ON THE SUMMABILITY OF THE CONJUGATE SERIES OF A FOURIER SERIES BY LOGARITHMIC MEANS.

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(Communicated by Dr. B. N. Prasad.)

(*Received October 10; read October 18, 1946.*)

1. Let

$$\sum_{n=1}^{\infty}(b_n \cos n\theta - a_n \sin n\theta) \tag{1.1}$$

be the conjugate series of the Fourier series associated with a function $f(\theta)$ which is integrable (L) over the interval $(-\pi, \pi)$ and defined outside by periodicity. The 'conjugate' function associated with the series (1.1) is

$$\bar{f}(\theta) = \frac{1}{2\pi}\int_0^{\pi}\{f(\theta+t)-f(\theta-t)\}\cot\frac{1}{2}t\,dt, \tag{1.2}$$

the integral being a Cauchy integral at the origin.

Prasad* proved that if the function $f(\theta)$ is bounded, the conjugate series (1.1) is summable (C, δ) for every positive δ to $\bar{f}(\theta)$ wherever the integral (1.2) exists. This theorem of Prasad was in a certain sense extended by Hardy and Littlewood † in the form of a simple necessary and sufficient condition for the Cesaro summability of the conjugate series corresponding to a bounded function as follows :—

Suppose that $f(t)$ is bounded in the neighbourhood of $t = 0$. Then the conjugate series of $f(t)$, for $t = 0$, is either summable by Cesaro means of every positive order or summable by no Cesaro mean. A necessary and sufficient condition for the summability is the convergence of the integral (1.2).

Now there are simple bounded functions for which the integral (1.2) is not convergent. For example, let $\theta = 0$ and let an odd function $f(t)$ be defined in $(0, \pi)$ by

$$\begin{cases} f(0) = 0 \\ f(t) = \dfrac{1}{t}\tan\dfrac{1}{2}t\sin\left(\log\dfrac{1}{t}\right), & \text{for } 0 < t \leq 1, \\ \quad\;\; = 0, & \text{for } 1 < t \leq \pi, \end{cases} \tag{1.3}$$

and by periodicity elsewhere. Then the integral for $\bar{f}(\theta)$ oscillates between 0 and $\frac{2}{\pi}$ in the neighbourhood of $t = 0$ and the conjugate series for this function $f(t)$ is not summable (C). This series is, however, summable by Riesz's logarithmic means of order unity as we shall see later.

Definition.—A series $\Sigma\, c_n$ is said to be summable by Riesz's logarithmic means of order $k > 0$, or summable (R, k), to the sum s, provided that

$$R_k(w) \equiv \frac{1}{(\log w)^k}\sum_{n<w}\left(\log\frac{w}{n}\right)^k c_n \tag{1.4}$$

tends to a limit s as $w \to \infty$.

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* Prasad, 4. † Hardy and Littlewood, 3.

The problem that naturally suggests itself is to find some general theorem, concerning Rieszian summability of the conjugate series of a bounded function, which may be of the same type as the theorem of Hardy and Littlewood given above. Theorem A, which we prove below, covers a class of functions much wider than that of bounded functions and is quite of the type desired. In what follows we use logarithmic integral means which are more general than fractional integral means of a function.

Let us write

$$\psi(t) = \frac{1}{2}\{f(\theta+t)-f(\theta-t)\}, \; g(t) = \frac{1}{\pi}\int_t^\pi \psi(t)\cot\frac{1}{2}t\,dt - s\,;$$

$$g_\alpha(t) = \frac{1}{\Gamma(\alpha)}\int_t^\pi \left(\log\frac{u}{t}\right)^{\alpha-1}\frac{g(u)}{u}\,du, \; \alpha>0,$$

$$g_0(t) = g(t).$$

It is known * that

$$g_{\alpha+\beta}(t) = \frac{1}{\Gamma(\beta)}\int_t^\pi \left(\log\frac{u}{t}\right)^{\beta-1} g_\alpha(u)\frac{du}{u}, \; \beta>0.$$

