ON THE HELIUM CONTENT OF STARS OF LARGE MASSES.

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ABSTRACT.

The effect of the assumption of a substantial proportion of helium together with hydrogen in the composition of certain stars of large masses has been worked out for the energy generation law of Bethe. It is found that the problem of determining the hydrogen and helium contents in stars of large masses with the observed values of their mass, radius and luminosity, though mathematically soluble, does not in general lead to a physically significant solution. The case in which a physically significant solution may be expected is indicated in this paper.

1. INTRODUCTION.

This paper is a continuation of a previous one* (Burman, 1946) where the internal constitution of stars of large masses with appreciable radiation pressure, was studied on the basis of Bethe's law of energy generation. The helium content of these stars was assumed to be negligible; it was then found that the Bethe formula of energy generation does not fit in satisfactorily with the mass-luminosity relation in these stars. It seems therefore reasonable to study the effect of helium on the constitution of these stars and it is the result of this study that we deal with in the present communication.

The models we consider here are of the convective-radiative type, and have assigned values for the ratio y_c of radiation to gas pressure at the centre. A number of *point source* stellar models for different values of y_c have been constructed by Henrich (1942), and we make use of some of them for our present purpose, introducing, however, the assumption that the energy generation in them is governed by Bethe's law. We consider a number of models for $y_c = 0{\cdot}01$, and $0{\cdot}10$ and for various values of X and Y, the hydrogen and helium contents respectively; so that y_c, X and Y serve as *three* parameters in these models, whereas in the previous paper we had y_c and X only as the *two* parameters in the models, the helium content there being assumed zero throughout. As explained in the previous paper we make use of the approximate power law representation of the exact exponential law of energy generation.

2. THE EQUATIONS.

The equilibrium equations in the convective core, in the usual notations are

$$\frac{dP}{dr} = -G\frac{M(r)}{r^2}\rho \qquad \ldots \quad (1)$$

$$\frac{dM(r)}{dr} = 4\pi r^2\rho \qquad \ldots \quad (2)$$

and

$$\frac{dP}{P} = \Gamma_1\frac{d\rho}{\rho} \qquad \ldots \quad (3)$$

* Referred to hereinafter as the 'previous paper'.

where the adiabatic exponent Γ_1 is given by

$$\Gamma_1 = \beta + \frac{(4-3\beta)^2(\gamma-1)}{\beta+12(\gamma-1)(1-\beta)} \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

In the radiative envelope, equation (3) is to be replaced by the equation

$$\frac{d}{dr}\left(\frac{1}{3}aT^4\right) = -\frac{\kappa\rho}{c}\frac{L}{4\pi r^2}, \quad \ldots \quad \ldots \quad \ldots \quad (5)$$

where the luminosity L is assumed constant in the envelope. The law of opacity (Henrich, 1942) is taken as in the previous paper as

$$\left.\begin{aligned} \kappa &= \frac{\kappa_0 \rho T^{-3\cdot5}}{10\theta} & \theta \geqslant \theta_\kappa \\ &= \frac{\kappa_0 \rho T^{-3\cdot5}}{10\theta_\kappa} & \theta < \theta_\kappa \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (6)$$

where $\theta = T/T_c$, T_c being the central temperature, and θ_κ, a certain value of θ in the envelope region where the opacity changes.

The luminosity equation is

$$L(r) = \int_0^r 4\pi r^2 \rho\epsilon \, dr \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (7)$$

with

$$\epsilon = EX\rho T^n, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (8)$$

the coefficient E and the exponent n being suitably chosen.

We had in the previous paper chosen different values of E and n for different values of y_c, and this choice was made to suit the temperature conditions in those models.

It is now found that the introduction of a not very high concentration of helium and a varying concentration of hydrogen in the models does not materially affect the mean value of n for the corresponding ranges of central temperature. We therefore retain in the present calculations the previous values of E and n for the same value of y_c.

Introducing the variables

$$r = \alpha\xi, \; \rho = \rho_c\sigma, \; T = T_c\theta$$

we obtain the total luminosity of the model, ignoring the slight generation of energy outside the convective core, as

$$L(\xi_i) = 4\pi EX\left(\frac{5k}{8\pi GH}\right)^{3/2}\frac{1}{\mu}\left(\frac{a}{3}\frac{H}{k}\right)^{1/2}\frac{T_c^{n+3}}{y_c^{1/2}} \cdot I(\xi_i, y_c) \quad \ldots \quad \ldots \quad (9)$$

where

$$I(\xi_i, y_c) = \int_0^{\xi_i} \sigma^2\theta^n\xi^2 \, d\xi, \quad \ldots \quad \ldots \quad \ldots \quad (10)$$

ξ_i, being the interface between the convective core and the radiative envelope. The method of evaluating the integral $I(\xi_i, y_c)$ has been shewn in the previous paper.

