ON A MODIFIED DEFINITION OF RIESZ POTENTIAL AND ITS CORRESPONDENCE TO THE WENTZEL POTENTIAL.

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In a recent paper Auluck and Kothari (Jr.) (1951) (later referred to as A-K. 1) have given, in the framework of the classical electro-magnetic theory, a modified definition of the Riesz potential. The usual definition of the Riesz potential (Fremberg (1946)) gives, on analytic continuation to $\alpha = 0$, the Maxwell potential, whereas the modified definition of the Riesz potential gives the Wentzel potential, and hence appears to be more suitable in quantum electrodynamics. The object of the present paper is to discuss explicitly this correspondence between the modified Riesz potential and the Wentzel potential.

We define the metric tensor $g_{\mu\nu}$ as

$$g_{00} = 1\,;\; g_{11} = g_{22} = g_{33} = -1\,;\; g_{\mu\nu} = 0 \text{ for } \mu \neq \nu,$$

and take the velocity of light as unity. Following Dirac (1947) the scalar product of two four-vectors A_μ, B_μ (the Greek suffixes take the values 0, 1, 2, 3 and the Latin suffixes the values 1, 2, 3) is denoted by

$$[A, B] = A_\mu B^\mu = A_0B_0 - A_1B_1 - A_2B_2 - A_3B_3 = A_0B_0 - (AB)$$

where (AB) is the scalar product of the space parts of A_μ and B_μ. The length (positive) of the space part of a vector A_μ is written as $|A|$.

The modified Riesz potential (A-K. 1) is defined at a space-time point x by the following equation (which represents a Fourier expansion)

$$A^\alpha_\mu(x) = H(\alpha)\int_D k^{\alpha-2}\{A_{k\mu}e^{i[k,x]} + \bar{A}_{k\mu}e^{-i[k,x]}\}\,d^4k, \quad \dots \quad \dots \quad (1)$$

where

$$A_{k\mu} = \frac{1}{4\pi^2 i}\int_S j_\mu(z')e^{-i[k,z']}d^4z', \quad \dots \quad \dots \quad \dots \quad \dots \quad (2)$$

and

$$H(\alpha) = \frac{2}{\Gamma(\alpha/2)\Gamma(1-\alpha/2)} = \frac{2}{\pi}\sin\frac{\pi\alpha}{2}. \quad \dots \quad \dots \quad \dots \quad \dots \quad (3)$$

D is the four-dimensional domain bounded by the light-cone $k_0 > 0$, $[k,k] = 0$, $j_\mu(z')$ in (2) is the current-density four-vector and the integral in (2) is over the whole of space-time. The parameter α is arbitrary: The integral in (1) converges for $\alpha < 0$.

We first calculate the potential at any point x, due to a general charge-distribution. Substituting (2) in (1), we have

$$A^\alpha_\mu(x) = \frac{H(\alpha)}{4\pi^2 i}\int_D\int_S k^{\alpha-2}j_\mu(z')\{e^{i[k,x-z']} - e^{-i[k,x-z']}\}d^4k\,d^4z'. \quad \dots \quad (4)$$

To evaluate this we consider a typical integral

$$I^{\alpha} = \frac{H(\alpha)}{4\pi^2 i}\int_D k^{\alpha-2} e^{i[k,\, x]} d^4k. \qquad (5)$$

Transforming the integration variables from k_0, k_1, k_2, k_3 to k, K_1, K_2, K_3, where

$$k^2 = k_0^2 + K^2 \text{ and } k_i = K_i,$$

and hence

$$d^4k = \frac{kdk}{\sqrt{k^2+K^2}}\, d^3K,$$

we have

$$I^{\alpha} = \frac{H(\alpha)}{4\pi^2 i}\int_k \int_{K_i} k^{\alpha-1} e^{ix_0\sqrt{k^2+K^2} - i(Kx)} \frac{dk\, d^3K}{\sqrt{k^2+K^2}}.$$