Accordingly we have

$$g_{\alpha+1}(t) = \int_t^\pi g_\alpha(u)\frac{du}{u},$$

and we define

$$g_\alpha(t) = -t\frac{d}{dt}g_{\alpha+1}(t), \text{ for } -1\leq\alpha<0,$$

so that

$$g_{-1}(t) = \frac{1}{\pi}t\cot\frac{1}{2}t\,\psi(t).$$

Theorem A. *Let*

$$\int_0^t |\, g_{\alpha-1}(t)\,|\, dt = O\left\{t\left(\log\frac{1}{t}\right)^{\alpha+1}\right\}, \; \alpha\geqslant 0, \tag{1.5}$$

and

$$g_\alpha(t) = o\left\{\left(\log\frac{1}{t}\right)^{\alpha+1}\right\}, \tag{1.6}$$

as $t\to 0$. *Then a necessary and sufficient condition that the conjugate series* (1.2) *be summable* $(R, \alpha+1)$ *for* $t=\theta$ *to the sum* s *is that*

$$g_{\alpha+1}(t) = o\left\{\left(\log\frac{1}{t}\right)^{\alpha+1}\right\}, \tag{1.7}$$

as $t\to 0$.

An analogous theorem for Fourier series was given by Wang † for integral α. He has also shown ‡ that if the condition (1.7) is satisfied, the conjugate series is

* Wang, 7. † Wang, 6. ‡ Wang, 8.

summable $(R, \alpha+2)$ and conversely if the conjugate series is summable $(R, \alpha+1)$ to the sum s, then

$$g_{\alpha+2}(t) = o\left\{ \left(\log\frac{1}{t}\right)^{\alpha+2} \right\},$$

as $t \to 0$.

The case $\alpha = 0$ of Theorem A gives a simple, elegant result, namely

Theorem B. *Let*

$$\int_0^t |\psi(u)|\, du = O\left(t \log\frac{1}{t}\right), \tag{1.8}$$

and

$$g(t) = o\left(\log\frac{1}{t}\right), \tag{1.9}$$

as $t \to 0$. *Then a necessary and sufficient condition that the conjugate series* (1.2) *be summable* $(R, 1)$ *for* $t = \theta$ *to the sum* s *is that*

$$\int_t^\pi \frac{g(u)}{u} du = o\left(\log\frac{1}{t}\right), \tag{1.10}$$

as $t \to 0$.

The condition (1.8) is satisfied wherever $\psi(t) = O\left(\log\frac{1}{t}\right)$ and in particular when $f(t)$ is bounded near $t = 0$, and the condition (1.9) holds in particular when $g(t)$ is bounded.

A result for Fourier series analogous to the Theorem B was given by Hardy and generalised by Takahasi and by Bosanquet and Offord.* Theorem B can also be deduced from a theorem of Bosanquet and Offord.

In § 3 we find a necessary and sufficient condition for the summability $(R, \alpha+1)$ of the conjugate series. In § 4 we prove Theorem A. We show in § 5 that the conjugate series for the function $f(t)$ defined by (1.3) is summable $(R, 1)$ to the sum $\frac{1}{\pi}$ although, as previously remarked, it is not summable (C). In § 6, we construct an example to show that the holding of only one of the conditions (1.8) and (1.9), namely

$$g(t) = o\left(\log\frac{1}{t}\right) \neq o(1)$$

is not sufficient to ensure the summability $(R, 1)$ of the conjugate series.

I am much indebted to Dr. B. N. Prasad for his kind interest and advice in the preparation of this paper.

2. We shall make use of functions $S_k(t)$ defined by

$$S_k(t) = \int_0^1 \left(\log\frac{1}{u}\right)^k \sin tu\, du = \frac{1}{t}\int_0^t \left(\log\frac{t}{u}\right)^k \sin u\, du,\ k > -1.$$

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* Hardy, 2; Takahasi, 5; Bosanquet and Offord, 1.

