NOTE ON A CLASS OF EXACT SOLUTIONS OF THE TWO-DIMENSIONAL FLOW PROBLEM FOR A VISCOUS INCOMPRESSIBLE FLUID.

By DR. N. L. GHOSH, *Research Fellow, N.I.S.I., Department of Applied Mathematics, Calcutta University.*

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PART I.

INTRODUCTION.

Problems on steady viscous liquid flow have been solved for a few cases where the non-linearity of the equations practically disappear as in the case of motion between two parallel plates or Poiseuille motion or motion between two cylinders, etc., mostly known as the Couette motions. Two other known exact solutions without any approximating assumptions are those of Hammel and Kármán (Goldstein, 1950) when the non-linear inertia-force terms exist. Stokes exact solutions apply to motions only very slow and Oseen's approximations are, after all, approximations and even as such are not free from defects (Lamb, 1932). The author, interested in the problem of motion of a viscous liquid past a solid of finite dimensions, has noted that the Couette types are only particular cases of a more general form. This present note seeks to construct such types of exact solutions. In part I of this note it has been shown that in the case where the vorticity is a function of the stream-function ψ_1, of the corresponding potential motion, it is possible to construct a general expression for the stream-function ψ, which will satisfy the boundary conditions. All flows, however, will not admit such types of solutions. The Couette motions on the other hand are all particular cases of this type.

Part 2 constructs the types of flow where the stream-function is a function of the same ψ_1 only. Some simple applications, by way of illustration, follow.

1. *The fundamental Equation.*

The stream-function for the steady two-dimensional Navier-Stokes equation for a liquid is given by

$$\psi_y \nabla^2\psi_x - \psi_x \nabla^2\psi_y = \nu \nabla^4\psi, \qquad \ldots \quad \ldots \quad \ldots \quad (1.1)$$

where the suffixes denote the corresponding partial derivatives and

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}.$$

We know, in such a case, the vorticity

$$\zeta \equiv \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}$$

is given by

$$\zeta = \nabla^2\psi, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1.2)$$

and hence (1.1) can be written as

$$\psi_y \zeta_x - \psi_x \zeta_y = \nu \nabla^2 \zeta. \qquad (1.3)$$

2. *Transformations.*

Confining our attention to those problems of specified boundaries, where the flow may extend to infinity if necessary, let us suppose that the corresponding potential flow of a perfect fluid with the same boundaries is known. Let ϕ_1, ψ_1, represent the velocity potential and the stream-function respectively for such a flow and let q_1 denote the magnitude of the velocity at any point due to such a flow. Then, we know that

$$q_1^2 = \psi_{1x}^2 + \psi_{1y}^2 = \phi_{1x}^2 + \phi_{1y}^2$$

$$= |\nabla \psi_1|^2 \text{ or } |\nabla \phi_1|^2 \qquad (2.1)$$

and

$$\nabla^2 \phi_1 = \nabla^2 \psi_1 = 0. \qquad (2.2)$$

Now, the two equations giving the values of ϕ_1, ψ_1 at any point (x, y) in the plane of flow, may be solved for x and y to give

$$x = x(\phi_1, \psi_1); \quad y = y(\phi_1, \psi_1). \qquad (2.3)$$

In other words, instead of the Cartesian co-ordinates (x, y) we can use the system of curvilinear co-ordinates (ϕ_1, ψ_1) and they are quite suitable for the purpose because we know that the system of curves $\phi_1 = \text{const.}$ and $\psi_1 = \text{const.}$ are orthogonal at any point and $\dfrac{\partial(\phi_1, \psi_1)}{\partial(x, y)} \neq 0$.

With such a system it can be easily shown that

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = q_1^2 \left(\frac{\partial^2}{\partial \phi_1^2} + \frac{\partial^2}{\partial \psi_1^2} \right)$$

i.e.,

$$\nabla^2 = q_1^2 \nabla_1^2, \qquad (2.4)$$

where

$$\nabla_1^2 = \frac{\partial^2}{\partial \phi_1^2} + \frac{\partial^2}{\partial \psi_1^2} \qquad (2.4a)$$

Hence, from (1.2)

$$\zeta = q_1^2 \nabla_1^2 \psi. \qquad (2.5)$$

Now, the basic equation (1.3) when transformed by (2.3) takes the form

$$\frac{\partial \psi}{\partial \psi_1} \cdot \frac{\partial \zeta}{\partial \phi_1} - \frac{\partial \psi}{\partial \phi_1} \cdot \frac{\partial \zeta}{\partial \psi_1} = \nu \nabla_1^2 \zeta. \qquad (2.6)$$

