A NOTE ON THE THEORY OF FOURIER SERIES.

By S. MINAKSHISUNDARAM, *Andhra University, Guntur.*

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1.

The scope of this paper is to generalise some results obtained by Szász (1938) and others (1925) on the determination of the jump $f(x+0)-f(x-0)$ of a function by its Fourier coefficients. We use the notations and formulæ of a paper by L. S. Bosanquet and J. M. Hyslop (1937).

Let $f(t)$ be integrable L in $(-\pi, \pi)$ and periodic with period 2π. We write for $t > 0$

$$\phi(t) = \frac{1}{2}\{f(x+t)+f(x-t)\}$$

$$\psi(t) = \frac{1}{2}\{f(x+t)-f(x-t)\}$$

$$\Phi_\alpha(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-u)^{\alpha-1}\phi(u)\,du \qquad \alpha > 0$$

$$\Phi_0(t) = \phi(t)$$

$$\phi_\alpha(t) = \Gamma(\alpha+1)t^{-\alpha}\Phi_\alpha(t)$$

$$\Phi^*_\alpha(t) = \int_0^t \left|\phi_\alpha(u)\right| du$$

$$\phi^*_\alpha(t) = \frac{1}{t}\int_0^t \phi_\alpha(u)\,du$$

and employ $\Psi_\alpha(t)$, $\psi_\alpha(t)$, $\Psi^*_\alpha(t)$, and $\psi^*_\alpha(t)$, with similar meaning. Without loss of generality we may assume the Fourier series of $f(t)$ to be

$$\sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt),$$

and then the Fourier series of $\phi(t)$ will be

$$\sum_1^{\infty} c_n \cos nt$$

where $c_n = a_n \cos nx + b_n \sin nx$. The allied series is

$$\sum_{n=1}^{\infty}(b_n \cos nx - a_n \sin nx)$$

and the Fourier series of $\psi(t)$ is

$$\sum_1^{\infty} \bar{c}_n \sin nt$$

where $\bar{c}_n = b_n \cos nx - a_n \sin nx$.

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We shall also write, for $\alpha > -1$, $\omega \geqslant 0$

$$C_\alpha(\omega) = \omega^{-\alpha} A_\alpha(\omega) = \sum_{n<\omega} \left(1-\frac{n}{\omega}\right)^\alpha c_n$$

$$B_\alpha(\omega) = \sum_{n<\omega} (\omega-n)^\alpha n c_n$$

and define $\bar{C}_\alpha(\omega)$, $\bar{A}_\alpha(\omega)$ and $\bar{B}_\alpha(\omega)$ in a similar way.

The following theorems were proved by Zygmund (1925) and Szász (1938) respectively.

THEOREM A.—$n\bar{c}_n$ converges $(c, 1+\delta)$ to $\frac{2}{\pi}\psi(+0)$ whenever $\psi(+0)$ exists.

THEOREM B.—If $\frac{1}{t}\int_0^t |\psi(u)|\,du = O(1)$ as $t \to 0$ and $\frac{1}{t}\int_0^t \psi(u)du \to d(x)$, as $t\to 0$, then

$$\bar{\sigma}_{2n} - \bar{\sigma}_n \to \frac{2}{\pi}\log 2\,.\,d(x), \text{ as } n \to \infty,$$

where $\bar{\sigma}_n$ is the sequence of the arithmetic means of the partial sums of the allied series.

This latter theorem has been generalised by H. C. Chow (1941) in the form of

THEOREM C.—With the same hypotheses as in theorem B

$$\lim_{n=\infty} \left\{\bar{\sigma}^\alpha_{2n} - \bar{\sigma}^\alpha_n\right\} = \frac{2}{\pi}\log 2\,.\,d(x)$$

for every $\alpha > 0$, where $\bar{\sigma}^\alpha_n$ is the cesaro mean of order α of the allied series.

Generalising theorems A and C we may state

THEOREM D.—$n\bar{c}_n$ converges $(c, \alpha+1+\delta)$ to $\frac{2}{\pi}\psi_\alpha(+0)$ provided the limit $\psi_\alpha(+0)$ exists, for every $\delta > 0$ and $\alpha \geqslant 0$.