It is known that *

$$\frac{d}{dt}\left\{tS_k(t)\right\} = kS_{k-1}(t),\ k>0, \tag{2.1}$$

$$S_{r+s+1}(t) = \frac{\Gamma(r+s+2)}{\Gamma(r+1)\Gamma(s+1)}\int_0^1 S_s(ut)\left(\log\frac{1}{u}\right)^r du,\ r>-1,\ s>-1. \tag{2.2}$$

We shall also require the following lemma :—

Lemma. *For* $\alpha>0$,

$$\int_0^\pi g(t)S_\alpha(\omega t)\,dt = \Gamma(\alpha+1)\int_0^\pi g_\alpha(t)S_0(\omega t)\,dt$$

$$= \frac{\Gamma(\alpha+1)}{\omega}\int_0^\pi g_\alpha(t)\frac{1-\cos\omega t}{t}\,dt.$$

This has also been shown by Wang. †

3. We may make the usual simplifications, supposing that $f(t)$ is odd and that $\theta=0$, $s=0$ so that

$$\psi(t)=f(t) \text{ and } g(t)=\frac{1}{\pi}\int_t^\pi f(t)\cot\frac{1}{2}t\,dt.$$

We have

$$s_0(\omega)=\sum_{n<\omega} b_n = \frac{1}{\pi}\int_0^\pi f(t)\cot\frac{1}{2}t(1-\cos\omega t)\,dt+o(1)$$

$$=\left[-(1-\cos\omega t)g(t)\right]_0^\pi+\omega\int_0^\pi g(t)\sin\omega t\,dt+o(1)$$

$$=\omega\int_0^\pi g(t)\sin\omega t\,dt+o(1),$$

as $\omega\to\infty$. And for $k>0$,

$$s_k(\omega)=\sum_{n<\omega}\left(\log\frac{\omega}{n}\right)^k b_n = k\int_1^\omega\left(\log\frac{\omega}{n}\right)^{k-1} s_0(x)\frac{dx}{x}$$

$$=k\int_1^\omega\left(\log\frac{\omega}{x}\right)^{k-1}dx\int_0^\pi g(t)\sin xt\,dt+o\{(\log\omega)^k\}$$

$$=k\int_0^\pi g(t)\,dt\int_1^\omega\left(\log\frac{\omega}{x}\right)^{k-1}\sin xt\,dx+o\{(\log\omega)^k\}$$

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* Wang, 7.

† Wang, 8.

$$= k\int_0^\pi g(t)\,dt\int_0^\omega \left(\log\frac{\omega}{x}\right)^{k-1}\sin xt\,dx - k\int_0^\pi g(t)\,dt\int_0^1 \left(\log\frac{\omega}{x}\right)^{k-1}\sin xt\,dx$$
$$+o\{(\log\omega)^k\}$$

$$= k\omega\int_0^\pi g(t)S_{k-1}(\omega t)\,dt + O\left\{(\log\omega)^{k-1}\int_0^\pi |g(t)|\,dt\right\} + o\{(\log\omega)^k\}$$

$$= k\omega\int_0^\pi g(t)S_{k-1}(\omega t)\,dt + o\{(\log\omega)^k\},$$

as $\omega\to\infty$, $g(t)$ being integrable (L) in $(0,\pi)$. Hence

$$R_k(\omega) = \frac{1}{(\log\omega)^k}s_k(\omega) = \frac{k\omega}{(\log\omega)^k}\int_0^\pi g(t)S_{k-1}(\omega t)\,dt + o(1).$$

Putting $k=\alpha+1$, we have by the lemma

$$R_{\alpha+1}(\omega) = \frac{(\log\omega)^{\alpha+1}}{(\alpha+1)\omega}\int_0^\pi g(t)S_\alpha(\omega t)\,dt + o(1)$$

$$= \frac{\Gamma(\alpha+2)}{(\log\omega)^{\alpha+1}}\int_0^\pi g_\alpha(t)\frac{1-\cos\omega t}{t}\,dt + o(1).$$

Thus the necessary and sufficient condition that the conjugate series be summable $(R,\alpha+1)$ *at* $t=\theta$ *to the sum* s *is that*

$$\int_0^\pi g_\alpha(t)\frac{1-\cos\omega t}{t}\,dt = o\{(\log\omega)^{\alpha+1}\}, \tag{3.1}$$

as $\omega\to\infty$.

