ADIABATIC PULSATIONS OF A PARTICULAR MODEL OF THE VARIABLE STAR.

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Introduction.

Eddington, Edgar (1933), Schwartzschild (1941) and others have considered the modes of small radial adiabatic oscillations of the standard model (Polytrope of index 3). Eddington's results show a period of oscillation which is short of the observed value by a factor near about 2. This fact points to the lower central concentration in actual Cepheids than the one given by the standard polytrope.

After this people tried different laws of variation of density. Dr. Sterne (1937) considered the small radial adiabatic oscillations for the following three models:

(1) The homogeneous density distribution throughout the star.
(2) A model in which the density varies inversely as the square of the distance from the centre of the star.
(3) A model with nearly all the mass concentrated at the centre and in the rest of the star the density varies inversely as the square of the distance from the centre.

Very recently Professor A. C. Banerji (1942) has given an entirely new and very interesting theory of the origin of solar system based on the instability of large radial oscillations of the following two models:

(1) The star with homogeneous density distribution.
(2) The star with small homogeneous core and the density in the envelope varying inversely as the p-th power of the distance from the centre, where p is any positive integer excluding 1 and 3.

This work of Prof. Banerji is very important as it throws an important light on the density concentration in the Cepheid variables. Dr. H. K. Sen (1942) also considered the stability of radial oscillations of similar models.

In the models where density varies inversely as some power of the distance from the centre we have singularity at the centre. To avoid this Prof. Banerji took a small central core. Another way to do it was to take the law for variation of density as $\rho_0 = \rho_c\left(1-\frac{r_0^n}{R^n}\right)$, where the notations have their usual meaning. Analytically the approximate solutions of the pulsation equation could be obtained (1949) only for $n = 2$, for which the model was found to be stable.

Knowing that the model is stable for homogeneous density distribution and also for the law $\rho_0 = \rho_c\left(1-\frac{r_0^2}{R^2}\right)$, it can very well be expected that for the law $\rho_0 = \rho_c\left(1-\frac{r_0}{R}\right)$, the model must be stable. The author has considered this model and found correct to our expectation.

Equations of the problem.—The differential equation for small adiabatic radial oscillations of a star as obtained by Eddington is

$$\frac{d^2\xi}{dr_0^2}+\frac{4-\mu}{r_0}\frac{d\xi}{dr_0}+\left\{\frac{\rho_0\sigma^2}{\gamma P_0}-\frac{\alpha\mu}{r_0^2}\right\}\xi=0, \quad \ldots \quad (1)$$

where

$$\frac{\delta r}{r_0}=\xi(r_0)\,e^{i\sigma t}, \quad \ldots \quad (2)$$

$$\alpha=3-\frac{4}{\gamma}, \quad \ldots \quad (3)$$

$$\mu=\frac{g_0\rho_0 r_0}{P_0}, \quad \ldots \quad (4)$$

and r_0, ρ_0, P_0, g_0, are the equilibrium values of the distance from the centre, density, pressure and gravity at a point. Here the boundary conditions are

$$\delta P=-\gamma P_0\left(3\xi+r_0\frac{d\xi}{dr_0}\right)=0 \text{ at } r_0=R, \quad \ldots \quad (5)$$

and ξ is finite everywhere in the region $0 < r < R$

For this model putting $x=\frac{r_0}{R}$ we have

$$\rho_0=\rho_c(1-x), \quad \ldots \quad (6)$$

$$g_0=4\pi G\rho_c Rx\left(\frac{1}{3}-\frac{x}{4}\right), \quad \ldots \quad (7)$$

$$P_0=4\pi G\rho_c^2R^2\left(\frac{5}{144}-\frac{x^2}{6}+\frac{7x^3}{36}-\frac{x^4}{16}\right). \quad \ldots \quad (8)$$

Substituting these values in (1) we get

$$x(1-x)(5+10x-9x^2)\frac{d^2\xi}{dx^2}+(20+20x-124x^2+72x^3)\frac{d\xi}{dx}$$
$$+\{f-12\alpha(4-3x)\}x\xi=0, \quad \ldots \quad (9)$$

where

$$f=\frac{36\sigma^2}{G\pi\gamma\rho_c}. \quad \ldots \quad (10)$$

The equation (9) has singularities at $x=0$ and $x=1$. We have to find the solution which is finite in the region $0 < x < 1$.