To analytically continue this to $\alpha = 0$, we may make use of the Riemann-Liouville integral

$$\underset{\alpha\to 0}{\mathrm{Lt}}\ \alpha \int x^{\alpha-1} f(x)\, dx = f(0) \qquad (6)$$

so that

$$I^0 = \frac{1}{4\pi^2 i}\int_{K_i} e^{ix_0|K| - i(Kx)} \frac{d^3K}{|K|}.$$

Transforming the above integral into polar co-ordinates $|K|$, θ, ϕ, and integrating over ϕ, we get

$$I^0 = \frac{1}{2\pi i}\int_{|K|=0}^{\infty}\int_{\theta=0}^{\pi} e^{ix_0|K| - i|K||x|\cos\theta} |K|\, d|K| \sin\theta\, d\theta$$

$$= \frac{1}{2\pi|x|}\int_{|K|=0}^{\infty} e^{ix_0|K|}\left\{e^{-i|x||K|} - e^{i|x||K|}\right\} d|K|$$

$$= \frac{1}{2|x|}\left\{\delta(x_0-|x|) - \delta(x_0+|x|)\right\}$$

or

$$I^0 = \tfrac{1}{2}\Delta(x), \qquad (7)$$

where

$$\Delta(x) = \frac{1}{|x|}\left\{\delta(x_0-|x|) - \delta(x_0+|x|)\right\} \qquad (8)$$

is the Heisenberg delta function.

In view of (7), (4) reduces to

$$A_{\mu}^0(x) \equiv A_{\mu}(x) = \int_S \frac{j_{\mu}(z')}{|x-z'|}\left[\delta\{(x_0-z_0')-|x-z'|\} - \delta\{(x_0-z_0')+|x-z'|\}\right] d^4z' \qquad (9)$$

For the case $x_0 > z_0'$, which is of physical importance, only the first delta function contributes and on integrating with respect to z_0', we get

$$A_{\mu}(x) = \iiint \frac{j_{\mu}(x_0-|x-z'|,\, z_i')}{|x-z'|}\, d^3z', \qquad (10)$$

which is the classical retarded potential. It will be noted that for $x_0 < z_0'$, (9) yields the advanced potential.

If the current is generated by a single point-particle of charge e, we have

$$j_\mu(z') = ev_\mu \delta^4(z-z')$$

where z is the space-time co-ordinate of the particle. In this case (2) reduces to

$$A_{k\mu} = \frac{e}{4\pi^2 i}\int_{-\infty}^{\tau_0} v_\mu e^{-i[k,z]}\,d\tau, \quad \ldots \quad (11)$$

τ being the proper time of the particle, and v_μ its velocity at the point z corresponding to the proper time τ. Making use of (11) we can write (1) as

$$A_\mu^\alpha(x) = \frac{H(\alpha)e}{4\pi^2 i}\int_{-\infty}^{\tau_0} v_\mu d\tau \int_D k^{\alpha-2}\left\{e^{i[k,x-z]}-e^{-i[k,x-z]}\right\} d^4z,$$

which, when integrated over k in the same manner as shown above for the case of a general charge distribution, reduces to

$$A_\mu(x) = e\int_{-\infty}^{\tau_0} v_\mu(z)\,\Delta(x-z)\,d\tau. \quad \ldots \quad (12)$$

It can easily be seen that (11) is a solution of the equations

$$\frac{\partial^2}{\partial x_\mu\,\partial x^\mu}\cdot A_\nu(x) = 0 \quad \ldots \quad (13a)$$

and

$$\frac{\partial A_\mu(x)}{\partial x_\mu} = -e\Delta(x-z) \quad \ldots \quad (13b)$$

which define the Wentzel potentials. This shows that the modified definition of the Riesz potential, when analytically continued to $\alpha = 0$, leads to the Wentzel potential.

We shall now consider (12) in a little more detail. If the point x lies inside the future part of the light cone of z at the proper time τ, i.e. if

$$[x-z, x-z] > 0, \quad x_0 - z_0 > 0 \quad \ldots \quad (14)$$

then for this case, since the Δ-function vanishes over the entire range of integration, $A_\mu(x) = 0$. In the case of the point x lying outside the light cone of z, i.e.

for
$$[x-z, x-z] < 0 \quad \ldots \quad (15)$$

there is just one point of τ, the retarded proper time for the field point x, where Δ-function does not vanish. Therefore, in this domain, on integration, one obtains

$$A_\mu(x) = e\left|\frac{v_\mu}{[v, x-z]}\right|_{\text{ret.}} \quad \ldots \quad (16)$$

where 'ret' denotes that the value of the function is to be taken at the retarded proper time. This is the classical Lienard-Wiechert potential.