4. *Proof of Theorem A.*

(i) To prove that the condition is sufficient, we have

$$\int_0^\pi g_\alpha(t)\frac{1-\cos\omega t}{t}\,dt = \int_0^{\lambda/\omega} g_\alpha(t)\frac{1-\cos\omega t}{t}\,dt + \int_{\lambda/\omega}^\pi \frac{g_\alpha(t)}{t}\,dt - \int_{\lambda/\omega}^\pi \frac{g_\alpha(t)}{t}\cos\omega t\,dt$$
$$= I+J-K, \text{ say,} \tag{4.1}$$

where λ is large but fixed. Now by (1.6)

$$I = O\left(\omega\int_0^{\lambda/\omega}|g_\alpha(t)|\,dt\right) = o\left\{\omega\cdot\frac{\lambda}{\omega}\left(\log\frac{\omega}{\lambda}\right)^{\alpha+1}\right\} = o\{(\log\omega)^{\alpha+1}\}, \tag{4.2}$$

as $\omega\to\infty$. By (1.7)

$$J = o\left\{\left(\log\frac{\omega}{\lambda}\right)^{\alpha+1}\right\} = o\{(\log\omega)^{\alpha+1}\}. \tag{4.3}$$

Also $g_\alpha(t)$ being an integral for $\lambda/\omega \le t \le \pi$, we have

$$K = \int_{\lambda/\omega}^{\pi} g_\alpha(t)\,\frac{\cos \omega t}{t}\,dt = \frac{1}{\omega}\left[g_\alpha(t)\,\frac{\sin \omega t}{t}\right]_{\lambda/\omega}^{\pi} - \frac{1}{\omega}\int_{\lambda/\omega}^{\pi} \sin \omega t\left[\frac{d}{dt}\,\frac{g_\alpha(t)}{t}\right]dt$$

$$= -\frac{1}{\lambda}\sin \lambda g_\alpha(\lambda/\omega) + \frac{1}{\omega}\int_{\lambda/\omega}^{\pi} g_\alpha(t)\,\frac{\sin \omega t}{t^2}\,dt + \frac{1}{\omega}\int_{\lambda/\omega}^{\pi} g_{\alpha-1}(t)\,\frac{\sin \omega t}{t^2}\,dt$$

$$= o\{(\log \omega)^{\alpha+1}\} + K_1 + K_2. \tag{4.4}$$

And by (1.6)

$$K_1 = \frac{1}{\omega}\int_{\lambda/\omega}^{\pi} o\left(\log \frac{1}{t}\right)^{\alpha+1}\frac{dt}{t^2} = \frac{1}{\omega}\left[o\left\{\frac{1}{t}\left(\log \frac{1}{t}\right)^{\alpha+1}\right\}\right]_{\lambda/\omega}^{\pi}$$

$$= o\{(\log \omega)^{\alpha+1}\}\,. \tag{4.5}$$

Also by (1.5),

$$|K_2| \le \frac{1}{\omega}\int_{\lambda/\omega}^{\pi} |g_{\alpha-1}(t)|\,\frac{dt}{t^2}$$

$$= \frac{1}{\omega}\left[\frac{1}{t^2}\int_0^t |g_{\alpha-1}(t)|\,dt\right]_{\lambda/\omega}^{\pi} + \frac{2}{\omega}\int_{\lambda/\omega}^{\pi}\left\{\int_0^t |g_{\alpha-1}(t)|\,dt\right\}\frac{dt}{t^3}$$

