A NOTE ON THE TRANSITION FROM VISCOUS TO PERFECT FLUID FLOW

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INTRODUCTION.

Elegant solutions of the potential motion of a perfect fluid past a solid obstacle have long been known. It is also well-known that the Navier-Stokes equations of motion of a viscous fluid reduce to the Eulerian equations of motion of a perfect fluid in the limit when μ, the coefficient of viscosity, tends to zero. One might therefore ask if a viscous shear flow ($\mu \neq 0$) can reduce to a potential flow (for $\mu = 0$) in the limit when μ ultimately vanishes. In other words, if we concieve of a series of similar flow-models with gradually diminishing viscosity, is it likely that for the fluids for which viscosity is vanishingly small the flow pattern is very nearly the stream-line perfect-fluid pattern of classical hydrodynamics? One prima facie objection seems to lie in the fact that the boundary conditions in the two cases are entirely (qualitatively) different and as such no such transition is obvious.

It is being gradually realised that the impossibility of a continuous transition from the one type to the other through $\mu \to 0$ is a fact (Hpoff, 1950; Truesdell, 1950) but the reason does not lie in the difference in the boundary conditions for the two cases. On the contrary the impossibility of the transition, except through the state of no motion, is inherent in the very nature of the Navier Stokes equations. In the present note we have sought to establish this point, for the steady two-dimensional motion of a viscous incompressible fluid.

THE SOLUTIONS OF THE TWO-DIMENSIONAL NAVIER-STOKES EQUATIONS.

The two-dimensional steady motion of a viscous incompressible fluid is represented by

$$\psi_y \nabla^2 \psi_x - \psi_x \nabla^2 \psi_y = \nu \nabla^4 \psi, \qquad \text{(1.1)}$$

where ψ stands for the stream-function and ν represents the coefficient of kinematic viscosity, (μ/ρ). Using

$$\zeta = \nabla^2 \psi \qquad \text{(1.2)}$$

we write (1.1) as

$$\psi_y \zeta_x - \psi_x \zeta_y = \nu \nabla^2 \zeta. \qquad \text{(1.1a)}$$

ζ being $\zeta(x,y)$ in general, we can, for the moment, regard (1.1*a*) as a first order differential equation in ψ and if

$$\zeta \neq 0,/\text{const.}, \qquad \text{(1.3)}$$

obtain the allied equation as

$$\frac{dy}{\zeta_x} = \frac{-dx}{\zeta_y} = \frac{d\psi}{\nu \nabla^2 \zeta}. \qquad \text{(1.4)}$$

One obvious integral of this equation being

$$\zeta = \text{const.}, \qquad (1.5)$$

another integral of (1.4) is obtained as

$$\psi = \nu \int\limits_{\zeta = \text{const.}} \frac{\nabla^2 \zeta}{\zeta_x} \, dy, \qquad (1.6)$$

where the integration is carried along any curve $\zeta = \text{const.}$ (c). Now, it is well-known that the general solution for ψ is obtained by replacing the constant c by the variable ζ after integration and adding any arbitrary function of ζ. Thus, we have, generally,

$$\psi = \nu U(\zeta, y) + F(\zeta), \qquad (1.7)$$

(where the meaning of $U(\zeta, y)$ is obvious) except in the case where ζ either vanishes or is a constant everywhere. [If $\zeta_x = 0$, we can write a similar integral in terms of ζ and x.] Thus we see that the solutions of the equation (1.1) fall into two different sets. The first set corresponds to the two conditions represented by the equations (1.3) and (1.2) and relates to the cases where the motion is either irrotational ($\zeta = 0$) or the vorticity is uniform throughout ($\zeta = \text{const.}$).

The second set corresponds to the two conditions represented by the equations (1.7) and (1.2) and relates to the cases where the vorticity is variable everywhere and is neither zero nor uniform over any finite region. Thus the two sets of solutions must be mutually exclusive.

Now if we suppose that we start with a motion represented by (1.7) [and, of course, with (1.2)] and proceed to the limit $\mu \to 0$, we shall naturally arrive at the type of motion represented by

$$\left.\begin{aligned} \psi &= F(\zeta) \\ \zeta &= \nabla^2 \psi \end{aligned}\right\} \qquad (1.8)$$

where ζ is neither a constant nor zero, but must be variable in every part of the fluid previously undergoing the second type of motion described above.

