ANHARMONIC PULSATIONS OF A POLYTROPIC MODEL OF INDEX UNITY.

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According to Rosseland the theory of anharmonic pulsations is a vital part of the pulsation theory in order to understand the form of the light and velocity curves, and it was due to him that the mathematical theory for the motion of an anharmonically pulsating gas sphere has been developed to explain the skewness of the radial velocity curve, and increase in the period of oscillation. Taking the case of a single excited mode for the homogeneous model, and retaining only the first term in the development for displacement, he was able to show that the velocity curve acquired a skewness and the period lengthened. But his results were only qualitative as it cannot account for the observed skewness and the increase in the period for comparatively small amplitude of pulsation of Cepheids. He expected that better agreement could be achieved by retaining higher order terms. But Bhatnagar and Kothari (1944), who have integrated the equation of a single excited mode for the homogeneous model for $\gamma = 5/3$, have shown that the theory of anharmonic pulsations for the homogeneous model 'cannot account for the observed skewness in the velocity-time curve of the Cepheid Variables'. Recently Schwarzschild and Savedoff (1949) have given the complete solution of the equations for the fundamental and the first mode for the standard model (polytropic index $n = 3$). Their results show that anharmonic pulsations yield practically the same period as harmonic pulsations do, though the skewness obtained is considerable yet it is smaller than that observed. Chandrika Prasad (1949) working on the same model by a different method has come to the same conclusion.

In the present paper anharmonic pulsations of a polytropic model of index unity have been studied. The results show that the effect of the first overtone gives a skewness to the velocity-time curve in the right direction, but the value obtained is smaller than that observed and the increase in the period of pulsation comes out to be slight.

The exact equation of motion for radial adiabatic pulsations of a non-rotating star, according to Rosseland (1949), is

$$\rho_0 r_0 \ddot{r}_1 = -(1+r_1)^2 \frac{\partial}{\partial r_0}\left[P_0(1+r_1)^{-2\gamma}(1+r_1+r_0 r_1')^{-\gamma}\right] + P_0'(1+r_1)^{-2}, \quad \ldots \quad (1)$$

where P_0, ρ_0 and r_0 refer to the equilibrium values of pressure, density and distance from the centre, γ is the ratio of specific heats and r_1 is the relative displacement defined by $r = r_0\,(1+r_1)$. The dashes denote differentiation with respect to r_0. Expanding the equation (1) and multiplying with $\frac{r_0^3}{\gamma}$ throughout, this equation, correct to quadratic terms, is

$$r_0^3\left[\left(\frac{1}{\gamma}\right) r_0\rho_0\ddot{r}_1 - \left(3-\frac{4}{\gamma}\right) P_0' r_1 - (r_0 P_0' + 4P_0) r_1' - r_0 P_0 r_1''\right]$$
$$+\frac{1}{2}\left[(3\gamma+1)\left(3-\frac{4}{\gamma}\right) P_0' r_0^3 r_1^2 - (3\gamma-1) P_0 r_0^4 r_1'^2 + \frac{\partial}{\partial r_0}\{(\gamma+1) P_0 r_0^5 r_1'^2\}\right.$$
$$\left. + \frac{\partial}{\partial r_0}\{(6\gamma-2) P_0 r_0^4 r_1 r_1'\}\right] = 0. \quad \ldots \quad (2)$$

This is the same equation as have been obtained by Schwarzschild and Savedoff (1949).

Following Rosseland's suggestion, the relative displacement may be written as

$$r_1 = \eta_1(r_0)q_1(t)+\eta_2(r_0)q_2(t), \quad \ldots \quad \ldots \quad (3)$$

where q_1 and q_2 are functions of the time which are to be determined, and η_1 and η_2 are chosen as the solutions for the fundamental mode and first overtone derived for small amplitudes which have been obtained in the tabular form in a previous paper by the author (1951). As the functions η_1 and η_2 are not normalised to unity at the surface of the star, the relative displacement will be $(\eta_1)_s q_1+(\eta_2)_s q_2$ where $(\eta_1)_s$ and $(\eta_2)_s$ are the values at the surface.

