A THEOREM IN ANALYTIC NUMBER THEORY.

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§ 1. Let p denote a prime such that p^t is a factor of n, while p^{t+1} is not; we then say that 'n contains the prime p to the power t'. Using a recent theorem of Selberg (*Skr. Norske Vid. Akad.*, Oslo, I, No. 5, 49 pp., 1942) I have proved the

Theorem 1. Let k and r be given positive integers. Then 'almost all' positive integers contain at least r different primes $\equiv -1 \pmod k$, each to an odd power.

From this and well-known congruence properties of Ramanujan's function $\tau(n)$ we derive, without difficulty,

Theorem 2. Let $\theta_1, \theta_2, \ldots, \theta_6$ be arbitrary positive integers; then the congruence

$$\tau(n) \equiv O \pmod{2^{\theta_1}\, 3^{\theta_2}\, 5^{\theta_3}\, 7^{\theta_4}\, 23^{\theta_5}\, 691^{\theta_6}}$$

is true for almost all n.

In Theorems 1 and 2, the words 'almost all' carry the usual sense in the analytic theory of numbers. The proof of Theorem 1 is based on the case $r = 1$, which is well-known (see Hardy's *Ramanujan*, Cambridge, 1940, p. 168). In the general case I have not been able to accomplish the proof without using the rather difficult results of Selberg.

This note contains the proof for the case $r = 2$. Let $\sigma(x)$ denote the number of positive integers n not exceeding x, such that every prime factor of n which is $\equiv -1 \pmod k$ is contained in n to an even power. Then we have (Hardy's *Ramanujan*, p. 168)

$$\sigma(x) = O\left(\frac{x}{\log^c x}\right) \tag{1}$$

where $0 < c < 1$ and $c = c(k)$.

It is easy to see that the sum

$$S(x) = \sum \sigma\left(\frac{x}{p}\right) \tag{2}$$

where p is subject to (3) and (4) below

$$p \text{ prime}, \; p \leq x \tag{3}$$

$$p \equiv -1 \pmod k \tag{4}$$

represents the number of numbers n not exceeding x and such that n contains exactly one prime factor $\equiv -1 \pmod k$ to an odd power [such n, may, naturally, contain any number of primes $\equiv -1 \pmod k$ to an even power].

We split $S(x)$ into 2 parts, thus:

$$S(x) = S_1(x) + S_2(x) \tag{5}$$

Here

(6) $$S_1(x) = \sum \sigma\left(\frac{x}{p}\right)$$

where p is subject to

α $$p \text{ prime, } p \equiv -1 \pmod{k}$$

β $$p \leqq \frac{x}{e^{\sqrt{\log x}}};$$

and

(7) $$S_2(x) = \sum \sigma\left(\frac{x}{p}\right)$$

where p is subject to

γ $$p \text{ prime, } p \equiv -1 \pmod{k}$$

δ $$\frac{x}{e^{\sqrt{\log x}}} < p \leqq x.$$

We now estimate $S_1(x)$. From (1) we have

$$S_1(x) = O\left(\frac{x}{\log^c\left(e^{\sqrt{\log x}}\right)}\right) \sum_{p \leqq x} \frac{1}{p}$$

where p runs through primes. Hence

(8) $$S_1(x) = O\left(\frac{x \log \log x}{\log^{\frac{c}{2}}(x)}\right) = O(x).$$

Again using the crude inequality $\sigma(x) \leqq x$ to estimate $S_2(x)$ we get :

(9) $$S_2(x) = O\Big(\sum_{y \leqq p \leqq x} \frac{x}{p}\Big)$$

where

(10) $$y = \frac{x}{e^{\sqrt{\log x}}}$$

and p runs through primes in (9).

To estimate (9) we use the classical result (in Prime Number Theory)

(11) $$\sum_{p \leqq x} \frac{1}{p} = \log \log x + B + O\left(\frac{1}{\log x}\right)$$

where p runs through primes. From (10) and (11) we get

$$\sum_{y < p \leqq x} \frac{1}{p} = -\log\left(\frac{\log y}{\log x}\right) + O\left(\frac{1}{\log y}\right)$$

$$= -\log\left(1 - \frac{1}{\sqrt{\log x}}\right) + O\left(\frac{1}{\log x}\right)$$

(12) $$= O\left(\frac{1}{\sqrt{\log x}}\right)$$

From (9) and (12) we get

$$S_2(x) = O\left(\frac{x}{\sqrt{\log x}}\right). \tag{13}$$

From (5), (8), (13) we have finally

$$S(x) = O(x). \tag{14}$$

From the definition of $S(x)$ it now follows that almost all positive integers $n \leq x$ have the property that n contains *at least two* prime factors $\equiv -1 \pmod{k}$ to an odd power.

Reference.

Walfisz, A. (1938). 'Zur additiven Zahlentheorie.' *Travaux de L'Institut Mathematique de Tbilissi*, **5**, 145–152.