3. *Solutions.*

We consider now that type of motion which is characterized by

$$\zeta = f(\psi_1). \qquad (3.1)$$

Obviously, (2.6) reduces to

$$-\frac{\partial \psi}{\partial \phi_1} \cdot f'(\psi_1) = \nu f''(\psi_1). \qquad (3.2)$$

This equation can obviously be integrated partially with respect to ϕ_1 giving

$$\psi = -\nu \frac{f''(\psi_1)}{f'(\psi_1)} \cdot \phi_1 + F(\psi_1), \qquad (3.3)$$

where $F(\psi_1)$ is any arbitrary function so far. The relation (3.3) is of course subject to the condition (1.2) which, in the new system of co-ordinates, has reduced to (2.5).

Putting

$$\frac{f''(\psi_1)}{f'(\psi_1)} = \xi(\psi_1) \qquad (3.4)$$

we have (3.3) as

$$\psi = -\nu\xi(\psi_1) \,.\, \phi_1 + F(\psi_1). \qquad (3.5)^*$$

From (3.5) it is clear that

$$\frac{\partial^2 \psi}{\partial \phi_1^2} = 0$$

and hence we have

$$\nabla_1^2 \psi = -\nu\xi''(\psi_1)\phi_1 + F''(\psi_1)$$

and hence from (2.5) and (3.1) we must have

$$q_1^2\{-\nu\xi''(\psi_1)\phi_1 + F''(\psi_1)\} = \zeta = f(\psi_1). \qquad (3.6)$$

Thus, for a given $f(\psi_1)$, $F(\psi_1)$ has to be determined from (3.6), provided a solution exists.

Equation (3.6) can be written as

$$\frac{1}{q_1^2} = -\nu \frac{\xi''(\psi_1)}{f(\psi_1)} \,.\, \phi_1 + \frac{F''(\psi_1)}{f(\psi_1)}$$

and hence generally it is of the form

$$\frac{1}{q_1^2} = A(\psi_1)\phi_1 + B(\psi_1). \qquad (3.7)$$

Now q_1^2 is assigned by the specifications of the boundary and may not be of the form (3.7); in such a case, the method cannot be applied.

If, however, q_1^2 admits of the form (3.7) formula (3.5) will represent the stream function if the value of $F(\psi_1)$ as given by (3.6) is substituted in (3.5).

Admitting such a possibility we notice that the boundary condition can be very easily adjusted.

For instance, the usually accepted boundary conditions are (Kampé de Fériét, 1948) $\psi = 0$ and $\Delta\psi = 0$ on the surface of the body.

Now, as, in the potential motion, ψ_1 vanishes on the boundary of the body, we may define the boundary in the viscous problem by $\psi_1 = 0$,
and hence, the boundary conditions must be

$$\psi = 0, \;\; \Delta\psi = 0 \;\text{ on }\; \psi_1 = 0. \qquad (3.8)$$

These are satisfied if

$$\left.\begin{aligned} \xi(0) = 0, \;\; F(0) = 0 \\ \text{and} \qquad \xi'(0) = 0, \;\; F'(0) = 0 \end{aligned}\right\} \qquad (3.9)$$

* A formula of the type (3.5) was arrived at independently by Prof. R. Ballav in course of his investigations on superposability, as reported in the Indian Science Congress, 1952.

Conditions (3.9) however can be relaxed. For the only physical condition rigorously necessary for a solution is that the velocity must vanish on the boundary. This is secured entirely if

$$\xi(0)=0,\quad \xi'(0)=(0),\quad F'(0)=0.$$

4. *Couette Motions.*

It may be easily seen that all Couette motions belong to the type for which $q_1^2 = q_1^2(\psi_1)$. This is a particular case of equation (3.7) where $A(\psi_1)$ vanishes. We shall discuss by way of illustration, some flows of this type at the end of part 2 of this note.

PART II.

