STABILITY OF STARS UNDER VARIABLE Γ.

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1. It is a well-known fact that if Γ, the effective ratio of specific heats of stellar material, is constant throughout the interior of the star, it becomes dynamically unstable for values of $\Gamma < \frac{4}{3}$, being in neutral equilibrium for $\Gamma = \frac{4}{3}$. Ledoux (1946) studied the effect of the variable Γ on the dynamical stability of homogeneous and standard models by taking a central core with $\Gamma = \frac{5}{3}$ and an envelope with $\Gamma = 1$. He found that for dynamical instability to set in the envelope should extend inside the star to the depth at which the temperature is of the order of half the central temperature, that is the ratio of the radius of the core to the radius of the star is approximately 0·727 for the homogeneous model and 0·36 for the standard model. Ledoux has pointed out that peculiar forms of $\Gamma(r_0)$ could increase the instability considerably. In the present note we have studied the stability of the fundamental mode of oscillation of the following two models taking Γ as a function of r_0 giving a continuous decrease in Γ from the centre to the surface of the star.

(i) Homogeneous model.

(ii) The model for which the density varies inversely as the square of the distance from the centre,

with $\Gamma = \Gamma_0 \left(1 - A \frac{r_0^2}{R^2}\right)$,

where r_0 stands for the distance of the point from the centre, Γ_0 for its value at the centre, and R for the radius of the star. A is dimensionless constant. The choice of law is made in such a manner that each model becomes amiable to mathematical analysis without resorting to numerical integration.

In other papers one of the authors (Kushwaha, 1951) has studied the stability of higher modes of oscillation of these two models and the stability of fundamental and higher modes of oscillation of the Roche-model and a model with a central point-mass equal to one-third of the total mass of the star with homogeneous density distribution throughout the star with the other suitable law for the variation of Γ.

2. We have taken continuous variation of Γ to see how the stability of a particular model is affected when in the outer part of the star Γ falls below $\frac{4}{3}$ and to know how the displacement function and the period of pulsation of the star vary with the mean value of Γ for the whole star. Besides, as is evident from the calculations of Fowler and Guggenheim (1925) for giant stars composed of mainly $Fe^{(26)}$ Γ decreases from 1·68 at the centre $z = 0$ to 1·295 at $z = 4$ on the scale when $z = 6·9011$ is the radius of the star, meaning thereby that Γ varies more or less continuously.

$\frac{d\Gamma}{dr_0}$ being proportional to A for a given value of r_0 the variation of Γ with r_0 can be arbitrarily chosen. For a small radial adiabatic deformation such that

$$\frac{\delta r}{r_0} = \xi(r_0)e^{i\sigma t}, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

σ^2 is given by

$$\sigma^2 \int_0^R \xi \, dI_0 = -\int_0^R 4\pi \xi r_0^3 \frac{d}{dr_0} [(3\Gamma - 4)P_0] \, dr_0, \quad \ldots \quad \ldots \quad (2)$$

where the suffix zero refers to equilibrium values. Further more

$$I_0 = \int_0^R r_0^2 \, dm, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

represents the moment of inertia with respect to the origin and ξ the solution of the differential equation

$$\frac{d^2\xi}{dr_0^2} + \left[\frac{4-\mu}{r_0} + \frac{1}{\Gamma}\frac{d\Gamma}{dr_0}\right]\frac{d\xi}{dr_0} + \left[\frac{\sigma^2 \rho_0}{\Gamma P_0} - \frac{\mu}{r_0^2}\left(3 - \frac{4}{\Gamma}\right) + \frac{3}{r_0 \Gamma}\frac{d\Gamma}{dr_0}\right]\xi = 0, \quad \ldots \quad (4)$$

with the boundary conditions

$$\xi \cdot r_0 = 0 \text{ at } r_0 = 0, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (5)$$

and

$$\delta P = -\Gamma P_0 \left[3\xi + r_0 \frac{d\xi}{dr_0}\right] = 0 \text{ at } r_0 = R, \ldots \quad (6)$$

where

$$\mu = G \frac{m r_0 \rho_0}{r_0 P_0}, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (7)$$

and Γ is the general adiabatic exponent defined by

$$dQ = dU - P\frac{d\rho}{\rho^2} = 0, \ldots \quad \ldots \quad \ldots \quad \ldots \quad (8)$$

the variation of the internal energy dU being expressed in terms of P and ρ. In a star Γ is, in general, a function of the ratio of the pressure of radiation to the total pressure, $(1-\beta)$, of the degree of ionisation and of the number of degrees of freedom of the particles.