$$= O\left(\frac{1}{\omega}\right) + \frac{1}{\omega}\cdot\frac{\omega}{\lambda}\,O\left\{\left(\log \frac{\omega}{\lambda}\right)^{\alpha+1}\right\} + \frac{2}{\omega}\int_{\lambda/\omega}^{\pi} O\left\{\left(\log \frac{1}{t}\right)^{\alpha+1}\right\}\frac{dt}{t^2}$$

$$= o(1) + \frac{1}{\lambda}\,O\left\{\left(\log \frac{\omega}{\lambda}\right)^{\alpha+1}\right\} + \frac{1}{\omega}\,O\left\{\frac{\omega}{\lambda}\left(\log \frac{\omega}{\lambda}\right)^{\alpha+1}\right\}$$

$$= o(1) + \frac{1}{\lambda}\,O\left\{(\log \omega)^{\alpha+1}\right\}, \tag{4.6}$$

as $\omega \to \infty$.

Hence combining the results from (4.1) to (4.6), we have

$$\int_0^{\pi} g_\alpha(t)\,\frac{1-\cos \omega t}{t}\,dt = o\{(\log \omega)^{\alpha+1}\} + \frac{1}{\lambda}\,O\{\log \omega)^{\alpha+1}\}$$

$$= o\{(\log \omega)^{\alpha+1}\}\,,$$

if $\lambda \to \infty$ after $\omega \to \infty$. This, by (3.1) proves the sufficiency part of Theorem A.

(ii) To prove that the condition is necessary, we assume (3.1), that is,

$$\int_0^{\pi} g_\alpha(t)\,\frac{1-\cos \omega t}{t}\,dt = o\{(\log \omega)^{\alpha+1}\}\,, \tag{4.7}$$

and deduce (1.7), provided that (1.6) is satisfied. Now

$$\int_0^\pi g_\alpha(t)\,\frac{1-\cos\omega t}{t}\,dt = \Big[-g_{\alpha+1}(t)(1-\cos\omega t)\Big]_0^\pi + \omega\int_0^\pi g_{\alpha+1}(t)\sin\omega t\,dt$$

$$= o(1)+\omega\int_0^\pi g_{\alpha+1}(t)\sin\omega t\,dt.$$

So we have, by (4.7),

$$\int_0^\pi g_{\alpha+1}(t)\sin\omega t\,dt = o\left\{\frac{(\log\omega)^{\alpha+1}}{\omega}\right\},$$

as $\omega\to\infty$.

Also after Hardy,* we take

$$g_{\alpha+1}(t)\sim\sum_{n=1}^{\infty} b_n^{(\alpha+1)}\sin nt,$$

for $0\le t\le\pi$, then

$$b_n^{(\alpha+1)} = \frac{1}{\pi}\int_0^\pi g_{\alpha+1}(t)\sin nt\,dt = o\left\{\frac{(\log n)^{\alpha+1}}{n}\right\},$$

as $n\to\infty$. Now

$$\frac{1}{t}\int_0^t g_{\alpha+1}(t)\,dt = \sum_{n=1}^{\infty} b_n^{(\alpha+1)}\frac{1-\cos nt}{nt}$$

$$= \sum_{n<\frac{1}{t}} b_n^{(\alpha+1)}\frac{\sin nt/2}{nt/2}\cdot\sin nt/2 + \sum_{n\ge\frac{1}{t}} b_n^{(\alpha+1)}\frac{1-\cos nt}{nt}$$

$$= \sum_{n<\frac{1}{t}} o\left\{\frac{(\log n)^{\alpha+1}}{n}\right\} + \sum_{n\ge\frac{1}{t}} o\left\{\frac{(\log n)^{\alpha+1}}{n}\right\} O\left(\frac{1}{nt}\right)$$

$$= o\left\{\left(\log\frac{1}{t}\right)^{\alpha+1}\right\}.$$

But

$$\frac{1}{t}\int_0^t g_{\alpha+1}(t)dt = g_{\alpha+1}(t)+\frac{1}{t}\int_0^t g_\alpha(t)\,dt.$$

Or, by (1.6)

$$o\left\{\left(\log\frac{1}{t}\right)^{\alpha+1}\right\} = g_{\alpha+1}(t)+o\left\{\left(\log\frac{1}{t}\right)^{\alpha+1}\right\}.$$

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* Hardy, 2.