5. With the same system of co-ordinates (ϕ_1, ψ_1) as in Art. 2 we now suppose that ψ, the stream-function for the viscous flow, is given by

$$\psi = f(\psi_1). \qquad \dots \quad \dots \quad \dots \quad \dots \quad (5.1)$$

We shall take the equation of motion to be given by (2.6) and (2.5) as before; but in this case (2.5) reduces to

$$\zeta = q_1^2 f''(\psi_1) \qquad \dots \quad \dots \quad \dots \quad \dots \quad (5.2)$$

and hence

$$\nabla_1^2\zeta = q_1^2 f^{\text{iv}}(\psi_1) + 2\,\frac{\partial q_1^2}{\partial\psi_1}\cdot f'''(\psi_1) + f''(\psi_1)\cdot\nabla_1^2 q_1^2, \qquad \dots \quad (5.2a)$$

where q_1^2 and ∇_1^2 are given, as before, by (2.1) and (2.4a). With (5.2a), (2.6) reduces to

$$f'(\psi_1)f''(\psi_1)\cdot\frac{\partial q_1^2}{\partial\phi_1} = \nu\left\{q_1^2 f^{\text{iv}}(\psi_1) + 2\,\frac{\partial q_1^2}{\partial\psi_1}\cdot f'''(\psi_1) + f''(\psi_1)\cdot\nabla_1^2 q_1^2\right\}. \qquad \dots \quad \dots \quad (5.3)$$

Integrability of the equation (5.3).

It may be noted that, in general, q_1^2 can be regarded as a function of both ϕ_1 and ψ_1. The function f being purely a function of ψ_1, it becomes apparent that the validity of the equation (5.3) for all values of ϕ_1 and ψ_1 will impose certain restrictions on q_1^2. These restrictions will really specify the types of flow which may admit of a solution of the form (5.1). For, for a given set of boundary specifications, the potential motion of the perfect-fluid flow is uniquely determined (Dirischlet flow) and so is q_1^2. When the value of q_1^2 thus obtained makes (5.3) integrable, the corresponding solution may be obtained. If not, there is no solution of the type for the assigned specifications.

It is obvious that when q_1^2 is either a constant or a function of ψ_1 alone, the equation (5.3) can always be integrated and hence costruction of a solution of the type (5.1) is always possible in these cases, provided the boundary conditions are satisfied by the resulting function.

6. *The Boundary Conditions.*

As before, we shall take the boundary of the solid to be defined by $\psi_1 = 0$ and hence for the no-slip condition (3.8) must hold. Now in this case, from (5.1), we have

$$\nabla\psi = f'(\psi_1).\,\nabla\psi_1, \qquad \dots \quad \dots \quad \dots \quad \dots \quad (5.4)$$

and hence, as $|\nabla\psi_1| \neq 0$ on $\psi_1 = 0$

$$f'(0) = 0. \qquad (5.4a)$$

If in addition, we consider that the fluid extends to infinity and flows there with a uniform velocity U, $|\nabla\psi_1| \to U$ when $\psi_1 \to \infty$. Hence, in order that the same condition may hold for the viscous flow we must have,

$$f'(\infty) = 1. \qquad (5.4b)$$

Thus, a function f, satisfying (5.3) and the conditions (5.4a) and (5.4b), (provided the integrability condition for the corresponding q_1^2 is satisfied) will lead to an exact solution of the viscous flow problem.

It has been noticed that the viscous flow past a circular cylinder does not admit of a solution of the type (5.1); for the corresponding q_1^2 for the potential flow does not render (5.3) integrable.

We shall discuss briefly, by way of illustration, two particular cases, the solutions for which are already well-known.

6. *Case* (1): $q_1^2 = U^2 =$ a const.

The viscous problem corresponding to this case is that of the motion of a viscous liquid past a plane wall, say $y = 0$. The stream function for the corresponding potential motion is given by $\psi_1 = Uy$ and hence (5.1) makes ψ a function of y alone. As, further, q_1^2 is a constant (5.3) reduces to

$$f^{\mathrm{iv}}(\psi_1) \equiv f^{\mathrm{iv}}(y) = 0, \qquad (6.1)$$

the general solution of which is

$$\psi = a_0 + a_1 y + a_2 y^2 + a_3 y^3. \qquad (6.2)$$

Now, as, $y = 0$ when $\psi_1 = 0$ and $y \to \infty$ when $\psi_1 \to \infty$, the boundary conditions (5.4a) and (5.4b) reduce in this case to

$$\psi(0) = 0, \; \psi'(\infty) = 1. \qquad (6.3)$$

The latter condition cannot obviously hold and hence the solution cannot apply to a flow extending to infinity. We shall therefore consider only the case of flow between two planes $y = 0$ and $y = h$. We may think of two cases; (i) when both planes are fixed and (ii) when one is fixed and the other moving with a uniform velocity V.