THEOREM E.—If $\Psi^*_\alpha(t) = O(t)$, as $t \to 0$, and $\psi^*_\alpha(+0)$ exists, then

$$\lim_{n=\infty} \left\{\bar{\sigma}^\beta_{2n} - \bar{\sigma}^\beta_n\right\} = \frac{2}{\pi}\log 2\,.\,\psi^*_\alpha(+0)$$

where $\beta = \alpha+\delta$, $\delta > 0$ and arbitrary.

A converse of theorem D that suggests itself is,

THEOREM F.—If $n\bar{c}_n$ converges $(c\alpha)$, $\alpha \geqslant 0$ to $\delta(x)$ then

$$\lim_{t=0} \psi_\beta(t) = \frac{\pi}{2}\delta(x) \text{ for } \beta > \alpha.$$

In what follows we shall prove these theorems using Rieszian means together with other results.

2.

We quote here a number of theorems as lemmas, some of which are classical while the others are either proved or referred by Bosanquet and Hyslop (1937).

LEMMA 1.—Let $s(\omega)$ be a function defined in (0∞) and finite in every finite interval. Let the kernel $k(t, \omega)$ be such that

$$\text{(i)} \lim_{t\to 0} k(t, \omega) = 0$$

uniformly

$$\text{(ii)} \int_0^\infty |k(t, \omega)|\,d\omega < \infty$$

uniformly in t

$$\text{(iii)} \lim_{t=0} \int_0^\infty k(t, \omega)\,d\omega = k \text{ exists.}$$

Then

$$\lim_{t=0}\int_0^\infty k(t,\,\omega)\;s(\omega)\;d\omega = k\,.\lim_{w=\infty} s(\omega)$$

whenever the limit on the right exists.

This is simply Toeplitz theorem and we can state an analogous lemma when $\omega\to 0$ and $t\to\infty$.

Lemma 2.—

$$C_\alpha'(\omega) = \alpha\omega^{-\alpha-1}B_{\alpha-1}(\omega) = \alpha\omega^{-1}\left\{C_{\alpha-1}(\omega)-C_\alpha(\omega)\right\} \text{ for } \alpha > -1.$$

Lemma 3.—The series Σc_n is summable $(C\alpha)$ to s, if and only if, $\lim_{\omega=\infty} C_\alpha(\omega) = s$, and the sequence nc_n converges $(C\alpha)$ to γ, if and only if

$$\lim_{\omega=\infty} \alpha\omega^{-\alpha} B_{\alpha-1}(\omega) = \gamma.$$

We require the properties of some auxiliary functions which have been obtained by Bosanquet and Hyslop.

We write

$$\gamma_\alpha(x)+i\bar{\gamma}_\alpha(x) = \int_0^1 (1-t)^{\alpha-1}e^{ixt}dt \qquad x\geqslant 0,\ \alpha>0$$

$$\gamma_\alpha^h(x) = \frac{d^h}{dx^h}\,\gamma_\alpha(x)$$

where h is an integer, with a similar meaning for $\bar{\gamma}_\alpha^h(x)$.

Lemma 4.—

$$\bar{\gamma}_{1+\alpha}(x)-\bar{\gamma}_\alpha(x) = \gamma_\alpha'(x).$$

Lemma 5.—If h is a positive integer and $\chi = \min\,(\alpha,\ h+2)$, $\lambda = \min\,(\alpha,\ h+1)$, then

$$\left|\gamma_\alpha^h(x)\right| \leqslant A(1+x)^{-\chi}$$

$$\left|\bar{\gamma}_\alpha^h(x)\right| \leqslant A(1+x)^{-\lambda}$$

where A is a positive constant.[1]

Lemma 6.—If $0\leqslant h = [\alpha]\leqslant\alpha<\beta-1\leqslant h+1$ and

$$J(\omega,\,u) = \frac{(-1)^{h+1}\omega^{h+1}u^{\alpha+1}}{\Gamma(\alpha+1)\Gamma(h+1-\alpha)}\int_u^\eta (t-u)^{h-\alpha}\gamma_\beta^{h+2}(\omega t)dt$$

where $\eta>0$, then for $\omega\geqslant 0$, $0\leqslant u\leqslant\eta$

$$|J(\omega,\,u)| \leqslant A\omega^\alpha u^{\alpha+1}(1+\omega u)^{-\beta}.$$

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[1] Here and elsewhere A denotes some positive number independent of the variables involved, which may be different at different places.