Thus we conclude that a viscous shear flow with a non-uniform vorticity cannot reduce to a potential flow, or for the matter of that, a flow with uniform vorticity in the limit when μ, the coefficient of viscosity tend to zero.

It would be useful to point out that the solutions corresponding to the first set do not depend upon μ explicitly and hence the question of taking such solutions to the limit $\mu \to 0$ does not arise.

2. The Geometrical Analysis.

It is possible to arrive at the same conclusion by making an explicit use of the condition of no-slip on the boundary. We shall illustrate this point by considering the case of a fixed circular cylinder in a flow where the velocity at infinity is U.

We know, that in the case of a *perfect fluid* streaming past the circular cylinder $r = a$ with the centre at origin the stream function is given by

$$\psi = Uy\left(1 - \frac{a^2}{r^2}\right). \qquad (2.1)$$

The stream-lines for the flow, are well-known, and the flow is characterised by the following features.

(i) The stream-lines are symmetrical about both axes and the flow on the upper half of the x-y plane gives the flow on the lower half by its reflection on the axis of x. Hence for the purpose of a description it is enough to consider only the flow on the upper half.

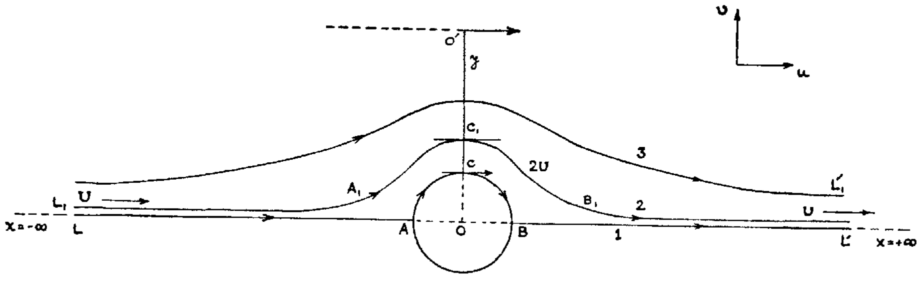

FIG. 1

(ii) At all points on the y-axis, $v = 0$ and u changes continuously from its maximum value $2U$ on the surface of the body at C (Fig. 1) to U at infinity (O').

(iii) Supposing we follow the stream-line $\psi = 0$ ($LACBL'$ in Fig. 1) from the point $L(-\infty, 0)$ to the point $L'(+\infty, 0)$, u diminishes continuously from U at $-\infty$ to 0 at A, the forward stagnation point, and $v = 0$ throughout this range: then, as we proceed along the cylindrical surface u, v both increase but by the time we reach C v vanishes again and u attains the maximum value $2U$. The same course is repeated backwards from C to B but v is negative throughout this range whereas u continues to be positive everywhere. At B again both u and v vanish and then whereas v remains zero, u increases continually to U at $L'(+\infty)$.