Hence

$$g_{\alpha+1}(t) = o\left\{\left(\log \frac{1}{t}\right)^{\alpha+1}\right\},$$

as $t \to 0$, which proves the necessary part of the theorem.

5. The conjugate series for the function $f(t)$ given by (1.3) is summable $(R, 1)$ to $\frac{1}{\pi}$. For

$$g(t) = \frac{1}{\pi}\int_t^\pi f(t) \cot \frac{1}{2} dt - \frac{1}{\pi}$$

$$= \frac{1}{\pi}\int_t^1 \sin\left(\log \frac{1}{t}\right)\frac{dt}{t} - \frac{1}{\pi}$$

$$= -\frac{1}{\pi} \cos\left(\log \frac{1}{t}\right) = o\left(\log \frac{1}{t}\right),$$

as $t \to 0$. Also

$$g_1(t) = \int_t^\pi \frac{g(u)}{u}\, du = o\left(\log \frac{1}{t}\right),$$

and $$f(t) = O(1),$$

as $t \to 0$. Thus $f(t)$ satisfies the conditions of Theorem B with $s = \frac{1}{\pi}$.

6. We shall now give an example* of a function $f(t)$ such that

$$g(t) = \frac{1}{\pi}\int_t^\pi \psi(t) \cot \frac{1}{2} t\, dt - s = o\left(\log \frac{1}{t}\right) \neq o(1),$$

but the conjugate series of the Fourier series of $f(t)$ is not summable $(R, 1)$ to s.

We take $\theta = o$, $s = o$, $\psi(t) = f(t) =$ an odd function.

Example. We choose sequences $\{t_n\}$, $\{\lambda_n\}$ and $\{c_n\}$ such that

$$\frac{\pi}{2} = t_0 > t_1 > t_2 > \ldots\ldots > t_n \to 0,\ 0 < \lambda_1 < \lambda_2 < \ldots\ldots < \lambda_n \to \infty$$

and

$$c_n = \frac{\log \lambda_n}{\sqrt{\log (4n+1)}},$$

and we take

$$\lambda_r = 1.5.9 \ldots . (4r+1),\ t_r = \frac{\pi}{2\lambda_r}.$$

* The corresponding example for Fourier series was given by Wang, 7.

We define an odd function $f(t)$ in (o, π) by

$$(6.1)\qquad \begin{cases} f(0)=0 \\ f(t)=-c_r\lambda_r \tan\dfrac{t}{2}\sin\lambda_r t, \text{ for } t_r<t\leq t_{r-1}, \ (r=1,2,3,\ldots), \\ f(t)=0, \text{ for } \dfrac{\pi}{2}<t\leq\pi. \end{cases}$$

and by periodicity elsewhere.
Then

$$\int_{t_r}^{t_{r-1}} |f(t)|\,dt = c_r\lambda_r\int_{t_r}^{t_{r-1}} \tan\frac{t}{2}\,|\sin\lambda_r t|\,dt$$

$$= c_r\lambda_r\int_{\frac{\pi}{2\lambda_r}}^{\frac{\pi}{2\lambda_{r-1}}} \tan\frac{t}{2}\,|\sin\lambda_r t|\,dt$$

$$= O\left(\frac{c_r}{\lambda_r}\int_{\frac{\pi}{2}}^{\frac{\pi}{2}(4r+1)} u\,|\sin u|\,du\right)$$

$$= O\left\{\frac{c_r}{\lambda_r}r(2r+1)\right\}$$

$$= O\left\{\frac{1}{\lambda_r-4}\right\},\ r\geq 4.$$

Thus

$$\int_0^{\pi} |f(t)|\,dt = \int_{t_1}^{\frac{\pi}{2}}+\int_{t_2}^{t_1}+\int_{t_3}^{t_2}+\sum_{r=4}^{\infty}O\left(\frac{1}{1.5.9\ldots(4r-15)}\right)$$
$$= O(1)$$

so that $f(t)$ is integrable (L) in $(0, \pi)$.