For a model with given y_c, we have the following relations (Henrich, 1942) between L, M, R, T_c, ρ_c and μ, the quantities L, M, R, being expressed in solar units.

$$\mu^2 M = 4{\cdot}420\, y_c^{1/2}\, \psi_R \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (11)$$

$$\log \rho_c = \log\left(\frac{1}{3}\frac{\alpha_2^3}{\psi_R}\right) + \log\frac{M}{R^3} + 0{\cdot}149 \quad \ldots \quad (12)$$

$$\log T_c = 6{\cdot}966 + \log\frac{\alpha_2}{\psi_R} + \log\frac{\mu M}{R} \quad \ldots \quad (13)$$

$$\log L = 28{\cdot}742 + \log\left(\frac{\alpha_2^{0\cdot5}\, Q_2}{\psi_R^{5\cdot5}}\right) + \log\left(\frac{\mu^{7\cdot5} M^{5\cdot5}}{\kappa_0 R^{0\cdot5}}\right), \quad \ldots \quad (14)$$

where α_2, ψ_R, Q_2 are known in terms of y_c alone. It is to be noted that the energy generation formula has not been used in the derivation of these relations.

The values of L as determined by equations (9) and (14) should agree and this requires that

$$\log\left\{4\pi E X\left(\frac{5k}{8\pi G H}\right)^{3/2}\frac{1}{\mu}\left(\frac{a}{3}\frac{H}{k}\right)^{1/2}\frac{T_c^{n+3}}{y_c^{1/2}}\cdot I(\xi_i, y_c)\right\}$$

$$= 28{\cdot}742 + \log\left(\frac{\alpha_2^{0\cdot5}\, Q_2}{\psi_R^{5\cdot5}}\right) + \log\left(\frac{\mu^7 M^5}{\kappa_0}\right) + \frac{1}{2}\log T_c - \log\frac{\alpha_2}{\psi_R}$$

$$-3{\cdot}483 + \log L_\odot \quad \ldots \quad (15)$$

where $L_\odot$ denotes the solar luminosity.

The opacity coefficient κ_0 and the mean molecular weight μ are given by

$$\kappa_0 = 4{\cdot}32\times 10^{25}(1+X)(1-X-Y) \quad \ldots \quad (16)$$

and

$$\mu = 2/(1+3X+\tfrac{1}{2}Y) \quad \ldots \quad (17)$$

We now explain how the configuration is determined by y_c, X and Y when the energy generation formula is taken into account. For given y_c, equation (11) determines the mass when X and Y are assigned. Equation (15) now determines T_c and the radius R is then obtained from equation (13). The central density and the luminosity are then given by equations (12) and (14) respectively, so that the configuration is completely determined. We saw in the previous paper how y_c and X, i.e. μ (when $Y = 0$) alone determined the configuration. Here, however, we do not put $Y = 0$ and therefore replace μ by the two parameters X and Y. The number of equations, however, is just sufficient to determine the configuration when y_c, X and Y are assigned.