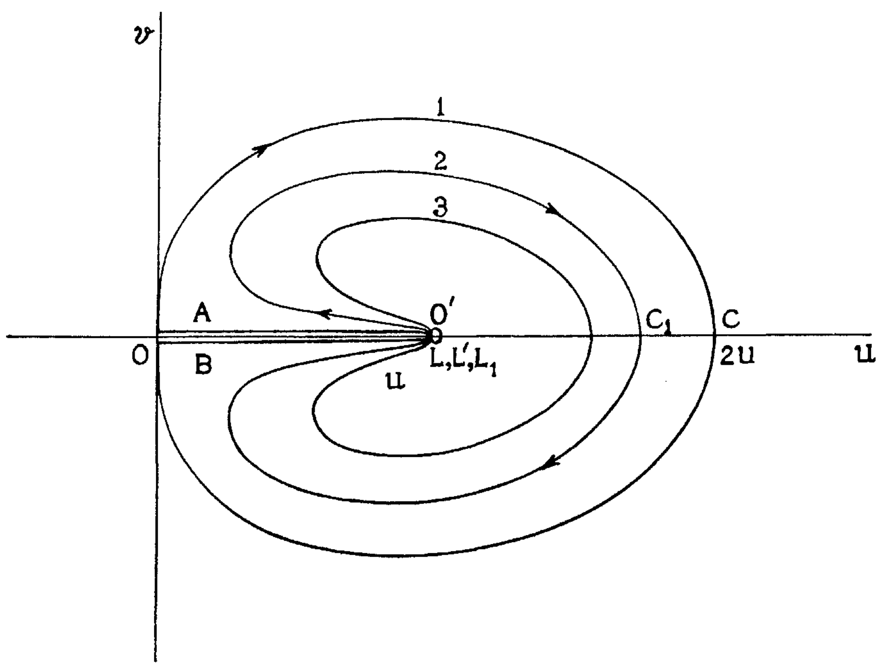

FIG. 2

(iv) A similar course is followed along any other stream-line $L_1A_1C_1B_1L_1'$ with u attaining its maximum value ($< 2U$) at C_1 and never vanishing anywhere; v vanishes only at $-\infty$, $+\infty$ and C_1. We now consider that the same phenomenon is represented in the u-v plane and we represent the stream-lines 1, 2, 3, of the x-y plane on this u-v plane by the lines 1, 2, 3. This result is roughly shown in Fig. 2.

The essential features are (Fig. 2).

(i) The entire motion on the upper half of the x-y plane is represented on the right half $u > 0$ of the u-v plane.

(ii) The line $\psi = 0$ starts from $O'(U, 0)$, moves to the origin ($u = 0$, $v = 0$) corresponding to A, runs along the arc $O\,1\,C$ and gets back to the origin for B and thence moves along the u-axis to O' again.

(iii) Every other stream-line starts from the same point O' and completes the loop at O' again.

(iv) The entire flow in the x-y plane is confined in the u-v plane within the curve $O'O\,1\,COO'$ corresponding to $\psi = 0$ and symmetrical about the u-axis.

(v) All stream-lines are heart-shaped loops touching each other at their common navel $O'(U, 0)$, the stream-line at infinity reducing itself to a point-curve about O'. Now, we consider that if the liquid instead of being perfect were a viscous one it should have had the following characteristics.

(1) Symmetry about the x-axis, as in Fig. 1.

(2) Symmetry about the y-axis, (at least far away from the body).

(3) The velocity at infinity is, still U everywhere.

(4) The velocity along the stream-line $\psi = 0$ changes, as before, between $-\infty$ and A but $u = 0$, $v = 0$ along the whole arc from A to B through C (Fig. 1). Hence whereas the body-surface in the perfect-fluid motion is represented by the loop $O\,1\,CO$ (Fig. 2) in the u-v plane, the same should reduce to merely the origin ($u = 0$, $v = 0$) in the case of the viscous flow. This means that the hodograph-plane representation of the viscous flow must be entirely outside the trace of the body in the same plane whereas in the perfect-fluid motion it is totally inside the same.

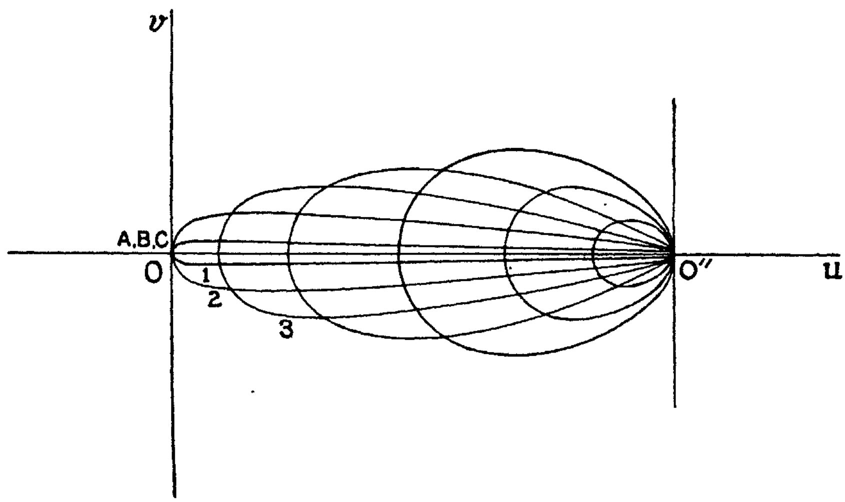

FIG. 3

(5) The velocity at all points on $x = 0$ (where $v = 0$) between C and $(O, +\infty)$ should now gradually increase from O to U and as such, the motion in the u, v

plane should be confined between $O < u < U$. Also v must have a finite maximum somewhere; assuming that it is $\pm V$ we can say that the motion must be confined within $O < u < U$, $-V < v < V$.

