ON SERIES OF THE LAMBERT TYPE WHICH ASSUME IRRATIONAL VALUES FOR RATIONAL VALUES OF THE ARGUMENT.

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(Communicated by Sir S. S. Bhatnagar, F.R.S.)

(*Read January 17, 1947.*)

Let

$$f(x) = \sum_{1}^{\infty} \frac{x^n}{1-x^n},$$

$$g(x) = \sum_{1}^{\infty} \frac{x^n}{1-x^n} \sin \frac{n\pi}{2},$$

where $|x| < 1$. It is not unlikely that $f(x)$ and $g(x)$ are irrational when x is a rational number different from 0. I am unable to prove anything about $f(x)$, but I can show that $g(x)$ is irrational when x is a rational number of the form $1/t$ where t is a positive integer $\geqslant 5$.

We have

Lemma 1. *We have*

$$1+4g(x) = \sum_{0}^{\infty} r(n)x^n$$

where $r(n)$ is the number of representations of n as a sum of two squares.

This is well-known.

Lemma 2. *Let ϵ denote an arbitrary positive number and m an arbitrary positive integer. Then we can find an integer x such that*

(i) $$r(x+t) = 0 \text{ for } 1 \leqslant t \leqslant m$$

(ii) $$m > (\tfrac{1}{2}-\epsilon) \frac{\log x}{\log \log x}$$

for all $m > m_0(\epsilon)$.

Proof. Let q_m denote the mth prime $\equiv 3 \pmod 4$. Then the system of congruences

$$x+1 \equiv q_1 \pmod{q_1^2}$$
$$x+2 \equiv q_2 \pmod{q_2^2}$$
$$x+m \equiv q_m \pmod{q_m^2}$$

is soluble, and in fact with

$$q_1^2 q_2^2 \ldots q_m^2 \leqslant x \leqslant 2q_1^2 q_2^2 \ldots q_m^2.$$

Now from the extended Prime Number Theorem,

$$q_m \sim 2m \log m$$

whence

$$\log x \sim 2\sum_{t=1}^{m} \log t \sim 2m \log m$$

$$\log\log x \sim \log m$$

$$\frac{\log x}{\log\log x} \sim 2m$$

so that for any $\epsilon > 0$ and $m > m_0(\epsilon)$ we have

$$m > (\tfrac{1}{2} - \epsilon)\frac{\log x}{\log\log x}.$$

Further $$r(x+t) = 0 \text{ for } 1 \leqslant t \leqslant m$$

since x satisfies the above m congruences.

Now it is known that

Lemma 3. *We have*

$$r(n) < 2^{(1+\epsilon)\frac{\log n}{\log\log n}}$$

where $\epsilon > 0$, *for all* $n > n_0(\epsilon)$.

Now consider

$$\sum_{g=m+1}^{\infty} \frac{r(x+g)}{t^{x+g}}$$

$$= \sum_{n=x+m+1}^{2x} \frac{r(n)}{t^n} + \sum_{2x+1}^{\infty} \frac{r(n)}{t^n}$$

$$\leqq \frac{2^{\frac{(1+\epsilon)\log x}{\log\log x}}}{t^{x+m+1}} + \sum_{2x+1}^{\infty} \frac{n}{t^n}$$

$$\leqq \frac{2^{\frac{(1+\epsilon)\log x}{\log\log x}}}{t^{x+m+1}} + 0\left(\frac{1}{t^{2x}}\right)$$

(1) $$\leqq \frac{t^{\frac{(1+\epsilon)\log 2\log x}{\log t\log\log x}}}{t^{x+m+1}} + 0\left(\frac{1}{t^{2x}}\right).$$

Let us represent

$$S = \sum_{1}^{\infty} \frac{r(n)}{t^n}$$

as a decimal in the scale of t.

Writing

$$S = \sum_{n=1}^{x} + \sum_{x+1}^{x+m} + \sum_{x+m}^{\infty}$$

$$= \sum\nolimits_1 + \sum\nolimits_2 + \sum\nolimits_3.$$

On account of $\Sigma_2 = 0$ it would follow that all the decimal places of x from the $(x+1)$th to the $(x+m)$th are zero, had Σ_3 not butted into this part of the decimal representation (in the scale of t) of S. But roughly

$$\frac{(1+\epsilon)\log 2 \log x}{\log t \log \log x} \tag{2}$$

decimal places to the left of the $(x+m)$th decimal places are affected by Σ_3 on account of (1). Now

$$m > \left(\frac{1}{2} - \epsilon\right)\frac{\log x}{\log \log x} \tag{3}$$

From (2) and (3) if

$$\frac{\log 2}{\log t} < \frac{1}{2}$$

i.e. $t > 5$, S has a block of at least

$$\left(\frac{1}{2} - \frac{\log 2}{\log t}\right)\frac{(1-\epsilon)\log x}{\log \log x}$$

decimal places all equal to 0. Since S has an infinity of decimal places $\neq 0$ it follows that S is irrational.—Q.E.D.

References.

Koksma, J. F. (1936). *Diophantische Approximationen.*

Siegel, C. L. (1930). *Über linige Anwendungen diophantischer Approximationen.*