Multiply equation (2) firstly by η_1 and then by η_2, integrate with respect to r_0 from the centre to the boundary R of the star. Due to the orthogonality of the η-functions, linear cross-terms drop out; the following equations are obtained after performing some partial integrations:

$$\left(\frac{1}{\gamma}\right)\ddot{q}_1+\frac{\sigma_1^2}{\gamma}q_1 = \frac{1}{\int_0^R \rho_0 r_0^4\eta_1^2\,dr_0}\Bigg[-\frac{1}{2}\left(3-\frac{4}{\gamma}\right)(3\gamma+1)\int_0^R P_0' r_0^3\eta_1 r_1^2\,dr_0$$
$$+\frac{1}{2}(3\gamma-1)\int_0^R P_0 r_0^4\eta_1 r_1'^2\,dr_0+\frac{1}{2}(\gamma+1)\int_0^R P_0 r_0^5\eta_1' r_1'^2\,dr_0$$
$$+(3\gamma-1)\int_0^R P_0 r_0^4\eta_1' r_1 r_1'\,dr_0\Bigg] \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

$$\left(\frac{1}{\gamma}\right)\ddot{q}_2+\frac{\sigma_2^2}{\gamma}q_2 = \frac{1}{\int_0^R \rho_0 r_0^4\eta_2^2\,dr_0}\Bigg[-\frac{1}{2}\left(3-\frac{4}{\gamma}\right)(3\gamma+1)\int_0^R P_0' r_0^3\eta_2 r_1^2\,dr_0$$
$$+\frac{1}{2}(3\gamma-1)\int_0^R P_0 r_0^4\eta_2 r_1'^2\,dr_0+\frac{1}{2}(\gamma+1)\int_0^R P_0 r_0^5\eta_2' r_1'^2\,dr_0$$
$$+(3\gamma-1)\int_0^R P_0 r_0^4\eta_2' r_1 r_1'\,dr_0\Bigg], \quad \ldots \quad \ldots \quad \ldots \quad (5)$$

where $\sigma_1 = \frac{2\pi}{\Pi_0}$, Π_0 being the period of the fundamental mode.

On substituting the value of r_1 from (3) on the right-hand side of equations (4) and (5) and arranging the terms in various products of the q's, we get

$$\frac{d^2q_1}{d\tau^2}+q_1 = \frac{1}{N_1\omega_1^2}\left[Aq_1^2+2Bq_1q_2+Cq_2^2\right] \quad \ldots \quad \ldots \quad \ldots \quad (6)$$

$$\frac{d^2q_2}{d\tau^2}+\frac{\omega_2^2}{\omega_1^2}q_2 = \frac{1}{N_2\omega_1^2}\left[Bq_1^2+2Cq_1q_2+Dq_2^2\right]; \quad \ldots \quad \ldots \quad (7)$$

here the independent variable t is replaced by

$$\tau = \frac{2\pi}{\Pi_0}t,$$

where Π_0 is the period of the fundamental mode for small amplitude, the coefficients ω_1 and ω_2 are the eigen-values corresponding to η_1 and η_2 in the pulsation problem for small amplitudes (Chatterji, 1951). The remaining six numerical constants are defined by the following equations:—

$$\left.\begin{aligned}
N_1 &= \int_0^R \theta r_0^4 \eta_1^2 \, dr_0. \qquad N_2 = \int_0^R \theta r_0^4 \eta_2^2 \, dr_0 \\
A &= -\left(3-\frac{4}{\gamma}\right)(3\gamma+1)\int_0^R \theta\theta' r_0^3 \eta_1^3 \, dr_0 + \frac{1}{2}(9\gamma-3)\int_0^R \theta^2 r_0^4 \eta_1 \eta_1'^2 \, dr_0 \\
&\quad + \frac{1}{2}(\gamma+1)\int_0^R \theta^2 r_0^5 \eta_1'^3 \, dr_0. \\
B &= -\left(3-\frac{4}{\gamma}\right)(3\gamma+1)\int_0^R \theta\theta' r_0^3 \eta_1^2 \eta_2 \, dr_0 + \frac{1}{2}(3\gamma-1)\int_0^R \theta^2 r_0^4 \eta_1'^2 \eta_2 \, dr_0 \\
&\quad + (3\gamma-1)\int_0^R \theta^2 r_0^4 \eta_1 \eta_1' \eta_2' \, dr_0 + \frac{1}{2}(\gamma+1)\int_0^R \theta^2 r_0^5 \eta_1'^2 \eta_2' \, dr_0. \\
C &= -\left(3-\frac{4}{\gamma}\right)(3\gamma+1)\int_0^R \theta\theta' r_0^3 \eta_1 \eta_2^2 \, dr_0 + \frac{1}{2}(3\gamma-1)\int_0^R \theta^2 r_0^4 \eta_1 \eta_2'^2 \, dr_0 \\
&\quad + (3\gamma-1)\int_0^R \theta^2 r_0^4 \eta_1' \eta_2 \eta_2' \, dr_0 + \frac{1}{2}(\gamma+1)\int_0^R \theta^2 r_0^5 \eta_1' \eta_2'^2 \, dr_0 \\
D &= -\left(3-\frac{4}{\gamma}\right)(3\gamma+1)\int_0^R \theta\theta' r_0^3 \eta_2^3 \, dr_0 + \frac{1}{2}(9\gamma-3)\int_0^R \theta^2 r_0^4 \eta_2 \eta_2'^2 \, dr_0 \\
&\quad + \frac{1}{2}(\gamma+1)\int_0^R \theta^2 r_0^5 \eta_2'^3 \, dr_0
\end{aligned}\right\} \quad \cdots \quad (8)$$