3. Results of Calculations.

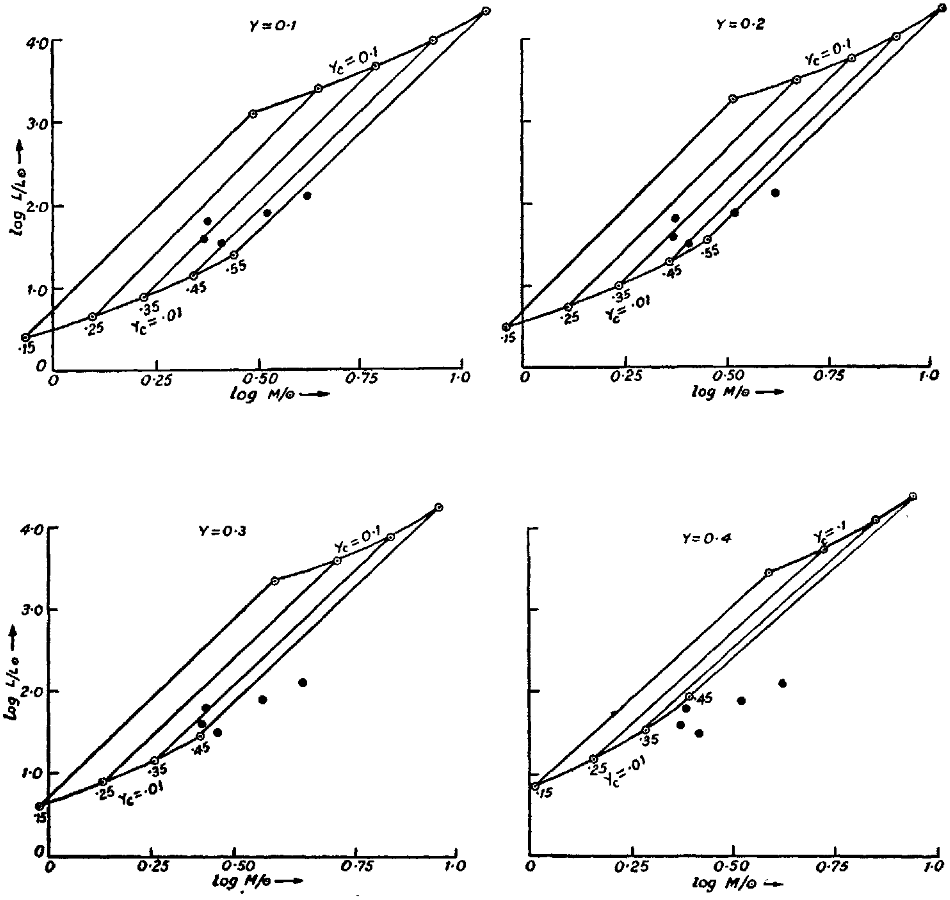
We have constructed a number of models for $y_c = 0{\cdot}01$ and $0{\cdot}10$ and for different values of X and Y, the hydrogen and helium contents respectively. For each value of y_c we take $Y = 0{\cdot}10$, $0{\cdot}20$, $0{\cdot}30$ and $0{\cdot}40$ and corresponding to each value of Y the values of X taken vary in general from 15 to 45 per cent. The stellar parameters L, M, R, T_c and ρ_c are calculated in each case according to the formulae in the last section, and the results are shewn in Table 1. As stated before, the coefficient E and the exponent n in the energy generation formula are taken to be different for different values of y_c and their values for a given y_c are taken to be the same as those in the previous paper. We shall now see how it is possible from a study of these results to calculate y_c, X, Y, T_c and ρ_c for a star whose L, M, R-values are observationally known.

TABLE I.

Luminosity, Mass, Radius, Central Temperature and Density of Configurations with assigned y_c (ratio of radiation to gas pressure at the centre) and composition.

