d)a^{2n}-(c-d)r^{2n}}\right]. \qquad (23.4)$$

It is worthy of notice that

$$8\pi p' = -\frac{(k+2)\,r^{-k-3}\,a^k}{\left(1+\frac{m}{2a}\right)^4 (1-y)^2}\left[y+\frac{k+\frac{k^2}{4}}{c+d}\right]^2, \qquad (25)$$

where
$$y = \frac{c-d}{c+d}\left(\frac{r}{a}\right)^{2n}. \qquad (26)$$

ρ and p will be non-negative and n real provided

$$0 \geqslant k \geqslant -2+\sqrt{2}. \qquad (27)$$

A solution thus obtained does not satisfy the additional criterion, which is the analogue of the Gauss theorem, that the field represented should be equivalent to that of a particle of mass m at infinity. This is true only if μ' is continuous which means the additional restraint, viz.,

$$\frac{m}{a} = -\frac{2k}{k+4}. \qquad (28)$$

In the case $n = 0$, the forms of ν and p differ. We have

$$e^{\frac{\nu}{2}} = \frac{1-\frac{m}{2a}}{1+\frac{m}{2a}} \left(\frac{r}{a}\right)^{\frac{1}{\sqrt{2}}} \left[1+\frac{\log a - \log r}{2\sqrt{2}}\right], \qquad (29.1)$$

$$8\pi p = \frac{1}{2} \frac{a^{-2+\sqrt{2}}\, r^{-\sqrt{2}}}{\left(1+\frac{m}{2a}\right)^4} \left[\frac{\log a - \log r}{\log a - \log r + 2\sqrt{2}}\right]. \qquad (29.2)$$

This implies the particular relation,

$$\frac{m}{a} = \frac{4-2\sqrt{2}}{2+\sqrt{2}}. \qquad (30)$$

In case n is not real we put

$$n_1 = \sqrt{-(1+2k+k^2/2)}. \qquad (31)$$

In this case also the forms of μ and ρ are the same as before. Only ν and p become different.

$$e^{\frac{\nu}{2}} = \frac{1-\frac{m}{2a}}{1+\frac{m}{2a}} \sec\phi \left(\frac{r}{a}\right)^{1+\frac{k}{2}} \cos\left[n_1 \log(r/a)+\phi\right], \qquad (32.1)$$

$$8\pi p = \frac{n_1(k+2)\, a^{\lambda}\, r^{-k-2}}{\left(1+\frac{m}{2a}\right)^4} \left\{\tan\phi - \tan\left[n_1 \log(r/a)+\phi\right]\right\}, \qquad (32.2)$$

where

$$\tan\phi = \frac{2+3k+3k^2/4}{n_1(k+2)}. \qquad (33)$$

For p and ρ to be non-negative it is necessary that

$$-2+\sqrt{2} \geqslant k \geqslant -2. \qquad (34)$$

It is worthy of notice that in all these cases p strictly diminishes from the centre to the surface without attaining a minimum.

We will now present the considerations that have lead us to regard μ' as continuous at $r = a$.

New Continuity Conditions on the Boundary, $r = a$.

If the material distribution represented by the internal solution

$$ds^2 = g_{\mu\nu}\, dx^{\mu}\, dx^{\nu} \qquad (35)$$

is to behave like a particle of mass m at great distances it is required that the total energy of the system be m. In the exposition of this point Tolman (1934*a*) has tacitly assumed, among other things, the continuity of

$$-g^{\alpha 4}\sqrt{-g}\,A^{\gamma}_{\mu\alpha}+\frac{1}{2}\,g^{4}_{\mu}\,g^{\alpha\beta}\,\sqrt{-g}\,A^{\gamma}_{\alpha\beta}$$

$$\mu=1,2,3,4\,;\;\gamma=1,2,3 \qquad \ldots \qquad (36)$$

on the boundary of the distribution. Here

$$A^{\gamma}_{\alpha\beta}=-\Gamma^{\gamma}_{\alpha\beta}+\delta^{\gamma}_{\alpha}\,\Gamma^{p}_{\beta p}+\delta^{\gamma}_{\beta}\,\Gamma^{p}_{\alpha p}\,. \qquad \ldots \qquad (37)$$

Let us see what this condition really amounts to for several familiar systems.