where θ is the Emden variable for the polytrope $n = 1$, and its values are taken from the *Mathematical Tables*, Vol. 2, of the British Association for the Advancement of Science (1932). In equation (8) the integrals have been taken from the centre to the surface of the star, and the dashes denote derivatives with respect to r_0. These integrals have been evaluated for the polytrope $n = 1$ taking $\gamma = 5/3$.

The values of the six constants are found to be as follows:

$$\begin{aligned}
N_1 &= 14{\cdot}766659 & N_2 &= 3{\cdot}483853 \\
A &= 11{\cdot}392750 & B &= -0{\cdot}863641 \\
C &= 14{\cdot}927878 & D &= -24{\cdot}594280
\end{aligned}$$

With these numerical values, equations (6) and (7) become

$$\frac{d^2 q_1}{d\tau^2} + q_1 = 3{\cdot}33991 q_1^2 - {\cdot}50637 q_1 q_2 + 4{\cdot}37627 q_2^2 \quad \cdots \quad (9)$$

$$\frac{d^2 q_2}{d\tau^2} + 6{\cdot}5671 q_2 = -1{\cdot}07315 q_1^2 + 37{\cdot}0985 q_1 q_2 - 30{\cdot}56063 q_2^2 \quad \cdots \quad (10)$$

In order to solve these equations, they are put in the form

$$\frac{d^2q_1}{d\tau^2}+q_1 = A_1q_1^2+2B_1q_1q_2+C_1q_2^2 \quad \ldots \quad (11)$$

$$\frac{d^2q_2}{d\tau^2}+\beta_2q_2 = A_2q_1^2+2B_2q_1q_2+C_2q_2^2 \quad \ldots \quad (12)$$

where $\beta_2 = 6{\cdot}5671$. $A_1 = 3{\cdot}33991$, $B_1 = -0{\cdot}253186$, $C_1 = 4{\cdot}37627$,
$A_2 = -1{\cdot}07315$, $B_2 = 18{\cdot}54925$, $C_2 = -30{\cdot}56063$.

Solutions of these equations are to be sought which are periodic with the same period for q_1 and q_2. Let

$$q_1 = a_0+a_1 \cos n\tau+a_2 \cos 2n\tau+a_3 \cos 3n\tau+ \quad \ldots \quad (13)$$

$$q_2 = b_0+b_1 \cos n\tau+b_2 \cos 2n\tau+b_3 \cos 3n\tau+ \quad \ldots \quad (14)$$

where $a_0, a_1, \ldots, b_0, b_1, \ldots$, and n are constants to be found out.

These values of q_1 and q_2 are substituted in the equations (11) and (12) and all the products of the cosines are expressed as sums of the cosines. Then putting the constant term and the coefficients of $\cos kn\tau$ for different k, separately equal to zero, the following equations are obtained :—

$$(k^2n^2-1)a_k+A_1\left[\frac{1}{2}\sum_{i=0}^{k} a_ia_{k-i}+\sum_{i=0}^{\infty} a_ia_{k+i}\right] +B_1\left[\sum_{i=0}^{k} a_ib_{k-i}+\sum_{i=0}^{\infty}(a_ib_{k+i}+a_{k+i}b_i)\right] +C_1\left[\frac{1}{2}\sum_{i=0}^{k} b_ib_{k-i}+\sum_{i=0}^{\infty} b_ib_{k+i}\right] = 0. \quad \ldots \quad (15)$$