LEMMA 7.—If $\alpha > 0$, $h = [\alpha] \leqslant \alpha < \beta \leqslant h+1$

$$P(v, t) = \frac{(-1)^{h+1}\beta t^h}{\Gamma(\alpha+1)\Gamma(h+1-\alpha)} \int_v^\infty (u-v)^{h-\alpha}\, \bar{\gamma}_\beta^{h+1}(ut)du$$

$$Q(v, t) = \int_0^v x^{\alpha-1} P(x, t)\, dx$$

and

$$R(v, t) = \int_0^v x^\alpha \frac{\partial}{\partial x} P(x, t)\, dx$$

then

$$R(v, t) = v^\alpha P(v, t) - \alpha Q(v, t)$$

and

$$|R(v, t)| \leqslant A t^{\alpha-1} v^\alpha (1+vt)^{-\beta}.$$

3.

THEOREM 1.—If for $\alpha \geqslant 0$ $\psi_\alpha(+0)$ exists then

$$n\bar{c}_n \to \frac{2}{\pi}\, \psi_\alpha(+0)\ (C\beta) \text{ for } \beta > \alpha+1.$$

PROOF.—By lemmas 2 and 3 we have to show that

$$\lim_{\omega=\infty} \omega \bar{C}'_\beta(\omega) = \frac{2}{\pi}\, \psi_\alpha(+0).$$

Now

$$\bar{C}_\beta(\omega) = \frac{2\omega}{\pi} \int_0^\infty \bar{\gamma}_{1+\beta}(\omega t)\, \psi(t)\, dt$$

and by lemmas 2 and 4

$$-\frac{\pi}{2}\, \beta^{-1}\, \bar{C}'_\beta(\omega) = \int_0^\infty \gamma'_\beta(\omega t) \psi(t) dt$$

$$= \int_0^\eta + \int_\eta^\infty = I_1(\omega) + I_2(\omega) \text{ say.}$$

If $\rho = \min(\beta, 3)$

$$|I_2(\omega)| \leqslant A \left\{ \int_\eta^\pi + \sum_1^\infty \int_{(2s-1)\pi}^{(2s+1)\pi} \right\} (\omega t)^{-\rho}\, |\psi(t)|\, dt$$

$$\leqslant A\omega^{-\rho} \left\{ \eta^{-\rho} + \sum_1^\infty (\overline{2s-1}\pi)^{-\rho} \right\} \int_{-\pi}^\pi |\psi(t)|\, dt$$

$$\leqslant A\omega^{-\rho},$$

so that

$$\omega I_2(\omega) = O(\omega^{-\rho+1})$$
$$= o(1) \qquad \text{since } \rho > 1.$$

$$I_1(\omega) = \int_0^\eta \gamma_\beta'(\omega t)\,\psi(t)\,dt$$

$$= \left[\sum_{\nu=1}^{h+1} (-1)^{\nu-1}\omega^{\nu+1}\Psi_\nu(t)\,\gamma_\beta^{\nu}(\omega t)\right]_0^\eta$$

$$+(-1)^{h+1}\omega^{h+1}\int_0^\eta \gamma_\beta^{h+2}(\omega t)\,\Psi_{h+1}(t)\,dt$$

$$= I_{1,1}(\omega)+I_{1,2}(\omega) \qquad \text{say.}$$

Now by lemma 5,

$$I_{1,1}(\omega)=O(\omega^{h-\beta}) = o\left(\frac{1}{\omega}\right) \text{ since } \beta > h+1,$$

and

$$I_{1,2}(\omega) = \frac{(-1)^{h+1}\omega^{h+1}}{\Gamma(h+1-\alpha)}\int_0^\eta \gamma_\beta^{h+2}(\omega t)\,dt\int_0^t (t-u)^{h-\alpha}\Psi_\alpha(u)\,du$$

$$= \frac{(-1)^{h+1}\omega^{h+1}}{\Gamma(h+1-\alpha)}\int_0^\eta \Psi_\alpha(u)\,du\int_u^\eta (t-u)^{h-\alpha}\gamma_\beta^{h+2}(\omega t)\,dt$$

$$= \int_0^\eta \frac{\psi_\alpha(u)}{u}\,J(\omega, u)\,du.$$

We have only to evaluate

$$\lim_{\omega=\infty} \omega I_{1,2}(\omega) = \lim_{\omega=\infty} \omega\int_0^\eta \frac{\psi_\alpha(u)}{u}\,J(\omega, u)\,du.$$