If the point x lies inside the past part of the light cone of z, i.e. if

$$[x-z, x-z] > 0, \quad x_0 - z_0 < 0, \quad \ldots \quad (17)$$

there are two values of τ for which the Δ-function does not vanish. One corresponds to the retarded proper time of and another to the advanced proper time of x. The contribution to the potential from the retarded proper time is the same as (16), while it can easily be proved that the contribution from the advanced proper time is

$$-e\left|\frac{v_\mu}{[v,\, x-z]}\right|_{\text{adv.}}$$

where 'adv' denotes that the value of the function is to be taken at the advanced proper time. Thus for the domain defined by (17)

$$A_\mu(x) = e\left|\frac{v_\mu}{[v,\, x-z]}\right|_{\text{ret.}} - e\left|\frac{v_\mu}{[v,\, x-z]}\right|_{\text{adv.}} \quad \ldots \quad (18)$$

Summing up our results we have

$$\left.\begin{aligned} A_\mu(x) &= 0 && \text{when (14) holds} \\ &= e\left|\frac{v_\mu}{[v,\, x-z]}\right|_{\text{ret.}} && \text{when (15) holds} \\ &= e\left|\frac{v_\mu}{[v,\, x-z]}\right|_{\text{ret.}} - e\left|\frac{v_\mu}{[v,\, x-z]}\right|_{\text{adv.}} && \text{when (17) holds} \end{aligned}\right\} . \quad \ldots \quad (19)$$

The potential on the world line of the electron will be given by

$$A_\mu^{\text{world line}} = \frac{e}{2}\left\{\left|\frac{v_\mu}{[v,\, x-z]}\right|_{\text{ret.}} - \left|\frac{v_\mu}{[v,\, x-z]}\right|_{\text{adv.}}\right\}, \quad \ldots \quad (20)$$

when we take the limit $x-z \to 0$. It can be shown that in the limit (20) reduces to

$$A_\mu^{\text{world line}} = e\dot{v}_\mu . \quad \ldots \quad (21)$$

This result can also be derived using Fremberg's definition for the Riesz potential.

The field tensor can be deduced from (19) by differentiation

$$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu} . \quad \ldots \quad (22)$$

The derivatives have to be taken with respect to the field point x_μ, but the quantities occurring in (19) are the functions of the co-ordinates corresponding to either the retarded or the advanced proper time. Advanced or the retarded proper times are defined by the relation

$$[x-z,\, x-z] = 0,$$

and from this it follows that

$$\frac{\partial \tau}{\partial x_\mu} = \frac{x_\mu - z_\mu}{[v,\, x-z]} . \quad \ldots \quad (23)$$

Using this relation we have

$$\left.\begin{aligned} F_{\mu\nu} &= 0 && \text{when (14) holds} \\ &= |F_{\mu\nu}|_{\text{ret.}} && \text{when (15) holds} \\ &= |F_{\mu\nu}|_{\text{ret.}} - |F_{\mu\nu}|_{\text{adv.}} && \text{when (17) holds} \end{aligned}\right\}, \quad \ldots \quad (24)$$

where

$$|F_{\mu\nu}|^{\text{ret.}}_{\text{adv.}} = e\left|\frac{(x_\mu - z_\mu)v_\nu - (x_\nu - z_\nu)v_\mu}{[v, x-z]^3}(1-[\dot{v}, x-z]) + \frac{(x_\mu - z_\mu)\dot{v}_\nu - (x_\nu - z_\nu)\dot{v}_\mu}{[v, x-z]^2}\right|^{\text{ret.}}_{\text{adv.}} \quad \ldots \quad (25)$$

The field on the world line of the electron is, therefore, given by the usual expression:

$$F^{\text{world line}}_{\mu\nu} = \tfrac{1}{2}\{|F_{\mu\nu}|_{\text{ret.}} - |F_{\mu\nu}|_{\text{adv.}}\}$$

$$= \tfrac{2}{3}e(\ddot{v}_\mu v_\nu - \ddot{v}_\nu v_\mu). \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (25)$$

Summary

It is known that the usual definition of the Riesz potential gives an analytic continuation to $\alpha = 0$, the Maxwell potential, whereas the modified definition given by Auluck and Kothari gives the Wentzel potential. In this paper the correspondence between the modified Riesz potential and the Wentzel potential is explicitly worked out.

References

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