$$(k^2n^2-\beta_2)b_k+A_2\left[\frac{1}{2}\sum_{i=0}^{k} a_ia_{k-i}+\sum_{i=0}^{\infty} a_ia_{k+i}\right] +B_2\left[\sum_{i=0}^{k} a_ib_{k-i}+\sum_{i=0}^{\infty}(a_ib_{k+i}+a_{k+i}b_i)\right] +C_2\left[\frac{1}{2}\sum_{i=0}^{k} b_ib_{k-i}+\sum_{i=0}^{\infty} (b_ib_{k+i})\right] = 0. \quad \ldots \quad (16)$$

These infinite number of infinite equations are solved by successive approximations by a method as given by Chandrika Prasad (1949). The value of $a_0, b_0, b_1, a_2, b_2, \ldots$, are obtained in terms of a_1.

A solution for the polytrope $n = 1$ of the equations (11) and (12) is found out by choosing $a_1 = 0{\cdot}06$, so as to make the surface amplitude equal to 0·08, that is 8% of the stellar radius, which is of the order observed. The solution is found to be :—

$$q_1 = 0{\cdot}006148+0{\cdot}06 \cos n\tau-0{\cdot}002076 \cos 2n\tau+0{\cdot}000052 \cos 3n\tau, \quad \ldots \quad (17)$$

$$q_2 = -0{\cdot}000373-0{\cdot}000398 \cos n\tau-0{\cdot}000803 \cos 2n\tau+0{\cdot}000298 \cos 3n\tau, \quad \ldots \quad (18)$$

where $n^2 = 0{\cdot}966161.$

If ζ denotes the surface amplitude due to the fundamental mode and the first overtone, then

$$\zeta = (\eta_1)_s q_1 + (\eta_2)_s q_2 .$$

Here Chatterji (1951) $(\eta_1)_s = 1{\cdot}218532$ and $(\eta_2)_s = -1{\cdot}251806.$

Therefore, $\zeta = 0{\cdot}007959 + 0{\cdot}073610 \cos n\tau - 0{\cdot}001525 \cos 2n\tau - 0{\cdot}000310 \cos 3n\tau.$ (19)

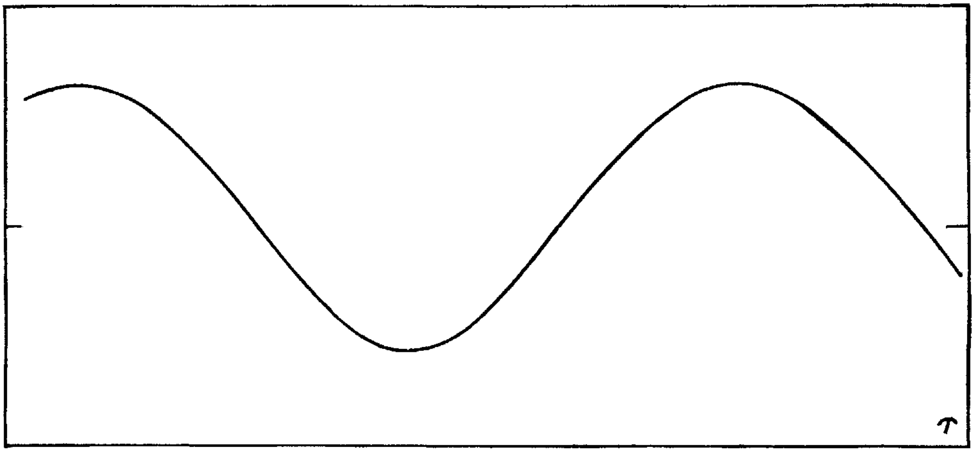

Fig. 1. Displacement curve for the polytrope $n = 1$.

The shape of the displacement curve obtained from this equation is shown in the diagram. If K (skewness) is the ratio of the time of decline from maximum to minimum to the time of rise to maximum, then K for the above curve is 1·176. The increase in the period is found to be slight, being only 1·74%.

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Summary.

The anharmonic pulsations of a polytropic model of index unity are investigated. It is found that the first overtone gives a skewness to the radial velocity curve. The value comes out to be small when compared with the observed value, and the increase in the period is found to be slight.

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