This limit is equal to

$$\psi_\alpha(+0)\ .\ \lim_{\omega=\infty}\omega\int_0^\eta \frac{J(\omega, u)}{\omega}\,du$$

provided $\psi_\alpha(+0)$ exists and the kernel $\frac{\omega}{u}J(\omega, u)$ satisfies Toeplitz conditions. Now by lemma 6,

$$\left|\frac{\omega}{u}J(\omega, u)\right| \leqslant A\omega^{\alpha+1}u^\alpha(1+\omega u)^{-\beta}$$

$$= o(1) \text{ as } \omega\to\infty \text{ since } \beta > \alpha+1,$$

$$\int_0^\infty \left|\frac{\omega}{u}J(\omega, u)\right| du \leqslant A\int_0^\infty \frac{\omega^{\alpha+1}u^\alpha}{(1+\omega u)^\beta}\,du$$

$$= A\int_0^\infty \frac{x^\alpha dx}{(1+x)^\beta} < \infty$$

since $\alpha \geqslant 0$ and $\beta > \alpha+1$, and

$$\omega \int_0^\eta \frac{J(\omega, u)}{u}\, du = \frac{(-1)^{h+1}\omega^{h+2}}{\Gamma(\alpha+1)\Gamma(h+1-\alpha)} \int_0^\eta u^\alpha du \int_u^\eta (t-u)^{h-\alpha} \gamma_\beta^{h+2}(\omega t)\, dt$$

$$= \frac{(-1)^{h+1}\omega^{h+2}}{\Gamma(\alpha+1)\Gamma(h+1-\alpha)} \int_0^\eta \gamma_\beta^{h+2}(\omega t)\, dt \int_0^t u^\alpha (t-u)^{h-\alpha} du$$

$$= \frac{(-1)^{h+1}\omega^{h+2}}{\Gamma(h+2)} \int_0^\eta t^{h+1} \gamma_\beta^{h+2}(\omega t)\, dt$$

$$= \frac{(-1)^{h+1}}{\Gamma(h+2)} \int_0^{\omega\eta} x^{h+1} \gamma_\beta^{h+2}(x)\, dx$$

so that

$$\lim_{\omega=\infty} \omega \int_0^\eta \frac{J(\omega, u)}{u}\, du = \frac{(-1)^{h+1}}{\Gamma(h+2)} \int_0^\infty x^{h+1} \gamma_\beta^{h+2}(x)\, dx$$

$$= \frac{(-1)^h}{\Gamma(h+1)} \int_0^\infty x^h \gamma_\beta^{h+1}(x)\, dx$$

$$= \quad . \quad . \quad .$$

$$= \int_0^\infty \gamma_\beta'(x)\, dx$$

$$= \gamma_\beta(\infty) - \gamma_\beta(0) = -\frac{1}{\beta}.$$

This shows that

$$\lim_{\omega=\infty} -\frac{\pi}{2\beta}\, \omega \overline{C}_\beta'(\omega) = -\frac{1}{\beta}\, \psi_\alpha(+0)$$

or

$$\lim_{\omega=\infty} \omega \overline{C}_\beta'(\omega) = \frac{2}{\pi}\, \psi_\alpha(+0),$$

which proves the theorem.