If we substitute the series

$$\xi=x^m\sum_{n=0}^{\infty}a_n x^n, \quad \ldots \quad (11)$$

in (9) we find that the roots of the indicial equation are zero and -3. To satisfy the physical conditions of the problem we have to choose the former one. The recurrence formula for the coefficients of the series is

$$(9n^2+27n-90+36\alpha)a_{n-2}-(19n^2+67n-86-f+48\alpha)a_{n-1}$$
$$+(5n^2+15n)a_n+5(n^2+5n+4)a_{n+1}=0. \quad \ldots \quad (12)$$

If the $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = l$

we find that this limit l will be the root of the equation

$$9-19l+5l^2+5l^3 = 0 \quad .. \quad .. \quad .. \quad .. \quad (13)$$

Obviously its roots are 1, 0·7 and −2·7. For $l = 1$ and −2·7 the series will be divergent. Only for $l = 0{\cdot}7$ it will be convergent. But the difficulty here comes in is how to get that value of f for which the limit is 0·7. This difficulty is overcome by Rayleigh-Ritz method (1941) for finding the approximate period for any oscillatory stellar model. After getting the approximate value of f, the pulsation equation has been integrated numerically to see actually if the amplitude remains finite throughout the star and also at the surface.

Process of integration:

Putting

$$\xi = \Psi e^{-\frac{1}{2}\int \frac{72x^3-124x^2+20x+20}{x(1-x)(5+10x-9x^2)}dx}, \quad .. \quad .. \quad .. \quad (14)$$

the differential equation reduces to

$$\frac{d^2\Psi}{dx^2}+I\Psi = 0, \quad .. \quad .. \quad .. \quad .. \quad (15)$$

where

$$I = \frac{f-48\alpha+36\alpha x}{(1-x)(5+10x-9x^2)} - \frac{50+150x-780x^2-920x^3+2376x^4-972x^5}{x^2(1-x)(5+10x-9x^2)^2}. \quad (16)$$

Equation (14) gives

$$\Psi = \xi x^2(1-x)\sqrt{5+10x-9x^2}. \quad .. \quad .. \quad .. \quad (17)$$

Put (15) in the form

$$\frac{d^2\Psi}{dx^2} = F(x), \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad (18)$$

where

$$F(x) = \xi x^2(1-x)\sqrt{5+10x-9x^2}\left\{\frac{50+150x-780x^2-920x^3+2376x^4-972x^5}{x^2(1-x)(5+10x-9x^2)^2} - \frac{f-48\alpha+36\alpha x}{(1-x)(5+10x-9x^2)}\right\}. \quad .. \quad (19)$$

To solve the equation (18) we apply Adam's method of integration as sketched by Von Zeipel in one of his papers (1924). Here the equidistant intervals were taken for x at a distance 0·02 ($\omega = 0{\cdot}02$). The difference formula

$$\Psi''(n\omega) = \omega^2F(n\omega)+\frac{\omega^2}{12}F''(n\omega)-\frac{\omega^2}{240}F^{iv}(n\omega), \quad .. \quad .. \quad (20)$$

helps us to integrate the equation (18) leading successively to the knowledge of the values of $\Psi(x)$ for $x = (n+1)\omega$, $(n+2)\omega$, $(n+3)\omega$, $(n+4)\omega$, .. etc.

Here ω being very small we neglected the difference $F^{iv}(n\omega)$ and by the process of extrapolation an approximate value of $\frac{\omega^2}{12}F''(n\omega)$ was obtained. By method of checking a more accurate value of $\frac{\omega^2}{12}F''(n\omega)$ was found. Hence the value of $\Psi(n\omega)$ was calculated with the knowledge of $\Psi(\overline{n-1}\omega)$ determined previously. Consequently the values of the amplitude were also calculated. These values are given in the table as ξ and Ψ for two values of f ($f = 13{\cdot}2$ and $f = 13{\cdot}3$).

TABLE.