Also for $r = 1, 2, 3, \ldots,$

$$\int_{t_r}^{t_{r-1}} f(u)\cot\frac{1}{2}u\,du = -c_r\lambda_r\int_{\frac{\pi}{2\lambda_r}}^{\frac{\pi}{2\lambda_{r-1}}} \sin\lambda_r t\,dt = c_r\Big[\cos\lambda_r t\Big]_{\frac{\pi}{2\lambda_r}}^{\frac{\pi}{2\lambda_{r-1}}}$$
$$=0.$$

Hence if $t_r<t\leq t_r-1$, then

$$g(t)=\frac{1}{\pi}\int_t^{\pi} f(u)\cot\frac{u}{2}\,du$$

$$= \frac{1}{\pi} \int_{t}^{t_{r-1}} f(u) \cot \frac{1}{2} u \, du$$

$$= -\frac{1}{\pi} c_r \cos \lambda_r t.$$

And

$$\left| \frac{g(t)}{\log \frac{1}{t}} \right| \leq \frac{c_r}{\log 1/t_{r-1}} = \frac{\log \lambda_r}{\sqrt{\log (4r+1)}} \cdot \frac{1}{\log \frac{2\lambda_{r-1}}{\pi}} = o(1),$$

as $r \to \infty$ so that $g(t) = o\left(\log \frac{1}{t}\right)$ as $t \to 0$.

Now from (3.1), the necessary and sufficient condition that the conjugate series of the Fourier series of $f(t)$ be summable $(R, 1)$ to zero is

$$\int_{0}^{\frac{\pi}{2}} g(t) \frac{1-\cos \omega t}{t} dt = o(\log \omega).$$

We shall now show that as ω takes successively the values of the sequence $\{\lambda_r\}$,

$$\frac{1}{\log \lambda_r} \int_{0}^{\frac{\pi}{2}} g(t) \frac{1-\cos \lambda_r t}{t} dt$$

is not $o(1)$, but tends to infinity as $r \to \infty$.
For

$$\int_{0}^{\frac{\pi}{2}} g(t) \frac{1-\cos \lambda_r t}{t} dt = \left\{ \int_{0}^{t_r} + \int_{t_r}^{t_{r-1}} + \sum_{k=1}^{r-1} \int_{t_k}^{t_{k-1}} \right\} g(t) \cdot \frac{1-\cos \lambda_r t}{t} dt$$

$$= \int_{0}^{t_r} g(t) \frac{1-\cos \lambda_r t}{t} dt - \int_{t_r}^{t_{r-1}} g(t) \frac{\cos \lambda_r t}{t} dt$$

$$+ \sum_{k=1}^{r} \int_{t_k}^{t_{k-1}} \frac{g(t)}{t} dt - \sum_{k=1}^{r-1} \int_{t_k}^{t_{k-1}} g(t) \frac{\cos \lambda_r t}{t} dt$$

$$= I + J + K + L, \text{ say.}$$

Now

$$I = \int_{0}^{\frac{\pi}{2\lambda_r}} y(t) \frac{1-\cos \lambda_r t}{t} dt = \lambda_r \int_{0}^{\frac{\pi}{2\lambda_r}} o\left(\log \frac{1}{t}\right) dt = o(\log \lambda_r).$$

