ON A PROBLEM OF ANALYTIC NUMBER THEORY.

By S. Chowla.

(Communicated by Sir S. S. Bhatnagar, F.R.S.)

(*Read January 17, 1947.*)

In his *Vorlesungen über Zahlentheorie* Landau raises the problem of an 'elementary' proof of the theorem :

Let p denote a prime $\equiv 3 \pmod 4$. *Then there are more quadratic residues than non-residues between* 0 *and* $\frac{p}{2}$. This note contains a reasonably elementary proof of this result. We have

Lemma 1 :

$$\textit{If}\quad 0 < x < 1 \quad \textit{we have}$$

$$\tfrac{1}{2} - x = \sum_{1}^{\infty} \frac{\sin 2n\pi x}{n\pi}.$$

Lemma 2 :

$$\sum_{n=1}^{p-1} \left(\frac{n}{p}\right) \sin \frac{2mn\pi}{p} = \left(\frac{m}{p}\right)\sqrt{p}$$

where $\left(\frac{n}{p}\right)$ *is Legendre's symbol.*

This follows from the Gaussian sum :

$$\sum_{1}^{\frac{p-1}{2}} \sin \frac{2n^2\pi}{p} = \tfrac{1}{2}\sqrt{p}$$

of which an elegant and simple proof was given by Estermann in *Journ. Lond. Math. Soc.* (1945).

From Lemmas 1 and 2 we immediately get

Lemma 3 :

(1) $$\sum_{1}^{\infty} \left(\frac{n}{p}\right) \frac{1}{n} = \frac{\pi \Sigma (b-a)}{p\sqrt{p}},$$

where a runs through all $n (1 \leqslant n < p)$ *with* $\left(\frac{n}{p}\right) = +1$,

b runs through all $n (1 \leqslant n < p)$ *with* $\left(\frac{n}{p}\right) = -1$.

Lemma 4 :

$$\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots = \frac{\pi}{4} \textit{ or } -\frac{\pi}{4}$$

according as $0 < x < \pi$ *or* $\pi < x < 2\pi$.

From Lemmas 1, 2, 4,

LEMMA 5:

$$\sqrt{p} \sum_{n \text{ odd}} \left(\frac{n}{p}\right) \frac{1}{n} = \frac{\pi}{2} \sum_{0}^{\frac{p}{2}} \left(\frac{n}{p}\right)$$

whence

$$\sqrt{p} \sum \left(\frac{n}{p}\right) \frac{1}{n} = \frac{\pi}{2} \left\{1 - \left(\frac{2}{p}\right) \frac{1}{2}\right\}^{-1} \sum_{0}^{\frac{p}{2}} \left(\frac{n}{p}\right). \tag{2}$$

From (1) and (2),

LEMMA 6:

$$\sum (b-a) = \frac{p}{\left\{2 - \left(\frac{2}{p}\right)\right\}} \sum_{0}^{\frac{p}{2}} \left(\frac{n}{p}\right). \tag{3}$$

From

$$\sum_{1}^{\infty} \left(\frac{n}{p}\right) \frac{1}{n^s} = \prod_{p} \left\{1 - \left(\frac{n}{p}\right) \frac{1}{n^s}\right\}^{-1} > 0 \qquad (s > 1)$$

it follows from considerations of continuity that

$$\sum_{1}^{\infty} \left(\frac{n}{p}\right) \frac{1}{n} \geqslant 0. \tag{4}$$

From (1) and (4),

$$\Sigma(b-a) \geqslant 0. \tag{5}$$

Now

$$\sum (b+a) = \frac{p(p-1)}{2} \equiv 1 \pmod{2}. \tag{6}$$

From (6),

$$\Sigma(b-a) \equiv 1 \pmod{2}. \tag{7}$$

From (5) and (7),

$$\Sigma(b-a) > 0. \tag{8}$$

From (3) and (8),

$$\sum_{0}^{\frac{p}{2}} \left(\frac{n}{p}\right) > 0. \tag{9}$$

q.e.d.

REFERENCE.

Landau, (1927), *Vorlesungen über Zahlentheorie*, Band 1.