3. *Homogeneous Model*

From (2) we shall have definitely a positive value of σ^2 for the fundamental mode ($\xi > 0$ throughout the whole star) if

$$\frac{d}{dr_0}[(3\Gamma - 4)P_0] < 0. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (9)$$

For this inequality to be true we must have

$$0 < A < \tfrac{1}{5}. \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (10)$$

Hence, we can safely predict that at least for values of A given by (10) the star will be stable. We shall consider here the higher values of A and see for what value of A the instability will set in by actually solving the pulsation equation (4).

Writing $x = \dfrac{r_0}{R}$, we have for this model

$$g_0 = 4\pi G \rho R x, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (11)$$

$$P_0 = 2\pi G \rho^2 R^2 (1 - x^2), \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (12)$$

$$\Gamma = \Gamma_0 (1 - A x^2). \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (13)$$

Substituting values for g_0, P_0 and Γ from (11), (12) and (13) in (4) we get

$$(1-x^2)(1-Ax^2)\xi''+[4-6(A+1)x^2+8Ax^4]\frac{\xi'}{x}+[F+12Ax^2]\xi = 0, \quad \ldots \quad (14)$$

where
$$F = \frac{3\sigma^2}{2\pi G\rho\Gamma_0}-2\alpha_0-6A, \quad \alpha_0 = 3-\frac{4}{\Gamma_0}, \quad \ldots \quad \ldots \quad (15)$$

dashes denoting differentiation with respect to x.

Equation (14) has regular singularities at $x = 1$ and at $x = \frac{1}{\sqrt{A}}$. We have to find out regular integrals which are finite in the range $0 < x < 1$ in order that boundary conditions may be satisfied.

The roots of the indicial equation are 0 and -3. To avoid singularity at the centre, we take the first root, and put

$$\xi = \sum_{k=0}^{\infty} a_{2k}x^{2k}. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (16)$$

Substituting it in (14) and equating the coefficients of various powers of x to zero we have

$$10a_2+Fa_0 = 0, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (17)$$

$$(2k+2)(2k+5)a_{2k+2}+[F-(A+1)2k(2k+5)]a_{2k}+[(2k-2)(2k+5)+12]Aa_{2k-2} = 0, \quad \ldots \quad \ldots \quad (18)$$

where $k = 1, 2, 3, \ldots.$

Writing

$$\frac{a_{2k+2}}{a_{2k}} = N_{k+1}$$

(18) can be put in the form

$$N_{k+1} = \frac{2k(2k+5)(A+1)-F}{(2k+2)(2k+5)}-\frac{A\{(2k-2)(2k+5)+12\}}{(2k+2)(2k+5)}\cdot\frac{1}{N_k}, \quad \ldots \quad (19)$$

$$= (A+1)-\frac{4k+10+F}{(2k+2)(2k+5)}-\left\{A-\frac{(8k+8)A}{(2k+2)(2k+5)}\right\}\frac{1}{N_k}. \quad \ldots \quad (19')$$

Suppose N_k tends to a limit λ when $k\to\infty$. Consequently N_{k+1}, N_{k+2}, ... will all tend to λ. Hence, from (19′) we get in the limit

$$\lambda = A+1-\frac{A}{\lambda}\cdot \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (20)$$

The roots of this equation are A and 1.

In case when $N_k\to 1$ the series for ξ diverges for $x = 1$. According to our physical conditions, we must choose the solution for which $N_k\to A$, where $A < 1$. This restricts us to a determinate set of values of F.