COROLLARY 1.—If $\psi_\alpha(+0)$ exists for $\alpha \geqslant 0$ then for $\beta > \alpha$

$$\lim_{\omega=\infty} \frac{\overline{C}_\beta(\omega)}{\log \omega} = \frac{2}{\pi}\, \psi_\alpha(+0).$$

Proof.—If $\beta > \alpha$, then by theorem 1

$$\frac{1}{\log \omega} \cdot \overline{C}_{\beta+1}(\omega) \simeq \frac{1}{\log \omega} \int^\omega C_{\beta+1}'(t)\, dt \simeq \frac{2}{\pi}\, \psi_\alpha(+0) \cdot \frac{1}{\log \omega} \int^\omega \frac{dt}{t}$$

and by lemma 2

$$\frac{1}{\log \omega}\, \overline{C}_\beta(\omega) = \frac{1}{\log \omega}\, \overline{C}_{\beta+1}(\omega) + \frac{1}{\beta+1} \cdot \frac{\omega \overline{C}_\beta'(\omega)}{\log \omega} \simeq \frac{2}{\pi}\, \psi_\alpha(+0) + o(1).$$

THEOREM 2.—If, for $\alpha \geqslant 0$, $\psi_\alpha(+0)$ exists, then for $\beta > \alpha$

$$\lim_{\omega=\infty} \left[\bar{C}_\beta(\omega') - \bar{C}_\beta(\omega)\right] = \frac{2}{\pi} \log \lambda \, . \, \psi_\alpha(+0)$$

where

$$\lim_{\omega=\infty} \frac{\omega'}{\omega} = \lambda > 1.$$

Proof.—If $\beta > \alpha$ and $\omega' > \omega$

$$\bar{C}_{\beta+1}(\omega') - \bar{C}_{\beta+1}(\omega) = \int_\omega^{\omega'} \bar{C}'_{\beta+1}(x)dx$$

$$\sim \frac{2}{\pi} \psi_\alpha(+0) \int_\omega^{\omega'} \frac{dx}{x} \text{ by theorem 1,}$$

$$\sim \frac{2}{\pi} \psi_\alpha(+0) \, . \log \frac{\omega'}{\omega}$$

$$= \frac{2}{\pi} \psi_\alpha(+0) \log \lambda.$$

But, since by lemma 2

$$\bar{C}_\beta(\omega) = \bar{C}_{\beta+1}(\omega) + \frac{\omega}{\beta+1} \bar{C}'_\beta(\omega)$$

we have

$$\bar{C}_\beta(\omega') - \bar{C}_\beta(\omega) = \bar{C}_{\beta+1}(\omega') - \bar{C}_{\beta+1}(\omega) + \left[\omega'\bar{C}'_\beta(\omega') - \omega\bar{C}'_\beta(\omega)\right] \frac{1}{\beta+1}$$

$$\to \frac{2}{\pi} \log \lambda \, . \, \psi_\alpha(+0)$$

which proves the theorem.

THEOREM 3.—If, for $\alpha \geqslant 0$,

$$\frac{1}{t} \int_0^t |\, \psi_\alpha(u) \,|\, du = O(1)$$

as $t \to 0$ and if $\psi_{\alpha+1}(+0)$ exists then

$$\lim_{\omega=\infty} \omega \bar{C}'_\beta(\omega) = \frac{2}{\pi} \psi_{\alpha+1}(+0)$$

where $\beta > \alpha + 1$.

Proof.—We first note the following

LEMMA 8.—If

$$\frac{1}{t} \int_0^t |\, \psi_\alpha(u) \,|\, du = O(1) \text{ then } \omega \bar{C}'_\beta(\omega) = O(1),$$

for $\beta > \alpha + 1$.

From the argument in the proof of theorem 1, it is sufficient for us to consider the integral

$$\omega \int_0^\eta \psi_\alpha(u) \frac{J(\omega, u)}{u} du = \omega \int_0^{\frac{1}{\omega}} + \omega \int_{\frac{1}{\omega}}^\eta$$

$$= J_1 + J_2, \text{ say.}$$

$$| J_1 | \leqslant A\omega \int_0^{\frac{1}{\omega}} | \psi_\alpha(u) | \omega^\alpha u^\alpha (1+\omega u)^{-\beta} du$$

$$\leqslant A\omega \int_0^{\frac{1}{\omega}} | \psi_\alpha(u) | du = O(1) \qquad \text{as } \omega \to \infty .$$

$$| J_2 | \leqslant A\omega^{\alpha+1} \int_{\frac{1}{\omega}}^\eta | \psi_\alpha(u) | \frac{u^\alpha}{(1+\omega u)^\beta} du$$

$$\leqslant A\omega^{\alpha+1-\beta} \int_{\frac{1}{\omega}}^\eta | \psi_\alpha(u) | u^{\alpha-\beta} du$$