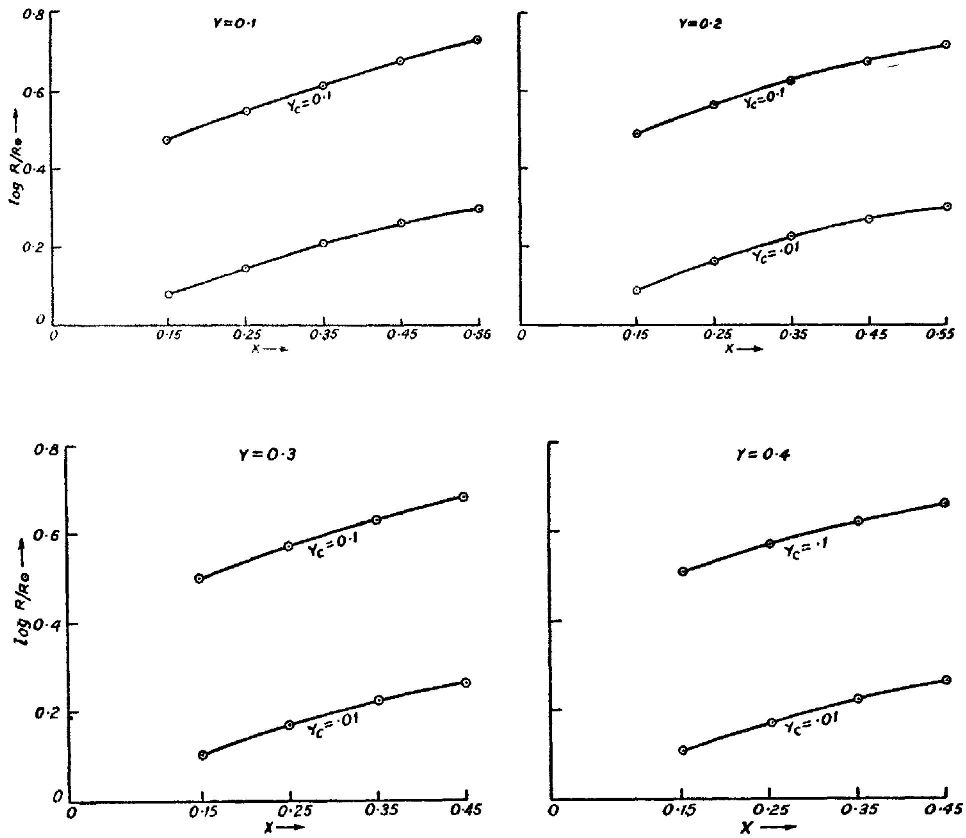
Y	X	y_c	$\log L/L_\odot$	$\log M/\odot$	$\log R/R_\odot$	$T_c 10^{-6}$	ρ_c
0·10	0·15	0·01	0·391	−0·072	0·083	22·7	47·4
		0·10	3·12	0·493	0·483	32·4	13·9
	0·25	0·01	0·659	0·087	0·153	23·1	42·0
		0·10	3·39	0·652	0·553	33·2	12·4
	0·35	0·01	0·903	0·220	0·211	23·6	38·3
		0·10	3·63	0·786	0·611	33·9	11·3
	0·45	0·01	1·14	0·337	0·261	24·1	35·5
		0·10	3·87	0·902	0·660	34·5	10·5
0·20	0·15	0·01	0·498	−0·043	0·093	22·9	47·3
		0·10	3·23	0·522	0·492	32·8	13·9
	0·25	0·01	0·770	0·110	0·160	23·4	42·3
		0·10	3·50	0·676	0·560	33·5	12·5
	0·35	0·01	1·02	0·241	0·216	23·9	38·8
		0·10	3·75	0·806	0·616	34·3	11·4
	0·45	0·01	1·28	0·355	0·263	24·4	36·4
		0·10	4·01	0·920	0·662	35·1	10·7
0·30	0·15	0·01	0·614	−0·016	0·102	23·2	47·4
		0·10	3·34	0·550	0·500	33·2	14·0
	0·25	0·01	0·895	0·134	0·166	23·7	42·8
		0·10	3·62	0·700	0·566	34·0	12·6
	0·35	0·01	1·17	0·261	0·220	24·3	39·6
		0·10	3·90	0·826	0·619	34·8	11·7
	0·45	0·01	1·45	0·372	0·264	24·9	37·7
		0·10	4·18	0·937	0·663	35·8	11·2
0·40	0·15	0·01	0·845	0·011	0·109	23·4	47·8
		0·10	3·47	0·576	0·509	33·7	14·1
	0·25	0·01	1·21	0·156	0·171	24·0	43·6
		0·10	3·77	0·721	0·570	34·5	12·9
	0·35	0·01	1·57	0·281	0·221	24·7	41·0
		0·10	4·08	0·845	0·620	35·5	12·2
	0·45	0·01	1·97	0·389	0·260	25·6	40·2
		0·10	4·39	0·954	0·659	36·7	11·9

From Table I, we construct a mass-luminosity diagram for each value of Y and the corresponding values of y_c and X as in Figs. 1–4. Each diagram consists of two sets of curves y_c = const. and X = const. We also plot $\log R/R_\odot$ against X for each value of Y as in Figs. 5–8.



FIGS. 1–4.

The mass-luminosity diagrams for $Y = 0{\cdot}1$, $0{\cdot}2$, $0{\cdot}3$ and $0{\cdot}4$, with $y_c = 0{\cdot}01$ and $0{\cdot}10$ in each case. The values of X for the various cases are indicated at the bottom of the curves.



FIGS. 5–8.

The variation of radius with X for $Y = 0{\cdot}1$, $0{\cdot}2$, $0{\cdot}3$, $0{\cdot}4$ and $y_c = 0{\cdot}01$ and $0{\cdot}1$.