In general, from (6.2)

$$u = \frac{\partial\psi}{\partial y} = a_1 + b_1 y + c_1 y^2, \quad \text{say.} \qquad (6.4)$$

Case (*i*). The boundary conditions are

$$u = 0 \quad \text{when} \quad y = 0 \quad \text{and} \quad y = h.$$

Hence the solution is

$$u = c_1 y(y-h). \qquad (6.5)$$

Case (*ii*). If $y = 0$ be fixed and $y = h$ moving with a uniform velocity V, we must have

$$\left.\begin{aligned} u &= 0 \quad \text{on} \quad y = 0 \\ u &= V \quad \text{on} \quad y = h \end{aligned}\right\} \qquad (6.6)$$

and hence (6.4) gives

$$u = \frac{Vy}{h} + \frac{c_1 y(y-h)}{h}, \quad \ldots \quad (6.7)$$

quite a well-known result. [Lamb, (1932), p. 583.]

7. *Case 2.* $q_1^2 \equiv q_1^2(\psi_1)$: *Motion round a circular cylinder.*

When a perfect fluid undergoes a motion of circulation round a circular cylinder, $r = a$, with the velocity vanishing at infinity, we know,

$$\psi_1 = k \log \frac{r}{a}. \quad \ldots \quad (7.1)$$

Hence,

$$q_1^2 = |\nabla \psi_1|^2 = \left(\frac{k}{a}\right)^2 e^{-2\psi_1/k} \quad \ldots \quad (7.2)$$

and so,

$$\frac{\partial}{\partial \phi_1} q_1^2 = 0. \quad \ldots \quad (7.2a)$$

The surface of the cylinder is given by $r = a$ and, hence, by

$$\psi_1 = 0, \quad \ldots \quad (7.3)$$

and hence, the *inner* boundary condition for the viscous flow, must be

$$f'(0) = 0. \quad \ldots \quad (7.4)$$

Now putting,

$$q_1^2 = Q(\psi_1) \quad \ldots \quad (7.5)$$

and

$$f''(\psi_1) = P(\psi_1), \quad \ldots \quad (7.6)$$

we have from (5.3)

$$\frac{d^2}{d\psi_1^2}(PQ) = 0, \quad \ldots \quad (7.7)$$

which gives, at once,

$$PQ = A + B\psi_1; \quad \ldots \quad (7.8)$$

and so,

$$f''(\psi_1) = (C + D\psi_1)e^{2\psi_1/k}, \quad \ldots \quad (7.9)$$

where

$$\left.\begin{aligned} C &= \left(\frac{a}{k}\right)^2 . A \\ D &= \left(\frac{a}{k}\right)^2 B. \end{aligned}\right\} \quad \ldots \quad (7.10)$$

Integrating (7.9) and applying (7.4) we have

$$f'(\psi_1) = \frac{k}{2} e^{\frac{2\psi_1}{k}} \left\{C + D\left(\psi_1 - \frac{k}{2}\right)\right\} - \frac{k}{2}\left\{C - \frac{k}{2} D\right\}. \quad \ldots \quad (7.11)$$

Now, the velocity at any point, ψ_1,

$$|\nabla\psi_1| = |f'(\psi_1)| \cdot |\nabla\psi_1| = \frac{k}{a} \cdot e^{-\frac{\psi_1}{k}} \cdot f'(\psi_1), \quad .. \quad .. \quad (7.12)$$

from (7.2), and hence we notice that the velocity at infinity can neither vanish nor tend to a finite limit, and so the motion considered cannot extend to infinity.

On the other hand, for appropriate values of C and D the solution can be fitted to the case of fluid rotating between two circular cylinders $r = a$, $r = b$ $(a < b)$, the inner of which is fixed and the outer rotating with a finite angular velocity. It is also possible to adjust the solution to the case when the inner cylinder is rotating and the outer is fixed but in such a case the boundary condition (7.4) has to be modified accordingly.

These cases have been treated otherwise so thoroughly and simply (Lamb, 1932, p. 588) that it will serve no useful purpose to go into the details of their discussion.

Abstract.

This paper contains two parts. In part I an exact solution of the boundary value problem for the two-dimensional viscous fluid motion has been formulated in the special case when the vorticity is a function of the stream function ψ_1, of the corresponding potential motion with the same boundary, alone. It has been found that such special cases are restricted by a particular form for the square of the velocity in the potential motion. The exact solutions without approximating hypothesis that have been so far obtained, viz., the Couette motions—are however covered by this general investigation.

In the second part, investigations have been carried on into another type of cases—the class of problems where the stream-function ψ of the viscous motion is, again, a function of ψ_1, of the corresponding potential motion, alone. The ordinary differential equation obtained is of the fourth order and generally non-linear and carries its own limitations of applicability. Two examples from the known Couette motions have been appended to illustrate its possible utility.

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