$$J = -\int_{t_r}^{t_{r-1}} g(t)\frac{\cos\lambda_r t}{t}\,dt = \frac{c_r}{\pi}\int_{\frac{\pi}{2\lambda_r}}^{\frac{\pi}{2\lambda_{r-1}}} \cos^2\lambda_r t\,\frac{dt}{t}$$

$$= \frac{1}{2\pi}c_r\int_{\frac{\pi}{2\lambda_r}}^{\frac{\pi}{2\lambda_{r-1}}} \frac{1+\cos 2\lambda_r t}{t}\,dt$$

$$= \frac{c_r}{2\pi}\log\frac{\lambda_r}{\lambda_{r-1}} + \frac{c_r}{2\pi}\int_{\pi}^{(4r+1)\pi}\frac{\cos u}{u}\,du$$

$$= \frac{1}{2\pi}\frac{\log\lambda_r}{\sqrt{\log(4r+1)}}\log(4r+1) + O(c_r)$$

$$= \frac{1}{2\pi}\log\lambda_r.\sqrt{\log(4r+1)} + o(\log\lambda_r).$$

$$K = \sum_{k=1}^{r}\int_{t_k}^{t_{k-1}}\frac{g(t)}{t}\,dt = -\sum_{k=1}^{r}\frac{1}{\pi}c_k\int_{\frac{\pi}{2\lambda_k}}^{\frac{\pi}{2\lambda_{k-1}}}\frac{\cos\lambda_k t}{t}\,dt$$

$$= -\sum_{k=1}^{r}\frac{1}{\pi}c_k\int_{\frac{\pi}{2}}^{(4k+1)\frac{\pi}{2}}\frac{\cos u}{u}\,du$$

$$= -\sum_{k=1}^{r}\frac{1}{\pi}\cdot\frac{2}{\pi}c_k\int_{\frac{\pi}{2}}^{\epsilon_k}\cos u\,du,\quad \frac{\pi}{2}<\epsilon_k<\frac{\pi}{2}(4k+1),$$

$$= \sum_{k=1}^{r}2c_k(1-\sin\epsilon_k) = \sum_{k=1}^{r}2(1-\sin\epsilon_k)\frac{\log\lambda_k}{\sqrt{\log(4k+1)}},$$

so that K is positive. Again

$$L = -\sum_{k=1}^{r-1}\int_{t_k}^{t_{k-1}} g(t)\frac{\cos\lambda_r t}{t}\,dt = \sum\frac{1}{\pi}c_k\int_{t_k}^{t_{k-1}}\frac{\cos\lambda_k t\cos\lambda_r t}{t}\,dt$$

$$= \frac{1}{2\pi}\sum_{k=1}^{r-1}c_k\int_{\frac{\pi}{2\lambda_k}}^{\frac{\pi}{2\lambda_{k-1}}}\frac{\cos(\lambda_r-\lambda_k)t+\cos(\lambda_r+\lambda_k)t}{t}\,dt$$

$$= \sum_{k=1}^{r-1}c_k\lambda_k O\left(\frac{1}{\lambda_r-\lambda_k}\right) = \sum_{k=1}^{r-1}\frac{\log\lambda_k}{\sqrt{\log(4k+1)}}O\left(\frac{\lambda_k}{\lambda_r-\lambda_k}\right)$$

$$= r \log \lambda_{r-1} O\left(\frac{\lambda_{r-1}}{\lambda_r - \lambda_{r-1}}\right)$$

$$= O(\log \lambda_{r-1}) = o(\log \lambda_r),$$

as $r \to \infty$.

Hence combining our results we have

$$\frac{1}{\log \lambda_r} \int_0^{\frac{\pi}{2}} g(t) \frac{1-\cos \lambda_r t}{t} dt \geq \frac{1}{2\pi} \sqrt{\log (4r+1)} + o(1)$$

$$\to \infty, \text{ as } r \to \infty,$$

which proves that the conjugate series for the function $f(t)$ defined by (6.1) is not summable $(R, 1)$ although

$$g(t) = o\left(\log \frac{1}{t}\right).$$

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