$$= A\omega^{\alpha+1-\beta} \Big[u^{\alpha-\beta} \Psi^*_\alpha(u) \Big]_{\frac{1}{\omega}}^\eta - A\omega^{\alpha+1-\beta}(\alpha-\beta) \int_{\frac{1}{\omega}}^\eta \Psi^*_\alpha(u) u^{\alpha-\beta-1} du$$

$$= \frac{A\eta^{\alpha-\beta}\Psi^*_\alpha(\eta)}{\omega^{\beta-\alpha-1}} - \omega A \psi^*_\alpha\left(\frac{1}{\omega}\right) + A\omega^{\alpha+1-\beta} \int_{\frac{1}{\omega}}^\eta 0(u^{\alpha-\beta}) du$$

$$= O\left(\frac{1}{\omega^{\beta-\alpha-1}}\right) + O(1)$$

$$= O(1) \text{ if } \beta > \alpha + 1.$$

Hence

$$\omega \int_0^\eta \psi_\alpha(u) \frac{J(\omega, u)}{u} du = O(1)$$

which proves the theorem.

We now proceed to the proof of theorem 3. By the first hypothesis and lemma 8,

$$n\bar{c}_n = O(1) \qquad (c, \alpha+1+\delta), \qquad \delta > 0.$$

By the second hypothesis and theorem 1

$$n\bar{c}_n \to \frac{2}{\pi} \psi_{\alpha+1}(+0) \qquad (c, \alpha+2+\delta), \qquad \delta > 0$$

Hence by a classical Tauberian theorem

$$n\bar{c}_n \to \frac{2}{\pi}\,\psi_{\alpha+1}(+0) \qquad (c\,\beta)$$

for every $\beta > \alpha+1$.

THEOREM 4.—If $\Psi^*_\alpha(t) = O(t)$ and $\psi^*_\alpha(+0)$ exists then

$$n\bar{c}_n \to \frac{2}{\pi}\,\psi^*_\alpha(+0) \qquad (c\,\beta) \text{ for every } \beta > \alpha+1.$$

This theorem follows from theorem 3 and the following

LEMMA 9.—If $\psi_{\alpha+1}(+0)$ exists then $\psi^*_\alpha(+0)$ also exists and conversely and the two limits are equal.

This lemma follows from the following two identities which are easily verified

$$\frac{1}{t}\int_0^t \psi_\alpha(u)\,du = \frac{1}{\alpha+1}\,\psi_{\alpha+1}(t) + \frac{\alpha}{\alpha+1}\,\frac{1}{t}\int_0^t \psi_{\alpha+1}(u)\,du$$

$$\psi_{\alpha+1}(t) = (\alpha+1)\,\psi^*_\alpha(t) - \frac{\alpha(\alpha+1)}{t^{\alpha+1}}\int_0^t u^\alpha \psi^*_\alpha(u)\,du.$$

THEOREM 5.—If $\Psi^*_\alpha(t) = O(t)$ and $\psi^*_\alpha(+0)$ exists then

$$\lim_{\omega=\infty}\left[\bar{C}_\beta(\omega') - \bar{C}_\beta(\omega)\right] = \frac{2}{\pi}\log\lambda\,.\,\psi^*_\lambda(+0)$$

where

$$1 < \lambda = \lim_{\omega=\infty}\frac{\omega'}{\omega}, \text{ and } \beta > \alpha.$$

This theorem follows from theorem 4 even as theorem 2 follows from theorem 1. It is clear that an analogue of cor. 1 holds good here also.

4.

We shall now prove a theorem which is a converse of theorem 1.

THEOREM 6.—If $n\bar{c}_n \to \delta(x)$ $(c,\ \alpha)$ $\alpha \geqslant 0$ then $\psi_\beta(+0)$ exists for every $\beta > \alpha$ and $\psi_\beta(+0) = \frac{\pi}{2}\,\delta(x)$.

Proof.—That, if $\psi_\beta(+0)$ exists, $\psi_\beta(+0) = \frac{\pi}{2}\,\delta(x)$ will follow from theorem 1 and consistency theorem for cesaro summability of series. Therefore it is enough if we prove the existence of $\psi_\beta(+0)$ for $\beta > \alpha$.