The stream-line at infinity is a pt. loop round $O'(U, O)$ (Fig. 3), the other lines also forming loops all touching at O' again. As, in this case, the surface of the body need not be a stream-line, much will depend on the disposition of the stream-line through C. If it coincides with $\psi = 0$ of the perfect fluid motion, i.e. it coincides with the surface of the body, there will be an obvious indeterminacy in drawing the stream-lines close to the body in the u-v plane, for the expanding and flattening loops (Fig. 3) must stop flattening somewhere, contract sidewise and get back to merge into the st. line $O'OO''$ (Fig. 3).

On the other hand, if this stream-line, i.e., the one through C, does not coincide with the surface of the body entirely (Fig. 4), say it is $PQCRS$. Then the loops

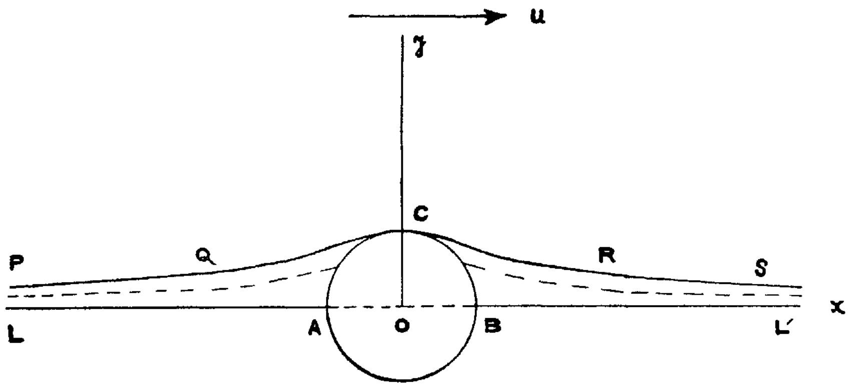

FIG. 4

expand until the largest one corresponding to $PQRS$ cut the u-axis at the origin perpendicularly. (Fig. 5). But in such a case the entire fluid between $PQRS$ (in

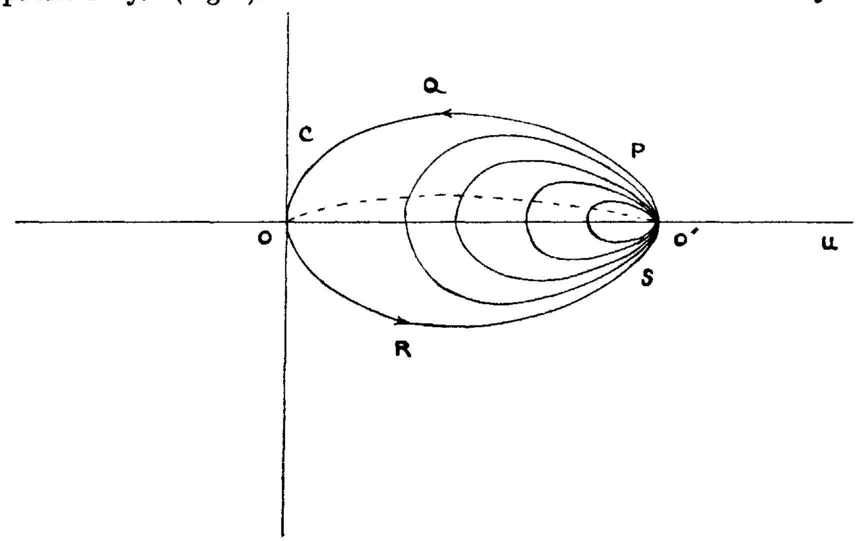

FIG. 5

the x-y plane) and the axis of y will perforce be rendered stagnant, relative to the body.

We have brought in this elaborate description to point out, incidentally, the lack of uniqueness in the adjustment of the no-slip condition.

However, the plain fact stands that whereas in the perfect fluid motion, the flow in the u-v plane is entirely confined within the trace of the boundary in the u-v plane, the no-slip condition throws the flow in the u-v plane outside the body and it is not possible to see how one system can be transformed into the other by a process of continuous transition.

ABSTRACT.

Observation of real fluid motions are admitted to reveal potential motion away from a solid obstacle. Prandl explained this on the hypothesis that away from the body viscosity does not come into play appreciably and the fluid behaves as a perfect one. This paper seeks to investigate the type of motion that is left when a viscous fluid with initial rotational motion ultimately loses its viscosity. It proves conclusively that a non-potential motion cannot reduce to a potential motion in the limit when $\mu \to 0$, a result that was not exactly unknown.

Some geometrical evidence has been brought to bear upon the same point, by considering the motion in the hodograph plane, i.e. the (u, v) plane.

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