(19) can also be written as

$$N_k = \frac{A\,\dfrac{(2k-2)(2k+5)+12}{(2k+2)(2k+5)}}{\dfrac{2k(2k+5)(A+1)-F}{(2k+2)(2k+5)}-N_{k+1}}, \quad \ldots \quad \ldots \quad (21)$$

and by successive application of (21) we can express N_k in the form of a convergent continued fraction,

$$N_k = \frac{A\dfrac{(2k-2)(2k+5)+12}{(2k+2)(2k+5)}}{\dfrac{2k(2k+5)(A+1)-F}{(2k+2)(2k+5)}-} \frac{A\dfrac{2k(2k+7)+12}{(2k+4)(2k+7)}}{\dfrac{(2k+2)(2k+7)(A+1)-F}{(2k+4)(2k+7)}-} \frac{A\dfrac{(2k+2)(2k+9)+12}{(2k+6)(2k+9)}}{\dfrac{(2k+4)(2k+9)(A+1)-F}{(2k+6)(2k+9)}-}\cdot \quad .. \quad (22)$$

In particular it will give N_1. Also from (17) we get

$$N_1 = -\frac{F}{10}. \quad .. \quad .. \quad .. \quad .. \quad (23)$$

Hence, equating the two values of N_1 we have

$$\frac{F}{10}+\frac{\frac{12}{28}A}{\dfrac{14(A+1)-F}{28}-}\frac{\frac{30}{54}A}{\dfrac{36(A+1)-F}{54}-}\frac{\frac{56}{88}A}{\dfrac{66(A+1)-F}{88}-}\cdots = 0. \quad .. \quad (24)$$

This equation determines the values of F, which ultimately gives the value of σ, the frequency of oscillations. The coefficients a_2, a_4, a_6, ... in (16) are given by

$$a_{2r} = \prod_{1}^{r}(N_s)a_0. \quad .. \quad .. \quad .. \quad .. \quad (25)$$

Here it is to be noticed that the successive constant coefficients for the series for ξ are not obtained from the relation (18) as this would require us to start with exactly the right value of F and to observe absolute accuracy in the subsequent stages of the work, otherwise calculation will lead to the values of N_k which approximate to the limit unity (Lamb, 1916).

The approximate solution of (24) was carried out as follows. Choosing different values of F we drew a graph for the relation (23). Also for different values of F we calculated N_5 from (22) by successive convergents approximately, and thereby obtained N_1 from (21). Again we drew an other graph for these values of N_1 and F. The point of intersection of these two graphs gives the approximate solutions of (24).

We have considered the variation of Γ given by $A = 0{\cdot}1$, $0{\cdot}2$, $0{\cdot}3$, $0{\cdot}4$ and $0{\cdot}5$. The results are collected in Table I in which x_c denotes the value of x such that $\Gamma < \frac{4}{3}$ for $x_c < x < 1$, and the displacement functions for these different values of A are given below.

$$A = 0{\cdot}1,\ \xi = a_0\{1+0{\cdot}08626x^2+0{\cdot}007215x^4+0{\cdot}0006137x^6+0{\cdot}0000532x^8+\ldots\}.$$

$$A = 0{\cdot}2,\ \xi = a_0\{1+0{\cdot}1735x^2+0{\cdot}02915x^4+0{\cdot}00497x^6+0{\cdot}000865x^8+\ldots\}.$$

$$A = 0{\cdot}3,\ \xi = a_0\{1+0{\cdot}262x^2+0{\cdot}0673x^4+0{\cdot}01705x^6+0{\cdot}004459x^8+\ldots\}.$$

$$A = 0{\cdot}4,\ \xi = a_0\{1+0{\cdot}3525x^2+0{\cdot}1197x^4+0{\cdot}053197x^6+0{\cdot}01865x^8+\ldots\}.$$

$$A = 0{\cdot}5,\ \xi = a_0\{1+0{\cdot}4458x^2+0{\cdot}19092x^4+0{\cdot}08283x^6+0{\cdot}03662x^8+\ldots\}.$$

TABLE I.

