MODULAR EQUATIONS AS SOLUTIONS OF ALGEBRAIC DIFFERENTIAL EQUATIONS OF THE SIXTH ORDER.

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In a recent paper (reviewed in *Mathematical Reviews*, Vol. 7, p. 243) I pointed out how identities of the Ramanujan-Rademacher-Zuchermann type can be proved by showing that both sides of the identity satisfy the same algebraic differential equation.

Let

$$K=\int_0^{\frac{\pi}{2}}\frac{d\phi}{\sqrt{1-k^2\sin^2\phi}},\quad K'=\int_0^{\frac{\pi}{2}}\frac{d\phi}{\sqrt{1-k'^2\sin^2\phi}}$$

$$L=\int_0^{\frac{\pi}{2}}\frac{d\phi}{\sqrt{1-l^2\sin^2\phi}},\quad L'=\int_0^{\frac{\pi}{2}}\frac{d\phi}{\sqrt{1-l'^2\sin^2\phi}}$$

where $k^2+k'^2=1,\ l^2+l'^2=1.$ If

(1) $$\frac{L'}{L}=n\,\frac{K'}{K}$$

where n is a positive integer, we have

$$\sqrt{kl}+\sqrt{k'l'}=1\quad(n=3)$$

$$\sqrt[4]{kl}+\sqrt[4]{k'l'}=1\quad(n=7)$$

such algebraic relations between k and l are called 'modular equations'. The object of this note is to point out that the 'modular equations' are solutions of algebraic differential equations of the sixth order.

Write

$$E=\int_0^{\frac{\pi}{2}}\sqrt{1-k^2\sin^2\phi}\,d\phi,\quad E'=\int_0^{\frac{\pi}{2}}\sqrt{1-k'^2\sin^2\phi}\,d\phi$$

$$M=\int_0^{\frac{\pi}{2}}\sqrt{1-l^2\sin^2\phi}\,d\phi,\quad M'=\int_0^{\frac{\pi}{2}}\sqrt{1-l'^2\sin^2\phi}\,d\phi.$$

Then we have

(2) $$EK'+KE'-KK'=\frac{\pi}{2}$$

(3) $$LM'+ML'-LL'=\frac{\pi}{2}$$

(4) $$\frac{dK}{dk}=\frac{E-k'^2K}{kk'^2}$$

(5) $$\frac{dE}{dk}=\frac{E-K}{k}$$

(6) $$\frac{dL}{dl}=\frac{M-l'^2L}{ll'^2}$$

(7) $$\frac{dM}{dl}=\frac{M-L}{l}$$

Differentiating (1) 6 times with respect to k and using (2), (3), (4), (5), (6), (7) to eliminate K, K', L, L', E, M from (1) and the 6 equations obtained by differentiation, we get

Theorem : If (1) *is true then*

$$f\left(k, l, \frac{dl}{dk}, \frac{d^2l}{dk^2}, \frac{d^3l}{dk^3}, \frac{d^4l}{dk^4}, \frac{d^5l}{dk^5}, \frac{d^6l}{dk^6}\right)=0$$

where $f(x_1, x_2, \ldots, x_8)$ *denotes a polynomial in the* x*'s with integral coefficients depending on* n *alone.*

References.

Chowla, S. (1945). Outline of a new method, etc. *Proc. Lahore Phil. Soc.*, Vol. 8. See *Mathematical Reviews*, Vol. 7, 1946, page 243.