Case (i):

$$g_{\mu\nu}=\delta_{\mu\nu}+h_{\mu\nu}\,. \qquad \ldots \qquad (38)$$

This is the case of a weak gravitational field, $\delta_{\mu\nu}$ having the usual meaning and the squares of h's being neglected. Here the continuity to be discussed is of

$$\frac{1}{2}\,\delta^{\gamma\gamma}\left[\frac{\partial h_{\mu\gamma}}{\partial x^{4}}+\frac{\partial h_{4\gamma}}{\partial x^{\mu}}-\frac{\partial h_{4\mu}}{\partial x^{\gamma}}\right]$$

$$-\frac{1}{4}\,g^{\gamma}_{\mu}\,\frac{\partial}{\partial x^{\gamma}}\,[h_{44}-h_{11}-h_{22}-h_{33}]$$

$$-\frac{1}{4}\,g^{\gamma}_{4}\,\frac{\partial}{\partial x^{\mu}}\,[h_{44}-h_{11}-h_{22}-h_{33}]$$

$$-\frac{1}{4}\,g^{4}_{\mu}\,\delta^{\alpha\alpha}\delta^{\gamma\gamma}\left(2\,\frac{\partial h_{\alpha\gamma}}{\partial x^{\alpha}}-\frac{\partial h_{\alpha\alpha}}{\partial x^{\gamma}}\right)$$

$$+\frac{1}{4}\,g^{4}_{\mu}\,\delta^{\gamma\gamma}\,\frac{\partial}{\partial x^{\gamma}}\,[h_{44}-h_{11}-h_{22}-h_{33}]. \qquad \ldots \qquad (39)$$

[*N.B.*—In the above γ, μ are real suffixes while α is a 'dummy' suffix so far as the summation convention goes.]

From the above the twelve expressions corresponding to $\mu=1,2,3,4$ and $\gamma=1,2,3$ can be easily obtained in concrete forms. If the cross terms $h_{\mu\nu}=0$ (when $\mu\neq\nu$) the following six expressions are required to be continuous on the boundary of the distribution:

$$(1)\;\frac{\partial}{\partial x^{1}}\,(h_{22}+h_{23})\,; \qquad (2)\;\frac{\partial}{\partial x^{4}}\,(h_{44}+h_{11}-h_{22}-h_{33})\,;$$

$$(3)\;\frac{\partial}{\partial x^{2}}\,(h_{11}+h_{33})\,; \qquad (4)\;\frac{\partial}{\partial x^{4}}\,(h_{44}-h_{11}+h_{22}-h_{33})\,;$$

$$(5)\;\frac{\partial}{\partial x^{3}}\,(h_{11}+h_{22})\,; \qquad (6)\;\frac{\partial}{\partial x^{4}}\,(h_{44}-h_{11}-h_{22}+h_{33}). \quad \ldots \quad (40)$$

Case (ii):

$$ds^2 = -A(dx^1)^2 - B(dx^2)^2 - C(dx^3)^2 + D(dx^4)^2. \quad .. \quad .. \quad (41)$$

The continuity of the following six expressions is demanded on the boundary:

$$(1)\ -\frac{A_4}{A}+\frac{B_4}{B}+\frac{C_4}{C}+\frac{D_4}{D}; \qquad (2)\ \frac{B_1}{B}+\frac{C_1}{C};$$

$$(3)\ \frac{A_4}{A}-\frac{B_4}{B}+\frac{C_4}{C}+\frac{D_4}{D}; \qquad (4)\ \frac{A_2}{A}+\frac{C_2}{C};$$

$$(5)\ \frac{A_4}{A}+\frac{B_4}{B}-\frac{C_4}{C}+\frac{D_4}{D}; \qquad (6)\ \frac{A_3}{A}+\frac{B_3}{B}. \quad .. \quad (42)$$

In the notation adopted in the above expressions a suffix denotes a differentiation with respect to the corresponding variable. Thus $D_2 \equiv \partial D/\partial x^2$.

Case (iii): For the line-element (1) which is a particular case of case (ii) no new condition is implied if λ and ν are continuous. But for (2) the condition turns out to be that μ' be continuous. Hence we understand how the usual boundary conditions are enough in the former case while they fall short of one in the latter. For (2) even ν' is continuous. This has a deeper significance which we intend to consider in another paper.

References.

Eddington, Sir A. S. (1924). The Mathematical Theory of Relativity, 144.

Tolman, R. C. (1934*a*). Relativity, Thermodynamics and Cosmology, 232.

——— (1934*b*). *Ibid.*, 247.

——— (1939). Static Solutions of Einstein's Field Equations for Spheres of Fluid. *Phys. Rev.*, **55**, 364