$x = \frac{r_0}{R}$	For $f = 13{\cdot}2$		For $f = 13{\cdot}3$	
	$\Psi = \xi x^2(1-x)\sqrt{5+10x-9x^2}$	ξ/a_0	$\Psi = \xi x^2(1-x)\sqrt{5+10x-9x^2}$	ξ/a_0
0·000	0·0	1	0·0	1
0·02	0·0008936966	1·00012164	0·0008936959	1·00012085
0·04	0·003566278	1·000474985	0·003566262	1·00047184
0·06	0·00799314	1·00104482	0·007993087	1·0010378
0·08	0·01413525	1·00181845	0·01413507	1·001806
0·10	0·02194038	1·0027855	0·02193996	1·002766
0·12	0·03134454	1·0039435	0·03134367	1·00391524
0·14	0·04227641	1·00536267	0·042274803	1·005324083
0·16	0·0546862	1·0076337	0·05468351	1·0075834
0·18	0·06844554	1·0098323	0·068441167	1·0097676
0·20	0·08344842	1·0120081	0·0834418	1·01192796
0·22	0·099584507	1·014231	0·09957493	1·014134
0·24	0·11673625	1·016309	0·11672286	1·01619
0·26	0·13477926	1·018959	0·13476106	1·018822
0·28	0·15358292	1·021495	0·153568799	1·0214015
0·30	0·17301099	1·024162	0·17299967	1·024095
0·32	0·19292197	1·026963	0·19291206	1·0269106
0·34	0·2131696	1·029899	0·2131595	1·029857
0·36	0·23360324	1·03298	0·2335912	1·032933
0·38	0·25468324	1·036214	0·25405238	1·036149
0·40	0·27440668	1·0395909	0·27438465	1·039507
0·42	0·29445696	1·043123	0·29442643	1·043012
0·44	0·31405495	1·046805	0·31401324	1·046667
0·46	0·333033973	1·050649	0·33297824	1·050474
0·48	0·35122524	1·054657	0·35115241	1·054438
0·50	0·368458186	1·0588335	0·36836494	1·0585656
0·52	0·38456077	1·0631809	0·38444364	1·062857
0·54	0·399360005	1·0677077	0·3992152	1·0673206
0·56	0·4126821	1·072418	0·4125058	1·071959
0·58	0·42435300	1·0773194	0·42414102	1·0767813

(TABLE—Continued).

$x = \frac{r_0}{R}$	For $f = 13{\cdot}2$		For $f = 13{\cdot}3$	
	$\Psi = \xi x^2(1-x)\sqrt{5+10x-9x^2}$	ξ/a_0	$\Psi = \xi x^2(1-x)\sqrt{5+10x-9x^2}$	ξ/a_0
0·000	0·0	1	0·0	1
0·60	0·4341987	1·082417	0·4339468	1·081789
0·62	0·4420456	1·087722	0·4417493	1·086993
0·64	0·44772097	1·093242	0·4473757	1·092399
0·66	0·45105334	1·098983	0·450655	1·097972
0·68	0·451873007	1·104961	0·45141595	1·103843
0·70	0·45001242	1·1111829	0·449492	1·109899
0·72	0·4453067	1·117666	0·44471923	1·116191
0·74	0·43759433	1·124418	0·43693485	1·122724
0·76	0·4267174	1·131459	0·4259817	1·129508
0·78	0·41252254	1·138806	0·4117053	1·13654
0·80	0·3948615	1·146482	0·39395914	1·143862
0·82	0·3735917	1·154496	0·3726033	1·151441
0·84	0·34857742	1·1628887	0·34750046	1·1592959
0·86	0·319690388	1·171689	0·31852458	1·167416
0·88	0·2868117	1·18094	0·28555855	1·17578
0·90	0·24983002	1·190689	0·24849362	1·18432
0·92	0·20864555	1·201022	0·2072337	1·192896
0·94	0·163170492	1·212093	0·161696887	1·201147
0·96	0·113330443	1·22432	0·111818356	1·207999
0·98	0·059064002	1·239302	0·05755772	1·207697
1·00	0·0003196	$+\infty$	$-0{\cdot}0010773$	$-\infty$

CONCLUSION.

The value of ξ from the value $\Psi(x)$ is calculated by the equation (17) from which it is clear that at $x = 1$, Ψ must tend to zero. But due to rough approximations it is impossible to get actually the value zero. Hence we see that for $f = 13{\cdot}2$, Ψ is positive at the surface giving $\xi = +\infty$ there, and for $f = 13{\cdot}3$ we get $\Psi(x)$ negative making $\xi = -\infty$. Hence we can quite safely conclude that the value of f which will make $\Psi = 0$ and consequently give a finite value for the displacement at the surface, must lie between these above two values. We therefore conclude that for the fundamental mode the value of f satisfying the equation (9) and keeping ξ finite at the surface is positive. Thus the radial oscillations for our model are proved to be stable. As we know that it is always the fundamental mode which becomes unstable first we can say in the present case that all the higher modes will also be stable. So we see here that the results obtained do not betray our expectation that a stellar model which lies between two stable models must also be stable.

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Summary.

The stability of a model in which the density varies in the interior of a star according to the law $\rho_0 = \rho_c\left(1-\frac{r_0}{R}\right)$, where ρ_0 is the density at any point, ρ_c the density at the centre, r_0 the equilibrium value of the distance of the point from the centre and R the radius of the star, has been studied in this paper. It is found to be stable, and the period of oscillation and the amplitude of displacement at the different points inside the star are obtained by numerical integration.

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