A	F	$y = \frac{3\sigma^2}{2\pi G\rho \Gamma_0}$	$\bar{\Gamma} = \int_0^1 \Gamma\, dx = \Gamma_0\left(1-\frac{A}{3}\right)$	Γ_{surface}	x_c
0·1	−0·8626	0·9374	1·61	1·5	—
0·2	−1·735	0·665	1·555	1·333	1
0·3	−2·62	0·38	1·5	1·166	0·817
0·4	−3·525	0·075	1·444	1·00	0·707
0·5	−4·458	−0·258	1·3889	0·8333	0·634

The instability sets in when $x_c \approx 0{\cdot}684$ and $\bar{\Gamma} \approx 1{\cdot}43$. Whereas the instability in the homogeneous two-phase model studied by Ledoux and described in §1 sets in when $\bar{\Gamma} \approx 1{\cdot}485$. It appears that by taking the continuous variation in Γ the stability is increased in the sense that the model remains vibrationally stable for much lower value of average Γ. In fact for constant Γ throughout the model it is vibrationally stable as long as $\Gamma > \frac{4}{3}$ but even if we take a small continuous variation in Γ as in the present case, the model becomes vibrationally unstable for much higher value of average Γ, namely $\bar{\Gamma} = 1{\cdot}43$ and instability is further increased if the variation in Γ is discontinuous as in the case of Ledoux's treatment.

4. *Inverse square model.*

In this case also we see from (2) that for fundamental mode σ^2 will be definitely positive for values of A in the range $0 < A < \frac{1}{3}$. We consider the effect of higher values of A on the dynamical stability.

For this case

$$\left.\begin{aligned} \Gamma &= \Gamma_0(1-Ax^2) \\ \rho_0 &= \frac{\bar{\rho}}{3x^2} \\ P_0 &= \frac{2\pi G R^2 \bar{\rho}^2}{9} \cdot \frac{1-x^2}{x^2} \\ g_0 &= \frac{4\pi G \bar{\rho} R}{3} \cdot \frac{1}{x} \end{aligned}\right\}. \qquad (26)$$

Substituting values of g_0, ρ_0, P_0 and Γ from (26) in (4) we get with the previous notations

$$(1-x^2)(1-Ax^2)\xi'' + [2-4(A+1)x^2+6Ax^4]\frac{\xi'}{x} + [Fx^2-2\alpha_0+6Ax^4]\frac{\xi}{x^2} = 0, \qquad (27)$$

where

$$F = \frac{3\sigma^2}{2\pi G\bar{\rho}\Gamma_0}. \qquad (28)$$

We shall again restrict ourselves to the solutions which are regular and finite in the region $0 < x < 1$.

The roots of the indicial equation are

$$m = \tfrac{1}{2}[-1 \pm \sqrt{1+8\alpha_0}]. \qquad (29)$$

Consistent with our physical conditions we choose positive sign before the radical and substitute

$$\xi = x^m \sum_{k=0}^{\infty} b_{2k}\, x^{2k}, \qquad \ldots \quad (30)$$

in (27) and as before we get the relations for different constant coefficients as

$$\{(m+2)(m+3)-2\alpha_0\}b_2-\{(A+1)m(m+3)-F\}b_0 = 0, \qquad \ldots \quad (31)$$

$$\{(m+2k+2)(m+2k+3)-2\alpha_0\}b_{2k+2}-\{(A+1)(m+2k)(m+2k+3)-F\}b_{2k} + \{(m+2k-2)(m+2k+3)+6\}A\, b_{2k-2} = 0, \quad (32)$$

where $k = 1, 2, 3, \ldots$.

Putting $\dfrac{b_{2k+2}}{b_{2k}} = N_{k+1}$ we get

$$N_{k+1} = \frac{(A+1)(m+2k)(m+2k+3)-F}{(m+2k+2)(m+2k+3)-2\alpha_0} - \frac{A\{(m+2k-2)(m+2k+3)+6\}}{(m+2k+2)(m+2k+3)-2\alpha_0}\cdot\frac{1}{N_k}. \quad (33)$$

Again we see here that N_k tends to either A or unity as k tends to infinity. We have to consider the solutions for which $N_k \to A(<1)$ only.