These curves would enable us to calculate in the manner explained below, y_c, X and Y for a model when the values of L, M and R are assigned. The central density and temperature can then be obtained from the equations (12) and (13) respectively, the quantities α_2 and ψ_R for a given y_c being obtained by interpolation from Henrich's (1942) values.

The position of a star in each of the mass-luminosity diagrams would give by interpolation the values of X and y_c corresponding to the different values of Y, that is to say, if we assume a given helium content in the composition of a star of given L, M, R, the position of the representative point of the star in the mass-luminosity diagram corresponding to the assumed value of Y, would give us a value for its hydrogen content X and a value for the central ratio y_c of radiation to gas pressure. A further interpolation in the R–X diagram for the appropriate Y would give for the radius of the star a value which is not necessarily the same as the observed value of the radius. In this manner we obtain different sets of values for y_c, X, Y and R for the same L, M-values of a star. Interpolation between these sets of values for a definite value of R (which is the observed value of the radius of the star whose L, M-values we have taken) would now fix the parameters y_c, X and Y for the star considered. The procedure is as follows:—

Consider, for example, any one of these sets of values, say y_c', X', Y' and R'. We note that for the given L, M and R' as parameters, only the simultaneous set of values y_c', X' and Y' is possible, as this set is the uniquely determined solution of our equations for the given L, M and R'. This will be true of every one of the (four) sets of y_c, X, Y and R determined by us, corresponding to given L, M-values. This being the situation, the values of y_c, X and Y for some intermediate value of R can be easily obtained by a one-parametric interpolation. The central density and temperature would then be obtained from equations (12) and (13), so that all the parameters of the star are determined by its L, M, R-values. From this point of view we have here examined some stars of Kuiper's table considered in the previous paper and the results are shewn in Table II.

TABLE II.

Solutions for X, Y, y_c, T_c and ρ_c for some stars whose L, M, R values are known from observation.

Star	$\log L/L_\odot$	$\log M/\odot$	$\log R/R_\odot$	X	Y	y_c	$T_c \cdot 10^{-6}$	ρ_c
α Aur A ..	2·08	0·62	1·20	no physically significant solutions				
α Aur B ..	1·90	0·52	0·82					
β Aur A ..	1·83	0·38	0·43					
ζ C Ma A ..	1·48	0·41	0·28	0·46	0·20	0·013	26·4	35·0
α C Ma A ..	1·59	0·37	0·25	0·45	0·45	0·015	24·8	24·2

The first 3 stars correspond to the case $R_{obs.} > R_{calc.}$ (for $Y = 0$) as may be seen from Table II of the previous paper. The other two correspond to the case $R_{obs.} < R_{calc.}$ (for $Y = 0$). It is only in the latter case that a physically significant solution is possible.

4. CONCLUSIONS.

It will be noticed from Table II of the previous paper that for an assumed zero helium content it is not possible to so choose a hydrogen content as to obtain an agreement in the observed and calculated values of L, M, R of a star in which the effect of radiation pressure is not negligible and in which energy generation takes place according to Bethe's law. We have in this paper attempted to obtain the desired agreement in the calculated and observed values of L, M, R of a star in which energy is generated according to Bethe's law, by introducing a suitable proportion of helium into its composition.

It is found that the effect of introducing helium into the composition of a star of given L, M is to cause a decrease in the calculated value of its radius as compared with its value calculated on the assumption of $Y = 0$. It is therefore not possible by giving suitable values to the helium content (Y) to obtain an agreement between the observed and calculated values of L, M, R in the case of those stars where the observed radius is greater than that calculated on the basis of $Y = 0$. This will be clear from Table II where the desired agreement has been obtained in the case of only two stars for which $R_{obs.} < R_{calc.}$ (with $Y = 0$). The other stars of the table correspond to the case $R_{obs.} > R_{calc.}$ (with $Y = 0$) and no agreement has been possible in their cases with any physically significant values of X and Y. It is thus permissible to conclude that Bethe's law will not generally explain the generation of energy in stars of large masses.

This result is of considerable importance in deciding whether a star of large mass with given L, M, R can be supplied with energy according to Bethe's scheme. Our previous paper gives a simple method of calculating the radius of a configuration of given L, M and zero helium content, assuming that energy is generated within it according to Bethe's law. Unless the calculated radius is larger than the given radius of the configuration, we may decide that Bethe's law will not be applicable to this configuration.

It has, however, been shewn that so far as accurate observations go Bethe's law satisfactorily accounts for the energy generation in stars whose masses are in the neighbourhood of the solar mass.

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