ON NUMBERS WHICH CAN BE EXPRESSED AS A SUM OF TWO SQUARES.

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§1. Denote by $b_1\ (=1)$, b_2, b_3, b_4, . . . the numbers, arranged in ascending order of magnitude, which can be expressed as a sum of two squares (of integers). It is known that (Landau 1)

$$\sum_{b_n \leqq x} 1 \sim \frac{Cx}{\sqrt{\log x}}$$

where C is a positive constant. In connection with the theory of lattice points in a circle it is known that

$$\sum_{\substack{u,\, v \\ u^2+v^2 \leqq x \\ u,\, v \gtreqless 0}} 1 = \pi x + P(x)$$

where

(1) $$P(x) = O(x^{\frac{27}{82}});$$

and it has been conjectured that, for every positive ϵ,

(2) $$P(x) = O(x^{\frac{1}{4}+\epsilon}).$$

We are concerned in this paper with the problem of the magnitude of the difference $b_{n+1} - b_n$. More precisely we wish to seek a function $f(x)$ such that between

$$x \text{ and } x + f(x)$$

there is at least one number expressible as a sum of 2 squares for all large x.

From (1) it follows at once that

$$f(x) = O(x^{\frac{27}{82}});$$

if (2) is true we would get

$$f(x) = O(x^{\frac{1}{4}+\epsilon})$$

for every positive ϵ.

We prove in this paper by a simple argument that (see the more precise result at the end of §2)

(3) $$f(x) = O(x^{\frac{1}{4}}).$$

It has been conjectured that if $(a, b) = 1$ there is (see Chowla 2) at least one prime $\equiv a \pmod{b}$ between x and $x+x^{\epsilon}$ when $x>x_0 = x_0\ (\epsilon, a, b)$. If this is true then taking $a = 1, b = 4$, it would follow that

$$f(x) = x^{\epsilon}$$

for any fixed positive ϵ. This shows that (3) is still very far from the probable truth. It would also be of interest to know whether (3) can be improved by elementary arguments.

We have to acknowledge here that the results (3) was found some years ago* by Dr. T. Vijayaraghavan by an argument not quite as simple as the one we give. This paper has its roots in this letter of Dr. Vijayaraghavan.

§2. In this section all letters denote positive real numbers.

Let ϵ be an arbitrary positive number and $x > x_0\ (\epsilon)$. Let $[g]$ denote the greatest integer contained in g. Write

$$t = [\sqrt{x}] = \sqrt{x}-\theta \tag{4}$$

where

$$0 \leqq \theta < 1. \tag{5}$$

Let (here x_1, x_2 are not necessarily integers)

$$x_1{}^2+t^2 = x$$

$$x_2{}^2+t^2 = x+2\sqrt{2+\epsilon}\ x^{\frac{1}{2}}$$

Then

$$x_2{}^2-x_1{}^2 = 2\sqrt{2+\epsilon}\ x^{\frac{1}{2}}$$

$$x_2-x_1 = \frac{2\sqrt{2+\epsilon}\ x^{\frac{1}{2}}}{x_1+x_2} \tag{6}$$

now

$$x_1 = \sqrt{x-t^2} = \sqrt{x-(\sqrt{x}-\theta)^2}$$

$$= \sqrt{2\,\theta\sqrt{x}-\theta^2} < \sqrt{2}\sqrt[4]{x}$$

similarly for $x>x_0(\epsilon)$,

$$x_2 = \sqrt{x+2\,(\sqrt{2+\epsilon})\ x_v{}^{\frac{1}{2}}-(\sqrt{x-\theta})^2}$$

$$\leqq \sqrt{2+\frac{\epsilon}{10}}\ x^{\frac{1}{4}}.$$

Hence, for $x>x_0(\epsilon)$,

$$x_1+x_2 < 2\sqrt{2+\frac{\epsilon}{10}}\ x^{\frac{1}{4}}. \tag{7}$$

From (6) and (7),

$$x_2-x_1 > \sqrt{\frac{2+\epsilon}{2+\frac{\epsilon}{10}}} > 1$$

Hence there exists an *integer* x_3 between x_1 and x_2.

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* In a letter addressed to one of us (S. C.), but unfortunately mislaid.

Hence

$$t^2+x_1^2 < t^2+x_3^2 < t^2+x_2^2$$

so that

$$x < t^2+x_3^2 < x+2\sqrt{2+\epsilon}\, x^{\frac{1}{4}}$$

(where t and x_3 are integers). Thus we have the

Theorem. Let ϵ denote an arbitrary positive number. Then there exists between x and $x+2\sqrt{2+\epsilon}\, x^{\frac{1}{4}}$ an integer which can be expressed as a sum of two squares (of integers) for all $x > x_0(\epsilon)$.

§3. References.

Chowla, S. (1935). 'On the representation of a large number as a sum of light almost equal cubes.' *Quart. Journ. of Maths.* (Oxford), **6**, 146–148.

Landau (Handbuch), Band 2.