Rewriting (33) as

$$N_k = \cfrac{A\,\dfrac{(m+2k-2)(m+2k+3)+6}{(m+2k+2)(m+2k+3)-2\alpha_0}}{\dfrac{(A+1)(m+2k)(m+2k+3)-F}{(m+2k+2)(m+2k+3)-2\alpha_0} - N_{k+1}}, \qquad \ldots \quad (34)$$

we express N_k in the form of a continued fraction.

$$N_k = \cfrac{A\,\dfrac{(m+2k-2)(m+2k+3)+6}{(m+2k+2)(m+2k+3)-2\alpha_0}}{\dfrac{(A+1)(m+2k)(m+2k+3)-F}{(m+2k+2)(m+2k+3)-2\alpha_0}-} \; \cfrac{A\,\dfrac{(m+2k)(m+2k+5)+6}{(m+2k+4)(m+2k+5)-2\alpha_0}}{\dfrac{(A+1)(m+2k+2)(m+2k+5)-F}{(m+2k+4)(m+2k+5)-2\alpha_0}-} \; \cfrac{A\,\dfrac{(m+2k+2)(m+2k+7)+6}{(m+2k+6)(m+2k+7)-2\alpha_0}}{\dfrac{(A+1)(m+2k+4)(m+2k+7)-F}{(m+2k+6)(m+2k+7)-2\alpha_0}-} \ldots \text{ etc.} \quad (35)$$

In particular obtaining N_1 from this and equating to its value from (31) we get

$$\frac{F-(A+1)m(m+3)}{(m+2)(m+3)-2\alpha_0} + \cfrac{A\,\dfrac{m(m+5)+6}{(m+4)(m+5)-2\alpha_0}}{\dfrac{(A+1)(m+2)(m+5)-F}{(m+4)(m+5)-2\alpha_0}-} \; \cfrac{A\,\dfrac{(m+2)(m+7)+6}{(m+6)(m+7)-2\alpha_0}}{\dfrac{(A+1)(m+4)(m+7)-F}{(m+6)(m+7)-2\alpha_0}-} \ldots = 0. \quad (36)$$

We solved this equation approximately as before and coefficients are obtained by the relation (25).

Here we considered the variation of Γ for $A = 0{\cdot}1$, $0{\cdot}3$, $0{\cdot}5$ and $0{\cdot}6$. The results are given in Table II. The displacement functions for these values of A are given below.

$$\left.\begin{aligned} A &= 0{\cdot}1,\ \xi = a_0 x^m\{1+0{\cdot}07869x^2+0{\cdot}006335x^4+0{\cdot}0005253x^6 \\ &\qquad\qquad +0{\cdot}0000448x^8+\ldots\}. \\ A &= 0{\cdot}3,\ \xi = a_0 x^m\{1+0{\cdot}2409x^2+0{\cdot}05932x^4+0{\cdot}01488x^6+0{\cdot}00385x^8+\ldots\}. \\ A &= 0{\cdot}5,\ \xi = a_0 x^m\{1+0{\cdot}4153x^2+0{\cdot}1754x^4+0{\cdot}07474x^6+0{\cdot}0328x^8+\ldots\}. \\ A &= 0{\cdot}6,\ \xi = a_0 x^m\{1+0{\cdot}5079x^2+0{\cdot}2606x^4+0{\cdot}1336x^6+0{\cdot}0706x^8+\ldots\}. \end{aligned}\right\} \quad (37)^*$$

TABLE II.

A	$F = \dfrac{3\sigma^2}{2\pi G\bar{\rho}\Gamma_0}$	$\bar{\Gamma} = \int_0^1 \Gamma\,dx = \Gamma_0\left(1-\dfrac{A}{3}\right)$	Γ_{surface}	x_c
0·1	2·175	1·61	1·5	—
0·3	1·2665	1·5	1·166	0·8
0·5	0·2508	1·3889	0·8333	0·63
0·6	−0·303	1·333	0·6667	0·577

The instability sets in when $x_c \approx 0{\cdot}606$ and $\bar{\Gamma} \approx 1{\cdot}364$.

A comparison of these values of x_c and $\bar{\Gamma}$ with those for the homogeneous model will show that for the same law of variation of Γ, this model is more stable than the homogeneous model.

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* *Note* :—In the similar equations of the other paper by the author quoted in the reference (*a*) the series within the brackets must be multiplied by x^m.