Now following the argument in the proof of theorem 3 of Bosanquet and Hyslop (1937) we observe that for $\beta > \alpha > 0$.

$$\psi_\beta(t) = t\int_0^\infty R(v,t)\,\bar{C}'_\alpha(v)\,dv$$

$$= t\int_0^\infty \frac{R(v,t)}{v}\,.\,v\bar{C}'_\alpha(v)\,dv.$$

Since $\lim\limits_{v=0} v\,\overline{C}'_{\alpha}(v) = \delta(x)$, the theorem is proved, if we show that the kernel $\frac{t}{v} R(v, t)$ fulfills Toeplitz conditions. By lemma 7

$$\left| \frac{t}{v} R(v, t) \right| \leqslant A \frac{t^{\alpha} v^{\alpha-1}}{(1+vt)^{\beta}}$$
$$= o(1)$$

either as $t \to 0$ or as $v \to \infty$ if $\alpha > 0$ and $\beta > \alpha$ and

$$\int_0^{\infty} \left| \frac{t}{v} R(vt) \right| dv \leqslant A \int_0^{\infty} \frac{t^{\alpha} v^{\alpha-1}}{(1+vt)^{\beta}} dv$$
$$= A \int_0^{\infty} \frac{x^{\alpha-1}}{(1+x)^{\beta}} dx$$
$$< \infty$$

if $\alpha > 0$ and $\beta > \alpha$. Again we shall show that

$$t \int_0^{\infty} \frac{R(v, t)}{v} dv$$

is independent of t, so that the limit of this function as $t \to 0$ will exist. In fact by lemma 7

$$P(v, t) = \frac{(-1)^{h+1} \beta t^h}{\Gamma(\alpha+1)\Gamma(h+1-\alpha)} \int_v^{\infty} (u-v)^{h-\alpha} \bar{\gamma}_{\beta}^{h+1}(ut) du$$
$$= (-1)^{h+1} \beta t^h v^{h+1-\alpha} \int_1^{\infty} (x-1)^{h-1} \bar{\gamma}_{\beta}^{h+1}(vtx)\, dx$$

so that

$$v^{\alpha-1} P(v, t) = \frac{(-1)^{h+1} \beta (vt)^h}{\Gamma(\alpha+1)\Gamma(h+1-\alpha)} \int_1^{\infty} (x-1)^{h-\alpha} \bar{\gamma}_{\beta}^{h+1}(vt\, x)\, dx.$$
$$= J(vt)$$

where

$$J(z) = \frac{(-1)^{h+1} \beta z^h}{\Gamma(\alpha+1)\Gamma(h+1-\alpha)} \int_1^{\infty} (x-1)^{h-\alpha} \bar{\gamma}_{\beta}^{h+1}(zx)\, dx.$$

$$Q(vt) = \alpha \int_0^{v} x^{\alpha-1} P(x, t)\, dx.$$
$$= \alpha \int_0^{v} J(xt)\, dx.$$
$$= \frac{\alpha}{t} \int_0^{vt} J(y)\, dy.$$

$$\frac{R(v, t)}{v} = v^{\alpha-1}P(v, t) - \frac{\alpha}{v}\int_0^v x^{\alpha-1}P(x, t)\,dx$$

$$= J(vt) - \frac{\alpha}{vt}\int_0^{vt} J(y)\,dy.$$

Hence

$$t\int_0^\infty \frac{R(v, t)}{v}\,dv = \int_0^\infty \left\{ J(z) - \frac{\alpha}{z}\int_0^z J(y)\,dy \right\} dz$$

on setting $z = vt$. The expression on the right is independent of t, and hence the theorem follows.

5.

The analogues of theorems 1 and 5 for absolute summability can be stated: viz.

(i) If $\psi_\alpha(t)$ is of bounded variation in the neighbourhood of the origin for $\alpha \geqslant 0$, then

$$n\bar{c}_n \to \frac{2}{\pi}\psi_\alpha(+0) \qquad | c, \beta | \text{ for } \beta > \alpha + 1.$$

(ii) If $n\bar{c}_n \to \delta(x) \mid c\alpha \mid \alpha \geqslant 0$ then $\psi_\beta(t)$ is of bounded variation in the neighbourhood of the origin for $\beta > \alpha$.

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